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ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF THE DIFFERENTIAL EQUATION OF THE FOURTH ORDER II

JOZEF MIKLO

In paper [7] the asymptotic behaviour of solutions of the linear differential equation of the fourth order of the form

\( y^{(iv)} + p(t) y'' + q(t) y' - (-1)^m r(t) y = 0, \quad m = 1, 2 \)

was investigated, where the functions \( p(t) \), \( q(t) \) and \( r(t) \) were supposed continuous and continuously differentiable to the order which stands in the Theorems and \( r(t) > 0 \) on the interval \([a, \infty)\).

In the paper presented an asymptotic behaviour of solutions of the equation of the form

\( y^{(iv)} + p(t) y'' - (-1)^m q(t) y' + r(t) y = 0, \quad m = 1, 2 \)

is studied, where the functions \( p(t) \), \( q(t) \) and \( r(t) \) have the same properties as in the equation (1) but \( q(t) > 0 \) is supposed instead of \( r(t) > 0 \).

Eight new asymptotic formulae for the linear differential equation of the fourth order are shown. The results in this paper generalize the results in [8].

Theorem 8.1 in [1], p. 92 (in [7] as Theorem I) and Corollary in [2] (in [7] as Theorem II) will be applied in this paper.

The equation (2) is equivalent to the system of linear differential equations of the first order

\( z'(t) = A(t) z(t), \)

where

\[
A(t) = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
- r(t) & (-1)^m q(t) & -p(t) & 0 \\
\end{pmatrix}
\]

and \( z(t) = (y(t), y'(t), y''(t), y'''(t))^T \).

Let \( T(t) = \text{diag}[q(t), q^{2/3}(t), q^{1/3}(t), 1] \) and let

\[ z(t) = T^{-1}(t) \mathbf{w}(t). \]
If \( z(t) \) is substituted in (3), then the system (3) has the form

\[
W(t) = [A_0q^1(t) + A_1r(t)q^{-1}(t) + A_2p(t)q^{-1,3}(t) + A_3q'(t)q^{-1}(t)] W(t),
\]

where \( A_3 = \text{diag}[1, 2/3, 1/3, 0], A_0 = (a_{ij}), A_1 = (b_{ij}) \) and \( A_2 = (c_{ij}) \) are matrices of the fourth degree such that \( a_{12} = a_{23} = a_{34} = 1, a_{42} = (-1)^m \) and all the others \( a_{ij} = 0; b_{ij} = 0 \) for \( i \neq 4, j \neq 1, b_{41} = -1; c_{ij} = 0 \) for \( i \neq 4, j \neq 3 \) and \( c_{41} = -1 \).

Let \( \int_a^t q^{1/3}(t) \, dt = \infty \), then the function \( s = \sigma(t) = \int_a^t q^{1/3}(u) \, du \) is defined on the interval \([a, \infty)\) and has an inverse function \( t = \alpha(s) \) defined on the interval \([0, \infty)\). By substituting \( t = \alpha(s) \) the system (4) has the form

\[
x'(s) = [A_0 + A_1f(s) + A_2g(s) + A_3h(s)] x(s),
\]

where

\[
x(s) = W(\alpha(s)), f(s) = r(\alpha(s))q^{-4,3}(\alpha(s)),
\]

\[
g(s) = p(\alpha(s))q^{-2/3}(\alpha(s)), h(s) = q'(\alpha(s))q^{-4,3}(\alpha(s)).
\]

In order to apply Theorem I (see [1], p. 92 or [7]) the system (5) will be considered in the form

\[
x'(s) = (A_0 + V(s) + R(s)) x(s).
\]

There are the following alternatives

(A1) \( V(s) = A_1f(s) + A_2g(s) + A_3h(s) \) and \( R(s) = 0 \),

(A2) \( V(s) = A_1f(s) + A_2g(s) \) and \( R(s) = A_3h(s) \),

(A3) \( V(s) = A_1f(s) + A_3h(s) \) and \( R(s) = A_2g(s) \),

(A4) \( V(s) = A_1f(s) \) and \( R(s) = A_2g(s) + A_3h(s) \),

(A5) \( V(s) = A_2g(s) + A_3h(s) \) and \( R(s) = A_1f(s) \),

(A6) \( V(s) = A_2g(s) \) and \( R(s) = A_1f(s) + A_3h(s) \),

(A7) \( V(s) = A_3h(s) \) and \( R(s) = A_1f(s) + A_2g(s) \),

(A8) \( V(s) = 0 \) and \( R(s) = A_1f(s) + A_2g(s) + A_3h(s) \).

