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# ALMOST-EVEN FUNCTIONS AS SOLUTIONS OF A LINEAR FUNCTIONAL EQUATION

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ABSTRACT. Let $f$ be an almost even function in $\mathcal{B}^{2+\eta}$, which is pointwise represented by its Ramanujan expansion. A (complicated) method is given in order to show a result which is easily accessible otherwise: If for all $n$ outside some "exceptional set" $\mathcal{E}$ with upper density 0 the function $n \mapsto g(n) = n \cdot f(n)$ satisfies the functional equation $g(n) = g(\ell) + g(n-\ell)$ for all $\ell$, $1 \le \ell \le n$, then $g(n) = \gamma \cdot n$ identically.

## 1. Introduction and notation

Having seen the paper [2] by Pham van Chung and the deep paper [4] by Claudia Spiro, where the multiplicative solutions of the functional equations $f(m^2+n^2) = f(m^2)+f(n^2)$ resp. $f(p+q) = f(p)+f(q)$, $p$, $q$ prime, are given, the author tried to obtain some results about solutions of functional equations by almost-even functions. This (complicated) method does not seem to work for the problems treated in [2] and [4], but a (trivial) result can be obtained. The author hopes for further, non-trivial applications of this method.

We need some notation.

**Note.** With the abbreviation $\mathbf{e}(\alpha) = \exp(2\pi \mathrm{i}\, \alpha)$, *Ramanujan's sum* is

$$c_r(n) = \sum_{d|(r,n)} d\mu\Big(\frac{r}{d}\Big) = \sum_{\substack{1\le a\le r\\ \gcd(a,r)=1}} \mathbf{e}\Big(\frac{a}{r}\cdot n\Big).$$

For an arithmetical function $f\colon \mathbb{N} \to \mathbb{C}$, define, if the limits involved do exist, the *mean-value*

$$M(f) = \lim_{x\to\infty} \frac{1}{x} \cdot \sum_{n\le x} f(n),$$

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the *Ramanujan coefficients*

$$a_r(f) = \frac{1}{\varphi(r)} \cdot M(f \cdot c_r), \qquad r = 1, 2, \dots,$$

and the *semi-norms*

$$\|f\|_q = \left\{ \limsup_{x\to\infty} \frac{1}{x} \cdot \sum_{n\le x} |f(n)|^q \right\}^{\frac{1}{q}}, \qquad q \ge 1.$$

The *closures of the space*

$$\mathcal{B} = \mathrm{Lin}_{\mathbb{C}}\{c_r, \ r = 1, 2, \dots\}$$

with respect to the norm $\|\cdot\|_q$ are the spaces $\mathcal{B}^q$ of $q$*-almost-even functions* $(q \ge 1)$.

## 2. Results and proofs

We start with a nearly trivial result.

**PROPOSITION 2.1.** *If an arithmetical function* $f\colon \mathbb{N} \to \mathbb{C}$ *in* $\mathcal{B}^2$ *satisfies*

$$|M(f)|^2 = M(|f|^2), \tag{1}$$

*and if the Ramanujan expansion* $\sum_{r=1}^{\infty} a_r(f) \cdot c_r(n)$ *of* $f$ *is pointwise convergent to* $f(n)$, *then* $f = M(f)$ *is constant.*

**Remark 1.** For functions in $\mathcal{B}^2$ the mean-values $M(f)$, $M(|f|)$, and $M(|f|^2)$ do exist. By the Cauchy-Schwarz inequality, $|M(f)|^2 \le M(|f|^2)$.

**Remark 2.** The Ramanujan expansion of additive or multiplicative functions $f \in \mathcal{B}^2$ (with mean-values $\ne 0$) is pointwise convergent to $f(n)$. See [1] and [3; Chapter VIII].

