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ON THE ω-PRIMITIVE

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ABSTRACT. In this paper we continue some results of [KOSTYRKO, P.: *Some properties of oscillation*, Math. Slovaca **30** (1980), 157–162]. It is shown that given a nonnegative, upper semicontinuous (USC) function $f\colon X \to \overline{\mathbb{R}}$ where X is a "massive" metric space, there is a function $F\colon X \to \mathbb{R}$ (which we call an ω-primitive for f) whose oscillation equals f everywhere on X. Moreover, F could always be found in at most Baire class two. In particular, the ω-primitive could be written in a simple form whenever f is finite. Namely, $F = f\varphi$, where φ is the characteristic function of an $\mathcal{F}_\sigma$-set or that of a $\mathcal{G}_\delta$-set. Except "massiveness", no other assumptions concerning metric spaces are made. Our main tool is Teichmüller-Tukey's lemma.

Some definitions and preliminaries

Let $X=(X,\rho)$ be a metric space. Given a function $F\colon X \to \mathbb{R}$, we let for each $x \in X$ and $\delta > 0$

$$M_\delta(F,x) = \sup\{F(z):\ z \in B(x,\delta)\}\,,$$
$$m_\delta(F,x) = \inf\ \{F(z):\ z \in B(x,\delta)\}\,,$$

where $B(x,\delta) := \{z \in X:\ \rho(z,x) < \delta\}$, and we let

$$M(F,x) = \lim_{\delta\to 0} M_\delta(F,x)\,,$$
$$m(F,x) = \lim_{\delta\to 0} m_\delta(F,x)\,.$$

The *oscillation* of F at the point x is defined as

$$\omega(F,x) = M(F,x) - m(F,x)\,.$$

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It is well known from elementary courses in Real Analysis that the oscillation

$$\omega(F,\cdot)\colon X \to \overline{\mathbb{R}}$$

is an upper semicontinuous (USC) and nonnegative function. In the present paper the following problem is studied.

Let $f\colon X \to [0,+\infty]$ be an USC-function. The question is whether there exists a function $F\colon X \to \mathbb{R}$ such that

$$\big(\forall x \in X\big)\big(\omega(F,x) = f(x)\big).$$

If such a function exists we call it an *oscillatory primitive* (or an *ω-primitive*) for f. We also ask of which minimal Baire class an ω-primitive could be. Trivial examples show that the ω-primitive might not exist if X contains isolated points. For this reason we consider only spaces dense in themselves in all statements on ω-primitives.

In particular, we shall make use of the following notions and notations. Let E be a nonempty subset of X. If E contains more than one point, we put

$$\Delta E := \inf\big\{\rho(x_1,x_2):\ x_1,x_2 \in E,\ x_1 \neq x_2\big\}; \tag{1}$$

and if E is a singleton, we put

$$\Delta E := +\infty. \tag{2}$$

By E^d we mean the *derived set*, i.e. the set of all accumulation points of E. By $\overline{E}$ we denote the closure of E. To avoid ambiguities, we specify the definition of extreme limits:

$$\limsup_{t\to x} F(t) := \lim_{r\to 0} \sup F\big|_{B(x,r)\setminus\{x\}}. \tag{3}$$

The lower limit is defined analogously. Our main tool in proofs will be the Teichmüller-Tukey's lemma. For convenience of the reader we remind its formulation.

Let P be a property related to subsets of a set $S \neq \emptyset$. We say that P is a property of finite character if the following holds:

E has the property $P \iff$ each finite set $A \subset E$ has the property P.

LEMMA 1. (Teichmüller-Tukey [1], [3]) *Let P be a property of finite character related to subsets of S. Then each subset $E \subset S$ with the property P is contained in a maximal (with respect to the inclusion relation) subset E_m of S which also has the property P.*

A maximal set E_m will be called a *P-maximal* set. We remind that a P-maximal set need not be unique.