The following designations will be used in Theorems of this paper

\[
E(t, t_0) = \exp\left[ -(-1)^m \int_{t_0}^t r(u)q^{-1}(u) \, du \right],
\]

\[
E_{1h}(t, t_0) = \exp\left[ - \int_{t_0}^t \left[ \mu_0 q^{1/3}(u) - \frac{(-1)^m}{3} (\mu_0^2 p(u)q^{-1/3}(u) + r(u)q^{-1}(u)) \right] \, du \right],
\]

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\[ E_{2k}(t, t_0) = \exp \left[ - \int_0^t \left( \mu_k q^{1/3}(u) - \frac{(-1)^m}{3} r(u) q^{-1}(u) \right) \, du \right], \]
\[ E_{3k}(t, t_0) = \exp \left[ - \int_0^t \left( \mu_k q^{1/3}(u) - \frac{(-1)^m}{3} \mu_k^2 p(u) q^{-1/3}(u) \right) \, du \right], \]
\[ E_{4k}(t, t_0) = \exp \left[ - \int_0^t \mu_k q^{1/3}(u) \, du \right], \]

where \( \mu_k, k = 1, 2, 3, 4 \) are the roots of the characteristic equation \( \mu^4 - (-1)^m \mu = 0, m = 1, 2 \) of the matrix \( A_0 \) and \( p_k = (1, \mu_k, \mu_k^2, \mu_k^3)^T \) are the characteristic vectors of the matrix \( A_0 \).

The symbol \( \mathcal{L}[a, \infty) \) will refer to the set of all complex-valued functions which are Lebesgue integrable on the interval \([a, \infty)\).

Applying Theorem 1 to the system (6) eight asymptotic formulae for the solutions of the equation (2) will be obtained.

**Theorem 1.** Let \( q''(t) q^{-4/3}(t), p'(t) q^{-2/3}(t), p^2(t) q^{-1}(t), r'(t) q^{-4/3}(t) \) and \( r(t) q^{-7/3}(t) \) be in \( \mathcal{L}[a, \infty) \). Then there is a fundamental system of solutions \( y_k(t), k = 1, 2, 3, 4 \) of the equation (2) and a number \( t_0 \geq a \) such that

\[ \lim_{t \to \infty} y_1(t) E(t, t_0) = 1, \quad \lim_{t \to \infty} y_1^{(0)}(t) q^{-4/3}(t) E(t, t_0) = 0, \quad j = 1, 2, 3 \]

(F1)

\[ \lim_{t \to \infty} y_k^{(0)}(t) q^{(2-j)/3}(t) E_{1k}(t, t_0) = \mu_k^j, \quad k = 2, 3, 4, \quad j = 0, 1, 2, 3. \]

If in addition it is supposed that \( q'(t) q^{-1}(t) \) is in \( \mathcal{L}[a, \infty) \), then there is a fundamental system of solutions \( y_k(t), k = 1, 2, 3, 4 \) of the equation (2) and a number \( t_0 \geq a \) such that

\[ \lim_{t \to \infty} y_1(t) q(t) E(t, t_0) = 1, \quad \lim_{t \to \infty} y_1^{(0)}(t) q^{(3-j)/3}(t) E(t, t_0) = 0, \quad j = 1, 2, 3, \]

(F2)

\[ \lim_{t \to \infty} y_k^{(0)}(t) q^{(3-j)/3}(t) E_{1k}(t, t_0) = \mu_k^j, \quad k = 2, 3, 4, \quad j = 0, 1, 2, 3. \]

**Theorem 2.** Let \( q''(t) q^{-4/3}(t), r'(t) q^{-4/3}(t), r^2(t) q^{-1/3}(t) \) and \( p(t) q^{-1/3}(t) \) be in \( \mathcal{L}[a, \infty) \). Then there is a fundamental system of solutions \( y_k(t), k = 1, 2, 3, 4 \) of the equation (2) and a number \( t_0 \geq a \) such that

\[ \lim_{t \to \infty} y_1(t) E(t, t_0) = 1, \quad \lim_{t \to \infty} y_1^{(0)}(t) q^{-4/3}(t) E(t, t_0) = 0, \quad j = 1, 2, 3, \]

(F3)

\[ \lim_{t \to \infty} y_k^{(0)}(t) q^{(2-j)/3}(t) E_{2k}(t, t_0) = \mu_k^j, \quad k = 2, 3, 4, \quad j = 0, 1, 2, 3. \]