*Proof of Proposition 2.1.* $c_1(n) = 1$, so the Ramanujan coefficient $a_1(f)$ is

$$a_1(f) = \lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} f(n) c_1(n) = M(f).$$

Parseval's equation (see, for example, [3]) states that

$$M(|f|^2) = \sum_{r=1}^{\infty} |a_r(f)|^2 \cdot \varphi(r) = |M(f)|^2 + \sum_{r=2}^{\infty} |a_r(f)|^2 \varphi(r),$$

therefore, by (1), $|a_r(f)| = 0$ for any $r \ge 2$. The convergence of the Ramanujan expansion implies $f(n) = M(f)$ for all $n \in \mathbb{N}$. □

**COROLLARY 2.1.1.** *If $f \in \mathcal{B}^2$ is multiplicative, and satisfies*

$$|M(f)|^2 = M(|f|^2) \neq 0,$$

*then $f = 1$ is constant.*

Proof. According to [3; Chapter VIII.5], the Ramanujan expansion of a multiplicative function with $M(f) \neq 0$ is pointwise convergent, and $f(1) = 1$ implies $f(n) = 1$ for all $n$. □

In order to give a trivial application of Proposition 2.1, we consider the often solved functional equation

$$g(n+m) = g(n) + g(m) \qquad (\text{for every } m, n). \tag{2}$$

Of course, this functional equation is trivially solved by $g(2) = 2 \cdot g(1)$, $g(3) = 3 \cdot g(1)$, etc. The aim of the paper is to present another method for solving functional equations.

**COROLLARY 2.1.2.** *Assume that $g$ satisfies the functional equation (2), and that $n \mapsto f(n) = \frac{g(n)}{n}$ is in $\mathcal{B}^2$ and is represented by its Ramanujan expansion. Then $f = M(f)$ identically, and $g(n) = M(f) \cdot n$.*

Proof. Without loss of generality, we may assume that $f$ is real-valued.[1] Put $f(0) = 0$ for simplicity. Then, using the functional equation (2) in the form $f(n+m) = f(n) \cdot \frac{n}{n+m} + f(m) \cdot \frac{m}{n+m}$, we calculate

$$\begin{aligned} M(f^2) &= \lim_{x\to\infty} \frac{1}{x} \cdot \sum_{n\le x} f(n) \cdot f(n) \\ &= \lim_{x\to\infty} \frac{1}{x} \times \sum_{n\le x} f(n) \cdot \frac{1}{n} \Bigg( \sum_{k+\ell=n} \left\{ \frac{k}{n} \cdot f(k) + \frac{\ell}{n} \cdot f(\ell) \right\} \Bigg) \\ &= \lim_{x\to\infty} \frac{1}{x} \cdot \sum_{n\le x} f(n) \cdot \frac{1}{n^2} \times 2 \cdot \sum_{\ell\le n} \ell \cdot f(\ell) \\ &= \lim_{x\to\infty} \frac{2}{x} \cdot \sum_{\ell\le x} \ell f(\ell) \cdot \sum_{\ell\le n\le x} \frac{f(n)}{n^2}. \end{aligned} \tag{3}$$

[1] If $f = u + \mathrm{i}\, v$, then

$$M(|f|^2) = M(u^2) + M(v^2), \quad \text{and} \quad |M(f)|^2 = |M(u) + \mathrm{i}\, M(v)|^2 = |M(u)|^2 + |M(v)|^2.$$

From $\sum_{n\le x} f(n) \sim M\cdot x$, with the abbreviation $M = M(f)$, we obtain by partial summation,

$$\begin{aligned}\sum_{\ell\le n\le x} \frac{f(n)}{n^2} &= \sum_{\ell\le n\le x} f(n)\cdot\frac{1}{x^2} + 2\int_{\ell}^{x} \sum_{\ell\le n\le u} f(n)\cdot\frac{\mathrm{d}u}{u^3} \\ &= o\left(\frac{1}{x}\right) + \frac{M}{x}\cdot\left(1-\frac{\ell}{x}\right) + 2\int_{\ell}^{x}\left(M(u-\ell)+o(u)\right)\frac{\mathrm{d}u}{u^3} \\ &= o\left(\frac{1}{\ell}\right) + \frac{M}{x}\cdot\left(1-\frac{\ell}{x}\right) + \frac{M}{\ell} - \frac{M}{x}\cdot\left(2-\frac{\ell}{x}\right) \\ &= o\left(\frac{1}{\ell}\right) + M\cdot\left(\frac{1}{\ell}-\frac{1}{x}\right).\end{aligned} \tag{4}$$