In what follows, given any metric space X, we define for each real $\alpha > 0$ the *property* $P(\alpha)$ *related to subsets* E of X as follows:

$$E \text{ has the property } P(\alpha) \iff \Delta E > \alpha\,, \tag{4}$$

where ΔE was defined in (1), (2). Clearly $P(\alpha)$ is the property of finite character (cf. [1; Vol. 2]). We shall also abbreviate

$$P_n := P(1/n) \qquad \text{for} \quad n \in \mathbb{N}\,. \tag{5}$$

DEFINITIONS.

1) A metric space X is called *σ-discrete at the point* $x \in X$ if there exists $\varepsilon > 0$ such that the ball $B(x,\varepsilon)$ is σ-discrete, i.e

$$B(x,\varepsilon) = \bigcup_{n=1}^{\infty} A_n\,,$$

where each A_n is a discrete subset of X (empty set is discrete by definition).

2) A metric space is said to be *locally σ-discrete* if it is σ-discrete at each of its points.

3) A metric space will be called *massive* if it is not σ-discrete at each of its points.

Our main result will be stated in Theorem 2, but first we shall show the existence of ω-primitives of type $f\varphi$ where φ is the characteristic function of an $\mathcal{F}_\sigma$-set (or that of a $\mathcal{G}_\delta$-set) whenever f is finite.

The following auxiliary assertion is valid in any metric space.

LEMMA 2. *Each σ-discrete subset A of a metric space X can be represented in the form*

$$A = \bigcup_{i \in I \subset \mathbb{N}} C_i\,, \tag{6}$$

where C_i are disjoint and $\Delta C_i > 0$ for each $i \in I \subset \mathbb{N}$. So in particular we have that A is an $\mathcal{F}_\sigma$-set.

Proof. It is easy to see that it suffices to consider the case A is discrete and $\Delta A = 0$. We may write

$$A = \bigcup_{n=1}^{\infty} A_n\,,$$

where

$$A_n := \{x \in A :\ \operatorname{dist}(x,\, A \setminus \{x\}) > 1/n\}\,.$$

Clearly $\Delta A_n \geq 1/n$ and $A_n \subset A_{n+1}$. Now it remains to write

$$A = \bigcup_{n=1}^{\infty} C_n\,,$$

where $C_1 := A_1$, $C_n := A_n \setminus A_{n-1}$ if $n > 1$, and we are done. ☐

THEOREM 1. *Let $X = (X, \rho)$ be a massive metric space and $f\colon X \to [0, \infty)$ be a USC-function. Then there exists an ω-primitive $F\colon X \to [0, \infty)$ for f which can be represented in the form $F = f\varphi$ where φ is the characteristic function of an $\mathcal{F}_\sigma$-set.*

Proof. Let $G(f)$ be the graph of f, which will be considered as a subspace of the metric space $X \times \mathbb{R}$ equipped with the metric

$$d\big((x,\xi),(y,\eta)\big) := \rho(x,y) + |\xi - \eta| \,. \tag{7}$$

Denote by $\pi\colon X \times \mathbb{R} \to X$ the natural projection. Now in the space $G(f)$ we consider the property $P(\alpha)$ (cf. (4)). We may assume, without loss of generality, that $\operatorname{diam} X > 1$, so that we have $\operatorname{diam} G(f) > 1$ too. Using Lemma 1 we conclude that there exists a P_1-maximal set Y_1 in $G(f)$ (cf. notation (5)). We claim that $\pi(Y_1)$ is a discrete subset of X. Indeed, if we assume the contrary, there will exist $x_0 \in \big(\pi(Y_1)\big)^d \cap \pi(Y_1)$ and a sequence (x_n), $x_n \in \pi(Y_1)$, such that

$$x_n \neq x_m \quad \text{for } n \neq m\,, \qquad \text{and} \qquad \rho(x_n, x_0) \to 0\,. \tag{8}$$