If in addition it is supposed that \( q'(t) q^{-1}(t) \) is in \( \mathcal{L}[a, \infty) \), then there is a
fundamental system of solutions $y_k(t)$, $k = 1, 2, 3, 4$ of the equation (2) and a number $t_0 \geq a$ such that

$$\lim_{t \to \infty} y_1(t) q(t) E(t, t_0) = 1, \quad \lim_{t \to \infty} y_1(t) q^{(3-j)/3}(t) E(t, t_0) = 0, \quad j = 1, 2, 3,$$

$$\lim_{t \to \infty} y_k^{(j)} q^{(3-j)/3}(t) E_{2k}(t, t_0) = \mu_k^j, \quad k = 2, 3, 4, \quad j = 0, 1, 2, 3.$$

**Theorem 3.** Let $q''(t) q^{-4/3}(t)$, $p'(t) q^{-2/3}(t)$, $p^2(t) q^{-1}(t)$ and $r(t) q^{-1}(t)$ be in $\mathcal{L}[a, \infty)$. Then there is a fundamental system of solutions $y_k(t)$, $k = 1, 2, 3, 4$ of the equation (2) and a number $t_0 \geq a$ such that

$$\lim_{t \to \infty} y_1(t) q(t) = 1, \quad \lim_{t \to \infty} y_1(t) q^{-j/3}(t) = 0, \quad j = 1, 2, 3,$$

$$\lim_{t \to \infty} y_k^{(j)} q^{(2-j)/3}(t) E_{2k}(t, t_0) = \mu_k^j, \quad k = 2, 3, 4, \quad j = 0, 1, 2, 3.$$

If in addition it is supposed that $q'(t) q^{-1}(t)$ is in $\mathcal{L}[a, \infty)$, then there is a fundamental system of solutions $y_k(t)$, $k = 1, 2, 3, 4$ of the equation (2) and a number $t_0 \geq a$ such that

$$\lim_{t \to \infty} y_1(t) = 1, \quad \lim_{t \to \infty} y_1(t) q^{(3-j)/3}(t) = 0, \quad j = 1, 2, 3,$$

$$\lim_{t \to \infty} y_k^{(j)} q^{(3-j)/3}(t) E_{3k}(t, t_0) = \mu_k^j, \quad k = 2, 3, 4, \quad j = 0, 1, 2, 3.$$

**Theorem 4.** Let $q''(t) q^{-4/3}(t)$, $r(t) q^{-1}(t)$ and $p(t) q^{-1/3}(t)$ be in $\mathcal{L}[a, \infty)$. Then there is a fundamental system of solutions $y_k(t)$, $k = 1, 2, 3, 4$ of the equation (2) and a number $t_0 \geq a$ such that

$$\lim_{t \to \infty} y_1(t) = 1, \quad \lim_{t \to \infty} y_1(t) q^{-j/3}(t) = 0, \quad j = 1, 2, 3,$$

$$\lim_{t \to \infty} y_k^{(j)} q^{(2-j)/3}(t) E_{4k}(t, t_0) = \mu_k^j, \quad k = 2, 3, 4, \quad j = 0, 1, 2, 3.$$

**Theorem 5.** Let $q'(t) q^{-1}(t)$, $r(t) q^{-1}(t)$ and $p(t) q^{-1/3}(t)$ be in $\mathcal{L}[a, \infty)$ and

$$\int_a^\infty q^{1/3}(t) \, dt = \infty.$$

Then there is a fundamental system of solutions $y_k(t)$, $k = 1, 2, 3, 4$ of the equation (2) and a number $t_0 \geq a$ such that

$$\lim_{t \to \infty} y_1(t) q(t) = 1, \quad \lim_{t \to \infty} y_1(t) q^{(3-j)/3}(t) = 0, \quad j = 1, 2, 3,$$

$$\lim_{t \to \infty} y_k^{(j)} q^{(3-j)/3}(t) E_{4k}(t, t_0) = \mu_k^j, \quad k = 2, 3, 4, \quad j = 0, 1, 2, 3.
Proof of Theorem 1. In this case all assumptions of Theorem I are satisfied. (The proof of this fact is analogous to the proof of Theorem 1 in [7]). Then there are four linearly independent solutions $x_k(s)$ of the system (6) in the case of the alternative (A1) and a number $s_0 \geq 0$ such that

$$\lim_{s \to \infty} x_k(s) \exp \left[ - \int_{s_0}^s \lambda_k(u) \, du \right] = p_k,$$

