Dao Rong Tong

The Galois connection between weak torsion and sub-product classes of l-groups

Mathematica Slovaca, Vol. 47 (1997), No. 5, 511--516

Persistent URL: http://dml.cz/dmlcz/133186

Math. Slovaca, 47 (1997), No. 5, 511–516

Mathematica Slovaca
©1997
Mathematical Institute
Slovak Academy of Sciences

THE GALOIS CONNECTION BETWEEN WEAK TORSION AND SUB-PRODUCT CLASSES OF L-GROUPS

DAO-RONG TON

(*Communicated by Tibor Katriňák*)

ABSTRACT. In this paper, we establish the Fundamental Connection Theorem between weak torsion classes and sub-product classes of l-groups, which generalizes the Fundamental Connection Theorem between torsion classes and torsion-free classes of l-groups in [Martinez, J.; *The fundamental theorem on torsion classes of lattice-ordered groups*, Trans. Amer. Math. Soc. **259** (1980), 311–317].

We use the standard terminologies and notations of [1], [2], [3]. Throughout the paper, G is an l-group. We use additive group notation. Let $\{G_\alpha \mid \alpha \in A\}$ be a family of l-groups, and let $\prod_{\alpha\in A} G_\alpha$ be their direct product. We denote the l-subgroup of $\prod_{\alpha\in A} G_\alpha$ consisting of the elements with only finitely many non-zero components by $\sum_{\alpha\in A} G_\alpha$. An l-group G is called a completely subdirect product of $\{G_\alpha \mid \alpha \in A\}$ if G is an l-subgroup of $\prod_{\alpha\in A} G_\alpha$ and $\sum_{\alpha\in A} G_\alpha \subseteq G$, we denote it by

$$\sum_{\alpha\in A} G_\alpha \subseteq G \subseteq \prod_{\alpha\in A} G_\alpha .$$

Let G be an l-group. If $G = G_1 \oplus G_2$, G_1 and G_2 are called cardinal summands of G. By $\mathcal{C}(G)$, $\mathcal{L}(G)$ and $\mathcal{S}(G)$ will be denoted the sets of all convex l-subgroups, all l-ideals and all cardinal summands of G, respectively. All classes of l-groups are assumed to be closed under l-isomorphisms. A class $\mathcal{T}$ of l-groups is said to be complete if $G \in \mathcal{T}$ whenever $H \in \mathcal{L}(G)$ and both $H \in \mathcal{T}$ and $G/H \in \mathcal{T}$. $\mathcal{T}$ is said to be weak complete if $G \in \mathcal{T}$ whenever $H \in \mathcal{T}$ and $G/H \in \mathcal{T}$. Let φ be an l-homomorphism from G onto G' such that the kernel $H = \varphi^{-1}(0) \in \mathcal{S}(G)$, then φ is called a strong l-homomorphism.

AMS Subject Classification (1991): Primary 06F15; Secondary 08B26, 08C99.
Key words: l-group, weak torsion class, sub-product class, strong l-homomorphism.

An l-isomorphism is always a strong l-homomorphism. The join in a lattice L is denoted by $\bigvee^{(L)}$. If G is l-isomorphic to G', we write $G \cong G'$.

LEMMA 1. *Strong l-homomorphisms are transitive.*

Proof. Suppose that φ is a strong l-homomorphism from G onto G', and φ' is a strong l-homomorphism from G' onto G''. Let $K_1' = \varphi'^{-1}(0)$ and $K_1 = \varphi^{-1}(0)$. Then $G' = K_1' \oplus K_2'$, $G = K_1 \oplus K_2$ and $G'' \cong G'/K_1' \cong K_2'$, $G' \cong G/K_1 \cong K_2$.

Therefore

$$G = K_1 \oplus H_1 \oplus H_2 ,$$
$$G'' \cong K_2' \cong H_2 \cong G/(K_1 \oplus H_1) ,$$

and $\varphi'\varphi$ is a strong l-homomorphism from G onto G'' with the kernel $K_1 \oplus H_1$. □

DEFINITION 1. A class $\mathcal{R}$ of l-groups is called a *weak torsion class* if it is closed under taking strong l-homomorphic images and forming joins of convex l-subgroups. Let $\mathcal{W}$ be the set of all weak torsion classes of l-groups.