Then from the P_1-property of Y_1 we obtain:

$$(\forall n)(\forall m)\big(n \neq m \implies \rho(x_n, x_m) + |f(x_n) - f(x_m)| > 1\big)\,. \tag{9}$$

Since f is USC and $f \geq 0$, we have that f is locally bounded. Hence there exist a ball $B(x_0, r)$, $r < 1/5$, and a natural N so that $\sup f|_{B(x_0,r)} < \infty$ and $x_n \in B(x_0, r)$, $n > N$. Therefore by (9) we get

$$(\forall n > N)(\forall m > N)\big(n \neq m \implies |f(x_n) - f(x_m)| > 1/2\big)\,,$$

which contradicts the boundedness of $f|_{B(x_0,r)}$. Thus $\pi(Y_1)$ is discrete whence $X \setminus \pi(Y_1)$ is massive since such is X. This implies that $\operatorname{diam}\big(X \setminus \pi(Y_1)\big) = \operatorname{diam} X > 1$ and therefore $\operatorname{diam}\big(G(f) \setminus Y_1\big) > 1$. So we may again apply Lemma 1 to $G(f) \setminus Y_1$ and find a P_2-maximal set $Y_2 \subset G(f) \setminus Y_1$. In the same way as for $\pi(Y_1)$, we prove that $\pi(Y_2)$ is a discrete subset of X, hence $\operatorname{diam}\big(X \setminus \big(\pi(Y_1) \cup \pi(Y_2)\big)\big) > 1$.

On repeating inductively this procedure, we obtain a sequence (Y_n) with the following properties:

(i) Y_1 is a P_1-maximal subset of $G(f)$;
(ii) Y_n is a P_n-maximal subset of $G(f) \setminus (Y_1 \cup \cdots \cup Y_{n-1})$, $n > 1$;
(iii) $\pi(Y_n)$ is a discrete subset of X.

We claim that the set

$$E := \bigcup_{n=1}^{\infty} Y_n \tag{10}$$

is dense in $G(f)$. Indeed, suppose that this is not the case. Then there is a ball $B(a,r) \subset G(f)$, disjoint from E. Then for each $n > 2/r$ we have

$$\inf\{d(z,a) : z \in Y_n\} \geq r > 1/n .$$

But since $\Delta Y_n > 1/n$ and $a \in G(f) \setminus (Y_1 \cup \cdots \cup Y_{n-1})$, we obtain that for $n > 2/r$ the set

$$Y_n \cup \{a\} \subset G(f) \setminus (Y_1 \cup \cdots \cup Y_{n-1})$$

has the P_n-property, contrary to the fact that Y_n is already a P_n-maximal subset of $G(f) \setminus (Y_1 \cup \cdots \cup Y_{n-1})$. We have thus proved that E is dense in $G(f)$. It follows from property (iii) of the sequence (Y_n) that

$$\pi(E) = \bigcup_{n=1}^{\infty} \pi(Y_n)$$

is a σ-discrete and dense subset of X. The space X being massive, we conclude that $X \setminus \pi(E)$ is also dense in X.

Now define the function

$$F = f\varphi , \tag{11}$$

where φ is the characteristic function of $\pi(E)$. Since $\pi(E)$ is an $\mathcal{F}_\sigma$-set (cf. Lemma 2) and f is USC, we conclude that F is at most in the Baire class two. It remains to check that F is an ω-primitive for f.

