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TOTAL DOMATIC NUMBER AND DEGREES OF VERTICES OF A GRAPH

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In [3] the interconnections between the domatic number of a graph and the degrees of its vertices were studied. Here we shall transfer those results to the total domatic number. All considered graphs will be undirected graphs without loops and multiple edges; they may be finite or infinite.

The domatic number of a graph was introduced by E. J. Cockayne and S. T. Hedetniemi [1], the total domatic number by the same authors and R. Dawes [2].

A subset D of the vertex set $V(G)$ of an undirected graph G is called dominating (or total dominating) if for each $v \in V(G) - D$ (or each $x \in V(G)$ respectively) there exists a vertex $y \in D$ adjacent to x. A partition $V(G)$ is called a domatic (or totally domatic) partition of $V(G)$ if all of its classes are dominating (or total dominating respectively) sets in G. The supremum of cardinalities of all domatic (or total domatic) partitions of G is called the domatic (or total domatic respectively) number of G and denoted by $d(G)$ (or $d_t(G)$ respectively). The domatic number is well-defined for every non-empty graph, the total domatic number for every non-empty graph without isolated vertices.

The minimum degree of a vertex of G will be denoted by $\delta(G)$. The quoted authors have proved that $d(G) \leqq \delta(G) + 1$, $d_t(G) \leqq \delta(G)$ for every graph G. We shall prove results for $d_t(G)$ which are analogous to the results from [3] for $d(G)$.

Theorem 1. *For each non-zero cardinal number $\mathfrak{a}$ there exist a graph G in which each vertex has the degree at least $\mathfrak{a}$ and whose total domatic number is 1. If $\mathfrak{a}$ is finite, then there exist both a finite graph with this property and an infinite one.*

Proof. We shall use a graph which is almost the same as that in [3]. Choose a cardinal number $\mathfrak{b} > 2\mathfrak{a}$. Let A be a set of the cardinality $\mathfrak{b}$, let B be the set of all subsets of A which have the cardinality $\mathfrak{a}$. The vertex set of G will be $A \cup B$. A vertex $u \in A$ will be adjacent to a vertex $v \in B$ in G if and only if v is a set which contains the vertex u. No two vertices of A and no two vertices of B will be adjacent. Evidently the degree of any vertex of B is $\mathfrak{a}$ and the degree of any vertex of A cannot be less than $\mathfrak{a}$.

Suppose that $d_t(G) \geqq 2$; then there exists a totally domatic partition $\{D_1, D_2\}$ of G. From the definition it is clear that a totally dominating set cannot be independent; this implies that none of the sets D_1, D_2 is a subset of A or of B. The sets $A_1 = A \cap D_1$, $A_2 = A \cap D_2$, $B_1 = B \cap D_1$, $B_2 = B \cap D_2$ are non-empty and pairwise disjoint. As $|A| = \mathfrak{b} > 2\mathfrak{a}$, at least one of the sets A_1, A_2 has the cardinality greater than $\mathfrak{a}$; without loss of generality let $|A_1| > \mathfrak{a}$. Then there exists a subset C of A_1 of the cardinality $\mathfrak{a}$. The set C is a vertex of B. As the set C contains no vertex of A_2, the vertex C is adjacent to no vertex of A_2. As B is independent, C is not adjacent either to a vertex of B_2; thus C is not adjacent to a vertex of D_1 and D_2 is not a totally dominating set in G, which is a contradiction. Hence $d_t(G) = 1$.

If $\mathfrak{a}$ is finite, then $\mathfrak{b}$ may be chosen as a finite number or a transfinite cardinal number. This implies that the required graph may be either finite, or infinite.

Now we shall prove a theorem giving a lower bound for $d_t(G)$ in terms of $\delta(G)$ and the number n of vertices.

Theorem 2. *Let G be a finite undirected graph with n vertices, let $\delta(G)$ be the minimum of degrees of vertices of G. Then*

$$d_t(G) \geqq [n/(n - \delta(G) + 1)]$$

and each subset of $V(G)$ with $n - \delta(G) + 1$ vertices is a totally dominating set in G.