A torsion class of l-groups is closed under taking l-homomorphic images and forming joins of convex l-subgroups, so every torsion class is a weak torsion class. If $\mathcal{U}$ is a weak torsion class of l-groups, and G is an l-group, let $\mathcal{U}(G)$ be the join of all the convex l-subgroup of G belonging to $\mathcal{U}$. $\mathcal{U}(G)$ is called a weak torsion radical of G. It is clear that $\mathcal{U}(G)$ is characteristic, and $\mathcal{U}(G)$ is the largest l-ideal of G belonging to $\mathcal{U}$.

PROPOSITION 2. *Suppose that $\mathcal{U}$ is a weak torsion class of l-groups and G is an l-group. Then*

(1) *If $C \in \mathcal{C}(G)$, then $\mathcal{U}(C) \subseteq \mathcal{U}(G)$.*
(2) *If $\varphi\colon G \to G'$ is a strong l-homomorphism, then $\varphi[\mathcal{U}(G)] \subseteq \mathcal{U}(G')$.*
(3) $\mathcal{U}(\mathcal{U}(G)) = \mathcal{U}(G)$.

Conversely, if we associate with each l-group G an l-ideal $\mathcal{T}(G)$ subject to (1), (2) *and* (3) *above, and let $\mathcal{U} = \{G \mid \mathcal{T}(G) = G\}$, then $\mathcal{U}$ is a weak torsion class of l-groups, and $\mathcal{U}(G) = \mathcal{T}(G)$ for each l-group G.*

The proof of Proposition 2 is similar to Lemma 1 of [4].

DEFINITION 2. A class of l-groups is called a sub-product class if it is closed under taking convex l-subgroups and forming completely subdirect products. Let $\mathcal{P}$ be the set of all sub-product classes of l-groups.

It is easy to see that $\mathcal{W}$ and $\mathcal{P}$ are complete lattices under inclusion.

A torsion-free class of l-groups is closed under taking convex l-subgroups and forming subdirect products, so every torsion-free class of l-groups is a sub-product class. Let $\mathcal{R}$ be a sub-product class of l-groups and G be an l-group. Put

$$\mathcal{H}_{\mathcal{R},G} = \{H \in \mathcal{S}(G) \mid G/H \in \mathcal{R}\}$$

and

$$\mathcal{R}(G) = \bigcap_{H \in \mathcal{H}_{\mathcal{R},G}} H .$$

$\mathcal{R}(G)$ is called a *sub-product radical* of G.

PROPOSITION 3. *A sub-product radical $\mathcal{R}(G)$ of an l-group G has the following properties:*

(1) *$\mathcal{R}(G)$ is the smallest cardinal summand of G such that $G/\mathcal{R}(G) \in \mathcal{R}$.*
(2) *$G \in \mathcal{R}$ if and only if $\mathcal{R}(G) = 0$.*
(3) *If $\mathcal{R}_1, \mathcal{R}_2 \in \mathcal{P}$, then $\mathcal{R}_1 \subseteq \mathcal{R}_2$ if and only if $\mathcal{R}_1(G) \supseteq \mathcal{R}_2(G)$ for each l-group G.*

Proof.

(1) Since $\mathcal{S}(G)$ is a subalgebra of the complete Boolean algebra of polar subgroups of G, $\mathcal{R}(G) \in \mathcal{S}(G)$. It is easy to see that G/H is a convex l-subgroup of $G/\mathcal{R}(G)$ for each $H \in \mathcal{H}_{\mathcal{R},G}$. Hence $G/\mathcal{R}(G)$ is a completely subdirect product of $\{G/H \mid H \in \mathcal{H}_{\mathcal{R},G}\}$. Therefore $G/\mathcal{R}(G) \in \mathcal{R}$. If $K \in \mathcal{S}(G)$ such that $G/K \in \mathcal{R}$, then $K \in \mathcal{H}_{\mathcal{R},G}$ and $K \supseteq \mathcal{R}(G)$.

