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CONSTRUCTIONS OF TRIANGULAR AND QUADRANGULAR POLYHEDRA OF INSCRIBABLE TYPE

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ABSTRACT. Procedures of construction of infinite families of triangular and quadrangular polyhedra of inscribable type are presented.

1. A polyhedron (i.e., the convex 3-dimensional hull of a finite number of points in the Euclidean 3-space) $\boldsymbol{P}$ is said to be *of inscribable type* if there exists a polyhedron $\boldsymbol{P}^*$ combinatorially isomorphic with $\boldsymbol{P}$ and a sphere γ such that all the vertices of $\boldsymbol{P}^*$ belong to γ. The problem of the existence of polyhedra of non-inscribable type goes back at least to Steiner [8]. (For references, see Grünbaum-Shephard [4].) Sufficient conditions for polyhedra to be of non-inscribable type defining large classes of polyhedra were first stated by Steinitz [9] and Grünbaum [3]. Recently, new results have been obtained toward characterizations of polyhedra of inscribable type. In Hodgson, Rivin and Smith [6], such a characterization for general polyhedra is given in terms of whether the polyhedra support certain edge-weightings. Another characterization for a special class of quadrangular polyhedra is contained in Jucovič-Ševec [7]. Essential sufficient conditions for a polyhedron to be of inscribable type have been presented by Dillencourt and Smith [1], [2].

The aim of this paper is to yield such conditions for triangular and quadrangular polyhedra which are not directly subsumed by the papers of Dillencourt and Smith. The proofs are performed by constructing infinite families of polyhedra having a certain structure.

The basic idea of our constructions is as follows: Let $\boldsymbol{P}$ be a polyhedron with all vertices belonging to a sphere γ, and S a subgraph of the graph $G(\boldsymbol{P})$ of the polyhedron $\boldsymbol{P}$ (i.e., the graph formed by the vertices and edges of $\boldsymbol{P}$). The vertices, edges and faces incident with S are transformed according to a

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rule τ so that another polyhedron $\boldsymbol{P}'$ with graph $G(\boldsymbol{P}') = \tau G(\boldsymbol{P})$ with all vertices belonging to γ is created. (The elements of $\boldsymbol{P}$ not incident with S remain unchanged.) So a new polyhedron of inscribable type is obtained. The polyhedron $\boldsymbol{P}'$ is constructed so that its graph $G(\boldsymbol{P}')$ contains a subgraph isomorphic to S which enables the construction to proceed in an analogous way.

2. We start with quadrangular polyhedra. The sufficient condition we intend to prove is the following:

THEOREM 1. *A quadrangular polyhedron is of inscribable type if its graph arises from the graph of the cube by the successive application of the transformation* τ_1 *in Fig.* 1.

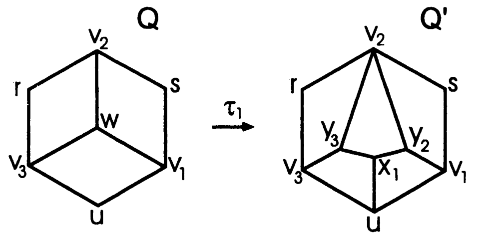

FIGURE 1.

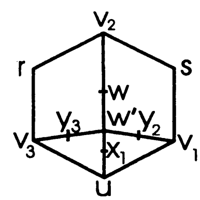

FIGURE 2.