where $\lambda_k(s), k = 1, 2, 3, 4$ are the roots of the characteristic equation

$$(7) \quad \lambda^4 + a_1 \lambda^3 + a_2 \lambda^2 + a_3 \lambda + a_4 = 0$$

of the matrix $A_0 + V(s)$, where

$$a_1 = -2h(s), \quad a_2 = \frac{11}{9} h^2(s) + g(s),$$

$$a_3 = -\frac{2}{9} h^3(s) - \frac{5}{3} h(s) g(s) - (-1)^m,$$

$$a_4 = \frac{2}{3} h^2(s) g(s) - (-1)^m h(s) + f(s), \quad m = 1, 2.$$

Similarly as in [7] it can be proved that the roots $\lambda_k(s)$ of the equation (7) can be expressed in the form

$$\lambda_1(s) = h(s) + (-1)^m f(s) + \gamma_1(s),$$

$$\lambda_k(s) = \mu_k + \frac{1}{3} [h(s) - (-1)^m (\mu_k^2 g(s) + f(s))] + \gamma_k(s),$$

$k = 2, 3, 4$, where $f(s) \to 0$, $g(s) \to 0$, $h(s) \to 0$ and $\gamma_k(s) \to 0$ as $s \to \infty$ and $\gamma_k(s)$ is in $L^0[0, \infty)$. Then

$$\lim_{s \to \infty} x_1(s) \exp \left[ - \int_{s_0}^s (h(u) + (-1)^m f(u) + \gamma_1(u)) \, du \right] = p_1$$

$$\lim_{s \to \infty} x_k(s) \exp \left[ - \int_{s_0}^s \left( \mu_k + \frac{1}{3} (h(u) - (-1)^m (\mu_k^2 g(u) + f(u))) + \gamma_k(u) \right) \, du \right] = p_k, \quad k = 2, 3, 4,$$

Denoting $\exp \left[ \int_{s_0}^{s \to \infty} \gamma_k(s) \, ds \right] = B_k, \quad k = 1, 2, 3, 4$ and putting $a(u) = v$, i.e., $u = \omega(v)$ and $du = q^{1/3}(v) \, dv$, $u \in [s_0, s]$, $v \in [t_0, t]$, (8) may be written as
\[ \lim_{t \to \infty} w_k(t) q^{-1}(t) E(t, t_0) = p_k B_k q^{-1}(t_0), \quad k = 2, 3, 4, \]

where the functions \( w_k(t), k = 1, 2, 3, 4 \) are solutions of the system (4).

Since \( w(t) = T(t) z(t) \) and the system (3) is linear there are linearly independent solutions \( z_k(t), k = 1, 2, 3, 4 \) of (3) such that

\[ \lim_{t \to \infty} T(t) z_k(t) q^{-1}(t) E(t, t_0) = p, \]

(9)

\[ \lim_{t \to \infty} T(t) z_k(t) q^{-1/3}(t) E_{1k}(t, t_0) = p_k, \quad k = 2, 3, 4. \]

Substituting \( T(t) = \text{diag}(q(t), q^{2/3}(t), q^{1/3}(t), 1) \) and

\[ z_k(t) = (y_k(t), y'_k(t), y''_k(t), y'''_k(t))^T, \quad k = 1, 2, 3, 4 \]

in (9), becomes

\[ \lim_{t \to \infty} \text{diag}(y_1(t), y'_1(t) q^{-1/3}(t), y''_1(t) q^{-2/3}(t), y'''_1(t) q^{-1}(t)) E(t, t_0) = (1, 0, 0, 0)^T \]

(10)

\[ \lim_{t \to \infty} \text{diag}(y_k(t) q^{2/3}(t), y'_k(t) q^{1/3}(t), y''_k(t), y'''_k(t) q^{-1/3}(t)) E_{1k}(t, t_0) = (1, \mu_k, \mu_k^2, \mu_k^3)^T, \quad k = 2, 3, 4. \]

Then the formula (F1) follows directly from (10). Therefore the first part of Theorem 1 is proved.

The formulae (F2)—(F8) may be proved analogously.