Proof. Consider the complement $\bar{G}$ of G. A subset D of the vertex set $V(G)$ of G is a totally dominating set in G if for each $x \in V(G)$ there exists a vertex y which is not adjacent to x in $\bar{G}$. If some vertex has the degree r in G, it has the degree $n - r - 1$ in $\bar{G}$. Hence the maximum of degrees of vertices of $\bar{G}$ is $n - \delta(G) - 1$. Let D be a subset of $V(G)$ having at least $n - \delta(G) + 1$ vertices. Then each vertex $x \in V(G)$ can be adjacent to at most $n - \delta(G) - 1$ vertices of D in $\bar{G}$; even if $x \in D$, then there exists a vertex $y \in D$ which is not adjacent to x in $\bar{G}$ and thus is adjacent to x in G. This implies that each subset of $V(G)$ with at least $n - \delta(G) + 1$ vertices is a totally dominating set in G. Consider a partition of $V(G)$ into classes having $n - \delta(G) + 1$ vertices each, with the exception of at most one which would have more vertices. Evidently there exists such a partition having $[n/(n - \delta(G) + 1)]$ classes; this is a totally domatic partition. Hence $d_t(G) \geqq [n/(n - \delta(G) + 1)]$.

Theorem 3. *Let G be an undirected graph whose vertex set has the infinite cardinality $\mathfrak{a}$, let $\bar{G}$ be its complement. If the supremum of degrees of vertices of $\bar{G}$ is less than $\mathfrak{a}$, then $d_t(G) = \mathfrak{a}$.*

Proof. In [3] a quite analogous theorem for $d(G)$ instead of $d_t(G)$ is proved. In [4] it is proved that if $d(G)$ is infinite, then $d_t(G) = d(G)$. This implies the assertion.

The bound from Theorem 2 is probably not the best. For $\delta(G) = n - 3$ we can express the following theorem.

Theorem 4. *Let G be a finite undirected graph with n vertices, where $n \geqq 9$, let $\delta(G) = n - 3$. Then $d_t(G) = [n/2]$.*

Remark. Here $[n/2]$ denotes the greatest integer less than or equal to $n/2$. This is the maximum value of $d_t(G)$ at the given n (see [4]).

Proof. Consider the complement $\bar{G}$ of G. In $\bar{G}$ the maximum degree of a vertex is 2. Therefore any connected component of $\bar{G}$ is either a circuit, or a path, or an isolated vertex. Let $\bar{G}_0$ be the subgraph of $\bar{G}$ consisting of all connected components which are paths or isolated vertices. If $\bar{G}_0$ has at least three vertices, then it is possible to add some edges to $\bar{G}$ (i.e. to delete some edges from G) in such a way that $\bar{G}_0$ becomes a circuit. The graph obtained from $\bar{G}$ (or of G) in this way will be denoted by $\bar{G}^*$ (or G^* respectively). If G_0 has at most two vertices, then we put $G^* = G$, $\bar{G}^* = \bar{G}$. The graph G^* is a spanning subgraph of G; if $d_t(G^*) = [n/2]$, then evidently also $d_t(G) = [n/2]$.

Thus we shall prove that $d_t(G^*) = [n/2]$. If a totally domatic partition of G^* has $[n/2]$ classes and n is even, then each class consists of two vertices (evidently no one-element set can be totally dominating); if n is odd, then one has three vertices and all the others two vertices. A two-element subset of $V(G^*)$ is totally dominating in G^*, either if its elements belong to distinct connected components of G^*, or if they belong to the same connected component of G^* and their distance in it is at least 3. (If this distance is 1, then they are not adjacent in G^*; if it is 2, then there exists a vertex non-adjacent to both of them in G.) In $\bar{G}^*$ there is at most one connected component having one or two vertices, while all the others are circuits. We shall construct a totally domatic partition $\mathscr{D}$ of G with $|\mathscr{D}| = \lfloor n/2 \rfloor$. Let $\bar{G}^{**}$ be the subgraph of $\bar{G}^*$ consisting of all connected components being circuits. In each connected component of G^* with an odd number of vertices choose one vertex; let the set of chosen vertices be M. Evidently $|M| \doteq |V(G)| \pmod 2$. Choose a partition of M into sets such that either anyone has two vertices, or one has three and any other has two vertices. All these sets will belong to $\mathscr{D}$. Now in each circuit of the length $k \geqq 6$ we may choose $[k/2]$ pairs of vertices having the distance $[k/2]$. Thus if there are no circuits in $\bar{G}^{**}$ of the length less than 6, the totally domatic partition of $\bar{G}^{**}$ is ready. Suppose that there are circuits $C_1, \ldots, C_p$ of $\bar{G}^{**}$ of lengths 4 or 5 for $p \geqq 2$. In anyone of them of length 5 there is one vertex of M. In each C_i denote the vertices not belonging to M by $u_1^i, u_2^i, u_3^i, u_4^i$. Then we take the pairs $\{u_3^i, u_1^{i+1}\}$, $\{u_4^i, u_2^{i+1}\}$ for all $i = 1, \ldots, p$, taking $i + 1$ modulo p, and put them into $\mathscr{D}$. Now suppose that there are circuits $C_1', \ldots, C_q'$ of the length 3 for $q \geqq 2$. In anyone of them one vertex is in M. In each C_i' we denote the vertices not belonging to M by v_1^i, v_2^i and the pairs $\{v_2^i, v_1^{i+1}\}$ will be added to $\mathscr{D}$. Then the case remains when there is exactly one circuit of the length 3, or exactly one circuit of the length 4 or 5. Distinguish particular subcases:

(a) One circuit C_3'' of the length 3 and one circuit C_4'' of the length 4.