(2) and (3) are the Theorem 2 of [5]. □

PROPOSITION 4. *Suppose that $\mathcal{R}$ is a sub-product class of l-groups and G is an l-group. Then*

(i) *If $A \in \mathcal{C}(G)$, then $\mathcal{R}(A) \subseteq \mathcal{R}(G)$.*
(ii) *If $H \in \mathcal{S}(G)$ and $\mathcal{R}(G/H) = 0$, then $H \supseteq \mathcal{R}(G)$.*
(iii) *$\mathcal{R}(G/\mathcal{R}(G)) = 0$.*

Conversely, suppose that we associate with each l-group G a $\mathcal{T}(G) \in \mathcal{S}(G)$ subject to (i), (ii) *and* (iii) *above. If $\mathcal{R} = \{G \mid \mathcal{T}(G) = 0\}$, then $\mathcal{R}$ is a sub-product class of l-groups and $\mathcal{R}(G) \supseteq \mathcal{T}(G)$ for each l-group G.*

The proof of this Proposition is similar to Theorem 3 of [5].

Now let $\mathcal{U}$ be a weak torsion class. Put

$$\hat{\mathcal{U}} = \{G \mid \mathcal{U}(G) = 0\} .$$

PROPOSITION 5. *Suppose that $\mathcal{U}$ is a weak torsion class of l-groups. Then $\hat{\mathcal{U}}$ is a weak complete sub-product class of l-groups.*

Proof. It is clear that $\hat{\mathcal{U}}$ is closed under taking convex l-subgroups. Suppose that $\{G_\lambda \mid \lambda \in \Lambda\} \subseteq \hat{\mathcal{U}}$ and G is a completely subdirect product of $\{G_\lambda \mid \lambda \in \Lambda\}$,

$$\sum_{\lambda\in\Lambda} G_\lambda \subseteq G \subseteq \prod_{\lambda\in\Lambda} G_\lambda .$$

If $\mathcal{U}(G) \neq 0$, then there exists $0 \neq H \in \mathcal{C}(G)$ such that $H \in \mathcal{U}$. For each $\lambda \in \Lambda$ put $\overline{G}_\lambda = \left\{g \in \prod_{\lambda\in\Lambda} G_\lambda \mid \lambda' \neq \lambda \implies g_{\lambda'} = 0\right\}$. Let $0 \prec h \in H$. Then there exists $\lambda_0 \in \Lambda$ such that $h_{\lambda_0} \succ 0$. Since $H \in \mathcal{C}(G)$ and $\overline{G}_{\lambda_0} \subseteq G$, $0 \neq (0,\dots,0,h_{\lambda_0},0,\dots,0) \in H \cap \overline{G}_{\lambda 0} \in \mathcal{C}(\overline{G}_{\lambda_0})$. That is, $H \cap \overline{G}_{\lambda_0} \neq 0$. It is clear that

$$H = (H \cap \overline{G}_{\lambda_0}) \oplus \Big(H \cap \prod_{\lambda\neq\lambda_0} G_\lambda\Big).$$

Hence $H \cap \prod_{\lambda\neq\lambda_0} G_\lambda \in \mathcal{S}(G)$ and $H \cap \overline{G}_{\lambda_0} \cong H/H \cap \Big(\prod_{\lambda\neq\lambda_0} G_\lambda\Big)$. Since $\mathcal{U}$ is closed under taking strong l-homomorphic images, $H \cap \overline{G}_{\lambda_0} \in \mathcal{U}$. This contradicts $U(\overline{G}_{\lambda_0}) \cong \mathcal{U}(G_{\lambda_0}) = 0$. Therefore $\mathcal{U}(G) = 0$, and $\hat{\mathcal{U}}$ is also closed under forming completely subdirect products.

Suppose $H \in \mathcal{S}(G)$ such that both $H \in \hat{\mathcal{U}}$ and $G/H \in \hat{\mathcal{U}}$. Then $\mathcal{U}(G) \subseteq H$ by (2) of Proposition 2. Since $\mathcal{U}(H) = 0$ and $\mathcal{U}(G) \in \mathcal{U}$, $\mathcal{U}(G) = 0$. Hence $G \in \hat{\mathcal{U}}$ and $\hat{\mathcal{U}}$ is weak complete. □

$\hat{\mathcal{U}}$ is called the *opposite sub-product class of* $\mathcal{U}$.

Let $\mathcal{R}$ be a sub-product class of l-groups. Put

$$\overline{\mathcal{R}} = \{G \mid \mathcal{R}(G) = G\}.$$

PROPOSITION 6. *Suppose that $\mathcal{R}$ is a sub-product class of l-groups. Then $\overline{\mathcal{R}}$ is a complete weak torsion class of l-groups.*

Proof. It is clear that $\overline{\mathcal{R}}$ is the class of l-groups having no nontrivial strong l-homomorphic images in $\mathcal{R}$. By Lemma 1, $\overline{\mathcal{R}}$ is closed under taking strong l-homomorphic images.