**Corollary.** The formulae (F1)—(F8) imply the corresponding formulae (F'1)—(F'8) for the general solution of the equation (2):

(F'1) \[ y = \left[ c_1 E^{-1}(t, t_0) + q^{-2/3}(t) \sum_{k=2}^4 c_k \mu_k E_{1k}^{-1}(t, t_0) \right] (1 + o(1)) \]

(F'2) \[ y = q^{-1}(t) \left[ c_1 E^{-1}(t, t_0) + \sum_{k=2}^4 c_k \mu_k E_{1k}^{-1}(t, t_0) \right] (1 + o(1)), \]

(F'3) \[ y = \left[ c_1 E^{-1}(t, t_0) + q^{-2/3}(t) \sum_{k=2}^4 c_k \mu_k E_{2k}^{-1}(t, t_0) \right] (1 + o(1)), \]

(F'4) \[ y = q^{-1}(t) \left[ c_1 E^{-1}(t, t_0) + \sum_{k=2}^4 c_k \mu_k E_{2k}^{-1}(t, t_0) \right] (1 + o(1)), \]

(F'5) \[ y = \left[ c_1 + q^{-2/3}(t) \sum_{k=2}^4 c_k \mu_k E_{3k}^{-1}(t, t_0) \right] (1 + o(1)), \]

(F'6) \[ y = \left[ c_1 + q^{-1/3}(t) \sum_{k=2}^4 c_k \mu_k E_{4k}^{-1}(t, t_0) \right] (1 + o(1)), \]

(F'7) \[ y = \left[ c_1 + q^{-1/3}(t) \sum_{k=2}^4 c_k \mu_k E_{5k}^{-1}(t, t_0) \right] (1 + o(1)), \]

(F'8) \[ y = \left[ c_1 + q^{-1/3}(t) \sum_{k=2}^4 c_k \mu_k E_{6k}^{-1}(t, t_0) \right] (1 + o(1)), \]
\[ y = q^{-1}(t) \left[ c_1 + \sum_{k=2}^{4} c_k \mu_k E_{3k}^{-1}(t, t_0) \right] (1 + o(1)), \]

\[ y = \left[ c_1 + q^{-2/3}(t) \sum_{k=2}^{4} c_k \mu_k E_{4k}^{-1}(t, t_0) \right] (1 + o(1)), \]

\[ y = q^{-1}(t) \left[ c_1 + \sum_{k=2}^{4} c_k \mu_k E_{4k}^{-1}(t, t_0) \right] (1 + o(1)) \]

where \( c_1, c_2, c_3, c_4 \) are arbitrary numbers and the symbol \( o(1) \) denotes a function which converges to zero as \( t \to \infty \).

Remark. The equation \( y^{(iv)} + a^2 y' = 0, \quad a > 0 \) satisfies the hypothesis of Theorems 1—5, thus from each formula (F'1)—(F'8) it follows that the general solution of this equation is of the form

\[ y = [c_1 + c_2 e^{\pm at} + e^{\mp at/2}(c_3 \cos (a \sqrt{3}t/2) + c_4 \sin (a \sqrt{3}t/2))](1 + o(1)) \]

This equation has constant coefficients and therefore its general solution is

\[ y = c_1 + c_2 e^{\pm at} + e^{\mp at/2}(c_3 \cos (a \sqrt{3}t/2) + c_4 \sin (a \sqrt{3}t/2)) \]

and so \( o(1) \equiv 0 \).

Example. If \( p(t) q^{-1/3}(t) \) and \( r(t) q^{-1}(t) \) are in \( \mathcal{L} [a, \infty) \), \( a > 0 \), where

\[ q(t) = \left( \frac{2t}{t + 1} \right)^3, \]

then the equation

\[ y^{(iv)} + p(t) y'' - q(t) y' + r(t) y = 0 \]

satisfies the assumptions of Theorem 5 and therefore its general solution has the form

\[ y = \left( \frac{t + 1}{2t} \right)^3 \left[ c_1 + c_2 \frac{e^{2t}}{(t + 1)^2} + (t + 1) e^{-t}(c_3 \cos 3(t - \ln (t + 1)) + c_4 \sin 3(t - \ln (t + 1))) \right] (1 + o(1)), \]

where \( c_1, c_2, c_3 \) and \( c_4 \) are arbitrary numbers.

From this example it may be seen that the coefficients do not satisfy the assumptions of theorems in [3], [4] and therefore this paper gives new results on the asymptotic behaviour of solutions of the linear differential equation of the fourth order.
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Received December 2, 1985

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АСИМПТОТИЧЕСКИЕ ПОВЕДЕНИЕ РЕШЕНИЙ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ ЧЕТВЕРТОГО ПОРЯДКА II

Jozef Miklo

Резюме

В работе рассматриваются асимптотические поведения решений уравнения (2) при \(t \to \infty\), если несобственные интегралы от некоторых дробей функций \(p, q\) и \(r\) являются конечными.