P r o o f. Let us consider a quadrangular polyhedron $\boldsymbol{P}$ all the vertices of which belong to the sphere γ and whose graph $G(\boldsymbol{P})$ contains a graph isomorphic to Q in Fig. 1 as its subgraph. (From now on we shall not distinguish the subgraph on the polyhedron and its image on the figures – this should cause no confusion.) Let $\boldsymbol{w}$ be a trivalent vertex of Q as indicated on Fig. 1. On the ray $\boldsymbol{v}_2\boldsymbol{w}$ outside γ near to $\boldsymbol{w}$ a point $\boldsymbol{w}'$ is chosen such that the segments $\boldsymbol{w}'\boldsymbol{u}$, $\boldsymbol{w}'\boldsymbol{v}_1$ and $\boldsymbol{w}'\boldsymbol{v}_3$ intersect γ in three mutually different points $\boldsymbol{x}_1$, $\boldsymbol{y}_2$ and $\boldsymbol{y}_3$, respectively. Fig. 2 depicts the changed subgraph. Notice that any choice of the point $\boldsymbol{w}'$ forces the 4-tuples $(\boldsymbol{v}_3, \boldsymbol{y}_3, \boldsymbol{x}_1, \boldsymbol{u})$ and $(\boldsymbol{u}, \boldsymbol{x}_1, \boldsymbol{y}_2, \boldsymbol{v}_1)$ to be coplanar. And moreover, because $\boldsymbol{w}'$ is on the ray $\boldsymbol{v}_2\boldsymbol{w}$, the 4-tuples $(\boldsymbol{v}_2, \boldsymbol{y}_3, \boldsymbol{v}_3, \boldsymbol{r})$ and $(\boldsymbol{v}_2, \boldsymbol{s}, \boldsymbol{v}_1, \boldsymbol{y}_2)$ are also coplanar. Then: The convex hull of

$$V(\boldsymbol{P}) - \{\boldsymbol{w}\} \cup \{\boldsymbol{x}_1, \boldsymbol{y}_2, \boldsymbol{y}_3\} \qquad (V(\boldsymbol{P}) \text{ is the vertex set of } \boldsymbol{P})$$

is a polyhedron $\boldsymbol{N}$ with all vertices belonging to γ. The edges and faces of the polyhedron $\boldsymbol{P}$ not incident to the vertex $\boldsymbol{w}$ are not changed, they are edges and faces of $\boldsymbol{N}$. We must still show that the graph of $\boldsymbol{N}$ is $\tau_1 G(\boldsymbol{P})$, i.e., that it contains Q' as subgraph. To do this, it clearly suffices to show that the vertices $\boldsymbol{v}_2$, $\boldsymbol{y}_3$, $\boldsymbol{x}_1$, $\boldsymbol{y}_2$ belong to the same face of $\boldsymbol{N}$, i.e., that vertices $\boldsymbol{y}_3$, $\boldsymbol{y}_2$ as well as $\boldsymbol{v}_2$, $\boldsymbol{x}_1$ are not joined by an edge.

This can be done in such a way that Steinitz's condition ([9]) for a polyhedron $\boldsymbol{X}$ not to be of inscribable type is used:

If the vertex-set $V(\boldsymbol{X})$ of the polyhedron $\boldsymbol{X}$ has a subset T with $|T| \geq \frac{|V(\boldsymbol{X})|}{2}$ such that no two of its vertices are joined by an edge, and in the case $|T| = \frac{|V(\boldsymbol{X})|}{2}$, in the set $V(\boldsymbol{X}) - T$, there exist two vertices joined by an edge – then $\boldsymbol{X}$ is not of inscribable type.

Let us return to our polyhedron $\boldsymbol{N}$. As the polyhedron $\boldsymbol{P}$ is quadrangular, all circuits of $G(\boldsymbol{P})$ have even lengths, and the graph $G(\boldsymbol{P})$ is bichromatic (cf. Harary [5]). As $\boldsymbol{P}$ is inscribed in γ, by the above theorem of Steinitz, both colour-classes of $G(\boldsymbol{P})$ have equal numbers of vertices. On a quadrangular face, adjacent vertices do not belong to the same colour-class. Let all vertices of $\boldsymbol{N}$, with the exception of $\boldsymbol{x}_1$, $\boldsymbol{y}_2$ and $\boldsymbol{y}_3$, retain the colours they have in $\boldsymbol{P}$. It is possible to colour the vertices $\boldsymbol{x}_1$, $\boldsymbol{y}_2$, $\boldsymbol{y}_3$ so that a (possibly non-regular) 2-colouring of $G(\boldsymbol{N})$ with equal numbers of vertices in both colour-classes appears. However, the existence of an edge $\boldsymbol{y}_3\boldsymbol{y}_2$ or $\boldsymbol{v}_2\boldsymbol{x}_1$ in the graph of the polyhedron $\boldsymbol{N}$ would by the theorem of Steinitz quoted mean the non-inscribability of $\boldsymbol{N}$ – a contradiction. So in fact, the graph of $\boldsymbol{N}$ has been obtained by performing operation τ_1 on the graph of the polyhedron $\boldsymbol{P}$, $\boldsymbol{N} = \boldsymbol{P}'$. In the graph of the polyhedron $\boldsymbol{P}'$, we have again got Q as its subgraph, therefore the procedure τ_1 can be performed once more, and so on. The proof of Theorem 1 is concluded by recalling that the cube is of inscribable type. □