Therefore

$$M(f^2) = \lim_{x\to\infty}\left[\frac{2}{x}\cdot\sum_{\ell\le x}\left(M\cdot f(\ell) - M\cdot f(\ell)\cdot\frac{\ell}{x}\right) + o\left(\frac{1}{x}\cdot\sum_{\ell\le x}|f(\ell)|\right)\right]. \tag{5}$$

Notice that $\lim_{x\to\infty} x^{-1}\sum_{\ell\le x}|f(\ell)| = \|f\|_1 \le \|f\|_2 < \infty$. By partial summation we obtain

$$\sum_{l\le x} f(l)\cdot l = M\cdot x^2 + o(x^2) - \int_1^x \left(M\cdot u + o(u)\right)\,\mathrm{d}u = \frac{1}{2}Mx^2 + o(x^2). \tag{6}$$

Therefore, we deduce from (5) and (6)

$$M(f^2) = \lim_{x\to\infty}\frac{2}{x}\cdot\left\{M^2\cdot x + o(x) - \frac{1}{2}M^2\cdot x\right\} + o(1) = \left(M(f)\right)^2,$$

and the result follows from Proposition 2.1. □

With the same proof, the result is easily extended.

**Proposition 2.2.** *Let $f\in\mathcal{B}^{2+\eta}$ for some $\eta>0$ be pointwise represented by its Ramanujan expansion, and let $M(f)\ne 0$. Define the function $g$ by $g(n) = n\cdot f(n)$. Assume that the functional equation*

$$g(n) = g(n-\ell) + g(\ell) \qquad \text{for all} \quad \ell,\ 1\le\ell\le n,$$

*holds for all $n\in\mathbb{N}\setminus\mathcal{E}$, where $\mathcal{E}\subset\mathbb{N}$ is a subset with upper density $\overline{d}(\mathcal{E}) = \limsup_{x\to\infty}\frac{1}{x}\cdot\sum_{n\le x,\, n\in\mathcal{E}} 1 = 0$. Then*

$$f = M(f) \qquad \text{is constant.}$$

We follow the same pattern of proof as before. Without loss of generality, $f$ is real-valued. Notice that (by Hölder's inequality and $\overline{d}(\mathcal{E}=0)$)

$$\sum_{n\le x,\, n\in\mathcal{E}} |f(n)|^2 \le \Big(\sum_{n\le x} |f(n)|^{2+\eta}\Big)^{\frac{2}{2+\eta}} \cdot \Big(\sum_{n\le x,\, n\in\mathcal{E}} 1\Big)^{\frac{\eta}{2+\eta}} = o(x)\,. \tag{7}$$

Paying attention to $\big|\sum_{\ell\le n} \ell\cdot f(\ell)\big| \le \sum_{\ell\le n} n\cdot f(\ell) \ll n^2$, and using the Cauchy-Schwarz inequality, we get

$$\sum_{n\le x,\, n\in\mathcal{E}} \frac{|f(n)|}{n^2} \times 2\cdot\sum_{\ell\le n} \ell\cdot f(\ell) \ll \Big(\sum_{n\le x} |f(n)|^2\Big)^{\frac12} \cdot \Big(\sum_{n\le x,\, n\in\mathcal{E}} 1\Big)^{\frac12} = o(x)\,. \tag{8}$$

Splitting the sum $\sum_{n\le x}$ in (3) into two sums $\sum_{n\le x,\, n\notin\mathcal{E}} + \sum_{n\le x,\, n\in\mathcal{E}}$, and using the two estimates (7) and (8) just deduced, we obtain

$$M(f^2) = \lim_{x\to\infty} \frac{2}{x}\cdot\sum_{\ell\le x} \ell f(\ell)\cdot \sum_{\ell\le n\le x} \frac{f(n)}{n^2}\,,$$

and then the proof is finished as earlier.

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