First we observe that since $X \setminus \pi(E)$ is dense in X, we have

$$(\forall x \in X)\big(m(F,x) = 0\big) . \tag{12}$$

(I) Let $x_0 \in \pi(E)$. Since f is USC, we get immediately from (11), (12) that

$$\omega(F,x_0) = M(F,x_0) = M(f,x_0) = f(x_0) .$$

(II) Let $x_0 \in X \setminus \pi(E)$. Then we have

$$\limsup_{x \to x_0} f(x) = f(x_0) . \tag{13}$$

Indeed, if this were not the case we would get

$$\limsup_{x \to x_0} f(x) < f(x_0) .$$

Then it would follow that $\big(x_0, f(x_0)\big)$ is an isolated point of $G(f)$. But as E is dense in $G(f)$, we infer immediately that $\big(x_0, f(x_0)\big) \in E$, which yields $x_0 \in \pi(E)$, contrary to the assumption of p. (II). Thus (13) holds whence it follows that there is a sequence (x_n), $x_n \in X$, $x_n \neq x_0$, such that

$$\lim f(x_n) = f(x_0) . \tag{14}$$

Moreover, since E is dense in $G(f)$, there exists a sequence (z_n), $z_n \in \pi(E)$, so that

$$(\forall n)\big(\rho(x_n, z_n) < 1/n \ \wedge \ |f(x_n) - f(z_n)| < 1/n\big).$$

This implies by (11), (14) and by f is USC that

$$f(x_0) = \lim f(z_n) = \lim F(z_n) \le M(F, x_0) \le M(f, x_0) = f(x_0)$$

whence, in view of (12), we get $\omega(F, x_0) = f(x_0)$. This completes the proof of Theorem 1. □

Remark. It is shown in [2; Corollary 1.2(c)] that each dense in itself metrizable Baire space is massive. On the other hand, there are massive spaces (metrizable or not) which are not Baire ([2; Examples 1.2, 1.3]). Thus our Theorem 1 extends the result of P. Kostyrko [4] obtained for metric Baire spaces.

Theorem 1 gives rise to our main result which follows. This time f will be allowed to take on the value $+\infty$.

THEOREM 2. *Let $X = (X, \rho)$ be a massive metric space and $f\colon X \to [0, +\infty]$ a USC-function. Then there exists an ω-primitive $F\colon X \to [0, +\infty)$ for f, which is at most in the Baire class two.*

We precede the proof by a simple auxiliary proposition.

LEMMA 3. *Given any massive metric space Z, there exists $m \in \mathbb{N}$ and a sequence $(W_n)_{n=m}^{\infty}$ of mutually disjoint subsets of Z such that*

(i) *W_m is P_m-maximal in Z, and for $n > m$ each W_n is P_n-maximal in $Z \setminus (W_m \cup \dots \cup W_{n-1})$;*

(ii) *the sets $W := \bigcup_{n=m}^{\infty} W_n$ and $Z \setminus W$ are both dense in Z.*

Proof. The main tool in the proof (same as in that of Theorem 1) is Lemma 1. With no loss of generality we may assume that $\operatorname{diam} Z > 1$ (and therefore we shall have in that case $m = 1$). By Lemma 1 we may find a P_1-maximal set $W_1 \subset Z$. This set being discrete, the set $Z \setminus W_1$ is again massive. Therefore by Lemma 1 we can find a P_2-maximal set W_2 in $Z \setminus W_1$ and so on. On the nth step we find a P_n-maximal set W_n in

$$Z \setminus (W_1 \cup \dots \cup W_{n-1})$$

(to note that this difference remains massive for each $n > 1$). Continuing this procedure, we obtain the required sequence (W_n). Indeed, write

$$W := \bigcup_{n=1}^{\infty} W_n .$$

Since $\Delta W_n > 1/n$, $n \in \mathbb{N}$, the set W is σ-discrete. We claim that W is dense in Z. But this can be shown in the same way as we proceeded to prove, in Theorem 1, that the set E (10) is dense in $G(f)$. So we omit the repetition of the argument. Finally, since Z is massive whereas W is σ-discrete, we conclude that $Z \setminus W$ is also dense in Z. Lemma 3 is thus proved. □

As an immediate corollary, we easily obtain the following analog of Theorem 1 involving $\mathcal{G}_\delta$-sets.