3. Analogous reasoning is fruitful if we intend to construct triangular polyhedra of inscribable type. Fig. 3 presents transformations of graphs of triangular polyhedra of inscribable type into more complicated ones retaining inscribability.

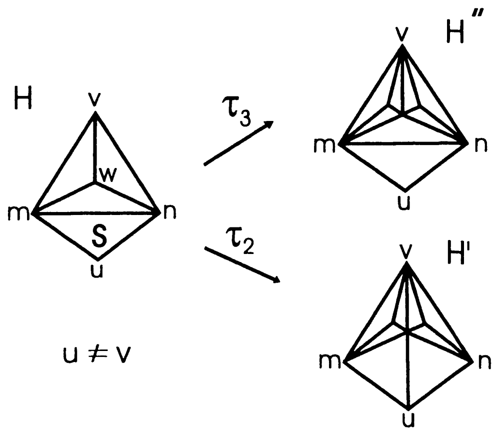

FIGURE 3.

THEOREM 2. *A triangular polyhedron is of inscribable type if its graph arises from the graph of the bipyramid with* 6 *faces* (*Fig.* 4) *by successive applying the transformations* τ_2 *or* τ_3 *depicted in Fig.* 3.

P r o o f . Let the graph of the triangular polyhedron $\boldsymbol{P}$ contain as proper subgraph the graph H in Fig. 3, and let all the vertices of $\boldsymbol{P}$ belong to the sphere γ. The plane containing the face s intersects γ in a circuit c decomposed by the vertices $\boldsymbol{m}$, $\boldsymbol{n}$ into two arcs. Inside the arc that does not contain the vertex $\boldsymbol{u}$ choose three points $\boldsymbol{y}_1$, $\boldsymbol{y}_2$, $\boldsymbol{y}_3$ and join them with the vertex $\boldsymbol{v}$ (Fig. 5). The vertex $\boldsymbol{w}$ and the edges incident with it are removed. A polyhedron $\boldsymbol{Q}$ with all vertices belonging to γ is obtained with vertex-set $V(\boldsymbol{Q}) = \{V(\boldsymbol{P}) - \{\boldsymbol{w}\}\} \cup \{\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3\}$ and a hexagonal face $f = \boldsymbol{uny_1y_2y_3m}$. In the next stages of the construction, the following is crucial:

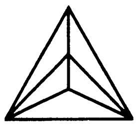

FIGURE 4.

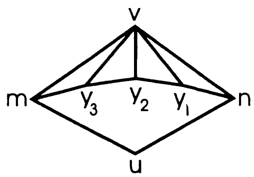

FIGURE 5.

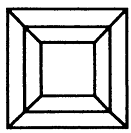

FIGURE 6.