Suppose that $\{G_\lambda \mid \lambda \in \Lambda\} \subseteq \mathcal{C}(G)$ and $\{G_\lambda \mid \lambda \in \Lambda\} \subseteq \overline{\mathcal{R}}$. Put $G' = \bigvee_{\lambda\in\Lambda} G_\lambda$. If $\mathcal{U}(G') \neq G'$, then there exists $0 \neq H \in \mathcal{H}_{\mathcal{R},G'}$ such that $H \neq G'$, and so

$$G' = H \oplus H'$$

and $G'/H \in \overline{\mathcal{R}}$. If $G_\lambda \cap H = G_\lambda$ for all $\lambda \in \Lambda$, then $G_\lambda \subseteq H$ for all $\lambda \in \Lambda$, and so $G' = \bigvee_{\lambda\in\Lambda} G_\lambda \subseteq H$, which is a contradiction. Hence there exists $\lambda_0 \in \Lambda$ such that $G_{\lambda_0} \cap H \neq G_{\lambda_0}$. Then

$$G_{\lambda_0} = (G_{\lambda_0} \cap H) \oplus (G_{\lambda_0} \cap H').$$

And we have

$$\frac{G_{\lambda_0}}{G_{\lambda_0} \cap H} \cong \frac{G_{\lambda_0} + K}{H} \in \mathcal{C}\Big(\frac{G'}{H}\Big).$$

Hence $G_{\lambda_0}/G_{\lambda_0} \cap H \in \mathcal{R}$ and $G_{\lambda_0} \cap H \in \mathcal{H}_{\mathcal{R},G_{\lambda_0}}$, $\mathcal{R}(G_{\lambda_0}) \subseteq G_{\lambda_0} \cap H \neq G_{\lambda_0}$, which is a contradiction. Therefore $\mathcal{U}(G') = G'$ and $G' \in \overline{\mathcal{R}}$. We have proved that $\overline{\mathcal{R}}$ is closed under forming joins of convex l-subgroups.

Suppose that G is an l-group, and $\mathcal{H} \in \mathcal{L}(G)$ such that both $H \in \overline{\mathcal{R}}$ and $G/H \in \overline{\mathcal{R}}$. Let $K \in \mathcal{S}(G)$ such that $G/K \in \mathcal{R}$. Then $H \cap K \in \mathcal{S}(H)$ and $H/H \cap K \cong (H+K)/K \in \mathcal{C}(G/K)$. Hence $H/H \cap K \in \mathcal{R}$. On the other hand, $H \in \overline{\mathcal{R}}$ and $H \cap K \in \mathcal{S}(H)$ infer $H/H \cap K \in \overline{\mathcal{R}}$. So we have $H = H \cap K$ or $H \subseteq K$. But $G/K \cong (G/H)/(K/H)$. $G/H \in \overline{\mathcal{R}}$ and $K/H \in \mathcal{S}(G/H)$ infer $G/K \in \overline{\mathcal{R}}$, and so $K = G$. Hence $G \in \overline{\mathcal{R}}$ and $\overline{\mathcal{R}}$ is complete. □

$\overline{\mathcal{R}}$ is called the *opposite weak torsion class of* $\mathcal{R}$.

Now We will give the main theorem — the Fundamental Connection Theorem between weak torsion classes and sub-product classes of l-groups, which generalizes the Connection Theorem between torsion classes and torsion-free classes of l-groups in [4]. If $\mathcal{U}$ (resp. $\mathcal{R}$) is a weak torsion class of l-groups (resp. sub-product class), let $\mathcal{U}^* = \overline{\hat{\mathcal{U}}}$ (resp. $\mathcal{R}^\circ = \hat{\overline{\mathcal{R}}}$).