THEOREM 1'. *Let $X = (X, \rho)$ be a massive metric space and $f\colon X \to [0, \infty)$ a USC-function. Then there exists an ω-primitive $F\colon X \to [0, \infty)$ for f which can be represented in the form $F = f\varphi$, where φ is the characteristic function of a $\mathcal{G}_\delta$-set.*

Proof. Let $E \subset G(f)$ be the set already defined by (10). Since $X \setminus \pi(E)$ is a massive subspace of X we may apply Lemma 3 according to which there exists a sequence (W_n), $W_n \subset X \setminus \pi(E)$, with properties (i), (ii). It follows that W and $X \setminus W$ are both dense in X, the set W obviously being of type $\mathcal{F}_\sigma$. Now as the set E is dense in $G(f)$, $\pi(E) \cap W = \emptyset$ and f is USC, we easily conclude that $F = f\varphi$, where φ is the characteristic function of the $\mathcal{G}_\delta$-set $X \setminus W$, is the ω-primitive for f, and we are done. □

It remains to prove our main result.

Proof of Theorem 2. We remind that by our definition an ω-primitive takes on only finite values. Suppose that the set

$$E_\infty := \{x \in X : \ f(x) = +\infty\}$$

(evidently closed) is nonempty for otherwise there, of course, would be nothing to prove. With no loss of generality, we may also assume that

$$Y := X \setminus E_\infty \neq \emptyset,$$
$$Z := \operatorname{Int} E_\infty \neq \emptyset.$$

We have thus the disjoint union

$$X = Y \cup Z \cup (E_\infty \setminus Z). \tag{15}$$

The space Y and the function $f|_Y$ obviously satisfy the assumptions of Theorem 1, hence there is an ω-primitive $F_y\colon Y \to [0, +\infty)$ for $f|_Y$, which is at most in the Baire class two. Next we are going to find an ω-primitive for $f|_Z$. Since the open set $Z \subset X$ is a massive space, there exists a sequence (W_n) having properties (i), (ii) stated in Lemma 3. It is therefore clear that the function $F_z\colon Z \to \mathbb{R}$ defined by

$$F_z(x) = \begin{cases} n & \text{for } x \in W_n, \\ 0 & \text{for } x \in Z \setminus W \end{cases}$$

is an ω-primitive for $f|_Z$.

Now let $T(x) := \big(\operatorname{dist}(x, E_\infty)\big)^{-1}$, $x \in Y$. We claim that the function $F\colon X \to [0,\infty)$ defined by

$$F(x) = \begin{cases} F_y(x) + T(x) & \text{if } x \in Y\,, \\ F_z(x) & \text{if } x \in Z\,, \\ 0 & \text{if } x \in E_\infty \setminus Z \end{cases}$$

is an ω-primitive for f. Indeed, since $T\colon Y \to [0,+\infty)$ is continuous and the "partial" ω-primitives F_y, F_z are already defined on disjoint open sets $Y, Z \subset X$, it is clear that the equality

$$\omega(F,x) = f(x)$$

needs to be checked only at points of $E_\infty \setminus Z$.

Let $x_0 \in E_\infty \setminus Z$. If $x_0 \in \overline{Z}$, then in view of the definition of F_z we have $\omega(F,x_0) = +\infty$ since $F(x_0) = 0$ and each neighborhood of x_0 intersects W_n for all sufficiently large n. On the other hand, if $x_0 \notin \overline{Z}$ then each neighborhood of x_0 intersects Y. Since F_y might be bounded, just the addition of the function $Y \ni x \mapsto T(x)$ which goes to $+\infty$ as $x \to x_0$, guarantees that $\omega(F,x_0) = +\infty$. We have thus shown that the function F defined above is an ω-primitive for f. Finally, since F_y, F_z are at most in the Baire class two, we conclude that F is so too, thereby completing the proof of Theorem 2. □

Remark. Instead of Theorem 1, we could use, of course, Theorem 1' in the proof of Theorem 2.

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