LEMMA. (D i l l e n c o u r t - S m i t h [2]) *Let* $\boldsymbol{M}$ *be a polyhedron of inscribable type with a non-bipartite graph. Add to this graph a new edge the diagonal of some face of* $\boldsymbol{M}$. *The resulting graph is isomorphic to the graph of a polyhedron of inscribable type.*

Let us return to polyhedron $\boldsymbol{Q}$ which clearly does not admit a bipartite graph. Add to the graph $G(\boldsymbol{Q})$ successively the diagonals $\boldsymbol{y}_2\boldsymbol{m}$, $\boldsymbol{y}_2\boldsymbol{n}$ and $\boldsymbol{y}_2\boldsymbol{u}$ of the hexagonal face f. Finally we get a graph containing H' as subgraph. Proceeding analogously just as before but inserting the diagonal $\boldsymbol{mn}$ instead of $\boldsymbol{uy}_2$ we get a graph containing H'' as subgraph. By the Lemma, both these graphs are realisable as graphs of polyhedra of inscribable type. The polyhedra are triangular and contain the graph H as subgraph. Thus the construction can proceed. □

4. Remarks.

1. We can show that Theorem 1 does not define all quadrangular polyhedra of inscribable type nor does Theorem 2 do so with the triangular ones. As examples for quadrangular polyhedra we can use the polyhedra constructed as follows: On the surface of the sphere, consider two mutually perpendicular great circles and divide each of the four half-circles obtained into $m \geq 3$ congruent arcs. The convex hull of the dividing points is a polyhedron of inscribable type. To make it quadrangular, replace the two quadruples of triangles by two quadrangles. For $m = 4$, Fig. 6 presents the polyhedron. All polyhedra obtained in this way can be employed for performing transformation τ_1 at the beginning of the construction. The transformation τ_1 can be employed for constructing other polyhedra of inscribable type having even faces only.

2. Further triangular polyhedra of inscribable type are constructed if the hexagonal face f of $\boldsymbol{Q}$ in the proof of Theorem 2 is otherwise decomposed by diagonals into triangles, or if on the arc of c with end-points $\boldsymbol{m}$, $\boldsymbol{n}$, any number $\neq 3$ of new vertices is set. Of course, non-triangular polyhedra of inscribable type can also be constructed in this way.

3. It is desirable to find analogous constructions of pentagonal polyhedra of inscribable type or to prove that the dodecahedron is the only such polyhedron.

REFERENCES

[1] DILLENCOURT, M. B.—SMITH, W. D.: *A linear time algorithm for testing the inscribability of trivalent polyhedra*, Internat. J. Comput. Geom. Appl. **5** (1995), 21–36.

[2] DILLENCOURT, M. B.—SMITH, W. D.: *A simple method for resolving degeneracies in Delaunay triangulations.* In: Automata, Languages, and Programming: Proc. 20th Internal. Coll., Lund Sweden ICALP, Lund, Sweden, July 1993. Lecture Notes in Comput. Sci. 700, Springer, Berlin-New York, 1993, pp. 177–188.

[3] GRÜNBAUM, B.: *On Steinitz's theorem about non-inscribable polyhedra*, Nederl. Akad. Wetensch. Proc. Ser. A **66** (1963), 452–455.

[4] GRÜNBAUM, B.—SHEPHARD, G. C.: *Some problems on polyhedra*, J. Geom. **29** (1987), 182–190.

[5] HARARY, F.: *Graph Theory*, Addison Wesley, Reading, 1969.

[6] HODGSON, C. D.—RIVIN, I.—SMITH, W. D.: *A characterization of convex hyperbolic polyhedra and of convex polyhedra inscribed in the sphere*, Bull. Amer. Math. Soc. (N.S.) **27** (1992), 246–251.

[7] JUCOVIČ, E.—ŠEVEC, S.: *Note on inscribability of quadrangular polyhedra with restricted number of edge-types*, J. Geom. **42** (1991), 126–131.

[8] STEINER, J.: *Systematische Entwicklung der Abhängigkeit geometrischer Gestalten von einander.* Reimer, Berlin, 1832; Jacob Steiner's Collected Works, Vol. 1, Berlin, 1881.

[9] STEINITZ, E.: *Über isoperimetrische Probleme bei konvexen Polyedern I; II*, J. Reine Angew. Math. **158; 159** (1927; 1929), 129–153; 133–143.

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