From M omit the vertex belonging to C_3'' and add a vertex of C_4''. Then choose a one-to-one correspondence between the vertices of C_3'' and the vertices of C_4'' not belonging to M. Add the pairs of corresponding vertices to $\mathscr{D}$.

(b) One circuit C_3'' of the length 3 and one circuit C_5'' of the length 5.

Choose a one-to-one correspondence between the vertices of C_3'' not belonging to M and two vertices of C_5'' not belonging to M and put pairs of corresponding vertices into $\mathscr{D}$. Now two vertices of C_5'' remain; thus we have not a totally domatic partition of G^{**}. But as $n \geqq 9$, there are further vertices of $\bar{G}^*$. If $n = 9$, the ninth vertex is isolated in $\bar{G}^*$ and we put the set consisting of it and of the mentioned two vertices into $\mathscr{D}$. If $n = 10$, then we add to $\mathscr{D}$ two pairs, each consisting of one vertex of C_5'' and one vertex not in $C_3'' \cup C_5''$. In the case $n = 11$ there would be two triangles in $\bar{G}^*$, which would not be this case. If $n \geqq 12$, then there is at least one set in $\mathscr{D}$ consisting of vertices outside $C_3'' \cup C_5''$; omit it from $\mathscr{D}$ and add two pairs, each of which consists of one vertex of this pair and one remaining vertex of C_5''. In all the cases we obtain a totally domatic partition of G^*.

(c) Exactly one circuit C'' of the length less than 6.

There exists at least one circuit of the length at least 6 and thus at least three pairs of vertices outside C'' which belong to $\mathscr{D}$. Choose one or two of them (according to the length of C''), omit them from $\mathscr{D}$ and choose a one-to-one correspondence between vertices of C'' not belonging to M and vertices of the chosen pairs. Put the pairs of corresponding vertices to $\mathscr{D}$.

Thus we have constructed a totally domatic partition of G^{**} (in the case (b) even of G^*). Now there are at most two vertices of G^* which are not in G^{**}. If there is none, then $G^{**} = G^*$ and we are ready. If there is one such vertex v and n is odd, we add this vertex (as the third) to an arbitrary pair from $\mathscr{D}$; if n is even, then from the (unique) three-element set being in $\mathscr{D}$ we omit one vertex and to $\mathscr{D}$ we add the pair consisting of this vertex and v. If there are two such vertices v_1, v_2, we omit one pair from $\mathscr{D}$ and add two pairs, each consisting of one of the vertices v_1, v_2 and one vertex of the omitted pair. In all the cases we obtain a totally domatic partition $\mathscr{D}$ of G^* and also of G.

Theorem 5. *Let n be an integer, $3 \leqq n \leqq 8$. Then there exists a graph G with n vertices for which $\delta(G) = n - 3$ and $d_t(G) < [n/2]$.*

Proof. For $n = 3$ it is the graph consisting of three isolated vertices. For $n = 4$ it is the graph consisting of two vertex-disjoint paths of the length 1. For $n = 5$ it is the circuit of the length 5. For $n = 6$ it is the wheel with six vertices. For $n = 7$ it is obtained from this wheel by adding one new vertex and joining it by edges with all vertices distinct from the centre. For $n = 8$ it is the graph obtained from the preceding one by adding a new vertex and joining it by edges with all vertices which are not centres.

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ТОТАЛЬНО ДОМАТИЧЕСКОЕ ЧИСЛО И СТЕПЕНИ ВЕРШИН ГРАФА

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Резюме

Тотально доминантным называется подмножество D множества $V(G)$ вершин графа G, обладающее тем свойством, что для каждого $x \in V(G)$ существует вершина $y \in D$, смежная с вершиной x. Максимальное число классов разбиения множества $V(G)$, все классы которого являются тотально доминантными множествами в G, называется тотально доматическим числом графа G и обозначается через $d_t(G)$. Изучаются отношения между $d_t(G)$ и степенями вершин графа G.