Connection Theorem. *The functions $\varphi\colon \mathcal{U} \to \hat{\mathcal{U}}$ and $\phi\colon \mathcal{R} \to \overline{\mathcal{R}}$ between $\mathcal{W}$ and $\mathcal{P}$ form a Galois Connection. In addition, $\mathcal{U}(G) \subseteq \overline{\hat{\mathcal{U}}}(G) = \mathcal{U}^*(G)$ for each l-group G and each weak torsion class $\mathcal{U}$, while $\mathcal{R}(G) \supseteq \hat{\overline{\mathcal{R}}}(G) = \mathcal{R}^\circ(G)$ for each l-group G and each sub-product class $\mathcal{R}$.*

Proof. It is clear that φ and ϕ are order-inverting. If $G \in \mathcal{U}$, it certainly has no strong l-homomorphic images in $\hat{\mathcal{U}}$ except $\{0\}$, which implies $G \in \mathcal{U}^*$. Thus $\mathcal{U}(G) \subseteq \mathcal{U}^*(G)$ for each l-group G. We should have $\mathcal{U}^*\big(G/\mathcal{U}^*(G)\big) = 0$ for each l-group G. Otherwise, there exists $G' \in \mathcal{C}(G)$ such that $\mathcal{U}^*(G) \subseteq G'$ but $\mathcal{U}^*(G) \neq G'$ and $G'/\mathcal{U}^*(G) \in \mathcal{U}^*$. Since $\mathcal{U}^*$ is complete, $G' \in \mathcal{U}^*$, which is a contradiction. Thus we have $\mathcal{U}\big(G/\mathcal{U}^*(G)\big) = 0$, so $G/\mathcal{U}^*(G) \in \hat{\mathcal{U}}$ and

$$\mathcal{U}^*(G) \supseteq \hat{\mathcal{U}}(G). \tag{1}$$

On the other hand, if $K \in \mathcal{S}(G)$ such that $G/K \in \hat{\mathcal{U}}$, then $\mathcal{U}^*(G) \cap K \in \mathcal{S}\big(\mathcal{U}^*(G)\big)$ and

$$\frac{\mathcal{U}^*(G) + K}{K} \cong \frac{\mathcal{U}^*(G)}{\mathcal{U}^*(G) \cap K}$$

is a strong l-homomorphic image of $\mathcal{U}^*(G)$. Hence $(\mathcal{U}^*(G)+K)/K$ belongs to $\mathcal{U}^*$. $(\mathcal{U}^*(G)+K)/K \in \mathcal{C}(G/K)$ implies $(\mathcal{U}^*(G)+K)/K$ also belongs to $\hat{\mathcal{U}}$. Therefore $\mathcal{U}^*(G)+K=K$, that is, $\mathcal{U}^*(G) \subseteq K$. By Proposition 4, we have

$$\mathcal{U}^*(G) \subseteq \hat{\mathcal{U}}(G). \tag{2}$$

Combining (1) and (2) we get $\mathcal{U}^*(G) = \hat{\mathcal{U}}(G)$.

The proof that $\mathcal{R}(G) \supseteq \overline{\mathcal{R}}(G) = \mathcal{R}^\circ(G)$ for all sub-product classes $\mathcal{R}$ of l-groups is analogous.

From the above, it follows that $\hat{\mathcal{U}}(G) = \mathcal{U}^*(G) = \hat{\hat{\mathcal{U}}}(G) = (\mathcal{U}^*)^\frown(G)$ for each l-group G. Hence $\mathcal{U} = (\mathcal{U}^*)^\frown$ for all weak torsion classes $\mathcal{U}$ of l-groups. Similarly, we have $\overline{\mathcal{R}} = (\mathcal{R}^\circ)^-$ for all sub-product classes $\mathcal{R}$ of l-groups. Therefore φ and ϕ form a Galois connection. □

REFERENCES

[1] ANDERSON, M.—FEIL, T.: *Lattice-Ordered Groups (An Introduction)*, D. Reidel Publishing Company, Boston, 1988.

[2] CONRAD, P.: *Lattice-Ordered Groups*. Tulane Lecture Notes, Tulane University, 1970.

[3] GLASS, A. M. W.—HOLLAND, W. C.: *Lattice-Ordered Groups (Advances and Techniques)*, Kluwer Academic Publishes, London, 1989.

[4] MARTINEZ, J.: *The fundamental theorem on torsion classes of lattice-ordered groups*, Trans. Amer. Math. Soc. **259** (1980), 311–317.

[5] TON, DAO-RONG: *Sub-product classes of lattice-ordered groups*, Acta Math. Sinica **2(37)** (1994), 224–229.

Received October 17, 1994

29-305
3 Xikang Road
Nanjing, 210024
P.R.CHINA