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Dedicated to Professor Tibor Šalát
on the occasion of his 70th birthday

MULTIPLICATIVE FUNCTIONS SATISFYING THE EQUATION $f(m^2+n^2) = f(m^2) + f(n^2)$

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ABSTRACT. In the present paper, we characterize multiplicative and completely multiplicative functions f which satisfy the equation $f(m^2+n^2) = f(m^2)+f(n^2)$ for all positive integers m and n.

1. Results

A *multiplicative function* is a function f defined on the set of positive integers such that $f(mn) = f(m)f(n)$ whenever the greatest common divisor of m and n is 1.

The function is called *completely multiplicative* if the condition $f(mn) = f(m)f(n)$ holds for all m and n.

Claudia A. Spiro [2] proved that if a multiplicative function f satisfies the condition $f(p+q) = f(p) + f(q)$ for all primes p, q and $f(p_0) \neq 0$ for at least one prime p_0, then $f(n) = n$ for each positive integer n.

Replacing the set of primes by the set of squares, we investigate the multiplicative functions satisfying the equation $f(m^2+n^2) = f(m^2) + f(n^2)$ for all positive integers m, n. We prove the following result.

THEOREM. *Let $f \neq 0$ be a multiplicative function. Then f fulfills the condition*

(E) $f(m^2+n^2) = f(m^2) + f(n^2)$

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for all positive integers m *and* n *if and only if either*

(E-1) $f(2^k) = 2^k$ *for all integers* $k \geq 0$,
(E-2) $f(p^k) = p^k$ *for all primes* $p \equiv 1 \pmod 4$ *and all integers* $k \geq 1$,
(E-3) $f(q^{2k}) = q^{2k}$ *for all primes* $q \equiv 3 \pmod 4$ *and all integers* $k \geq 1$,

or

(E'-1) $f(2) = 2$ *and* $f(2^k) = 0$ *for all integers* $k \geq 2$,
(E'-2) $f(p^k) = 1$ *for all primes* $p \equiv 1 \pmod 4$ *and all positive integers* k,
(E'-3) $f(q^{2k}) = 1$ *for all primes* $q \equiv 3 \pmod 4$ *and all integers* $k \geq 1$.

COROLLARY. *Let* $f \neq 0$ *be a completely multiplicative function. Then* f *satisfies the condition* $f(m^2+n^2) = f(m^2) + f(n^2)$ *for all positive integers* m *and* n *if and only if* $f(2) = 2$, $f(p) = p$ *for all primes* $p \equiv 1 \pmod 4$ *and* $f(q) = q$ *or* $f(q) = -q$ *for all primes* $q \equiv 3 \pmod 4$.

Proof. By the theorem, if a function f is completely multiplicative and (E) holds, then we have $f(2) = f(1)+f(1) = 2$. So the complete multiplicativity of f gives $f(2^k) = \big(f(2)\big)^k = 2^k$ for all positive integers k. In this case, the completely multiplicative function f satisfies (E) if and only if the conditions (E-1), (E-2) and (E-3) hold. By (E-1) and (E-2), we have $f(2) = 2$ and $f(p) = p$ for all primes $p \equiv 1 \pmod 4$. By (E-3) and the complete multiplicativity of f, $f(q^2) = \big(f(q)\big)^2 = q^2$ follows for all primes $q \equiv 3 \pmod 4$. These prove the corollary. □

For the proof of the theorem we need some auxiliary results.

2. Lemmas

In the following, f denotes a multiplicative function for which there exists a positive integer m_0 with $f(m_0) \neq 0$ and

$$f(m^2+n^2) = f(m^2) + f(n^2) \tag{1}$$

holds for all positive integers m and n.

LEMMA 1. *If* f *satisfies* (1), *then* $f(2) = 2$, $f(9) = 2f(4)+1$ *and* $f(25) = 6f(4)+1$.

Proof. Since $f \neq 0$ is multiplicative, we have $f(1) = 1$. Therefore, (1) implies that $f(2) = f(1^2+1^2) = f(1)+f(1) = 1+1 = 2$. Moreover, by (1) and $f(2) = 2$ and the multiplicativity of f, we have $f(9) = f(10)-f(1) = 2f(5)-1 = 2\big(f(4)+f(1)\big) - 1 = 2f(4)+1$, which implies that $f(25) = f(26) - f(1) = 2f(13) - 1 = 2\big(f(9)+f(4)\big) - 1 = 6f(4)+1$.

So the lemma is proved. □

Lemma 2. *If f satisfies* (1), *then*

$$f(q^{2k}) = f(q^{2k-2})f(q^2) \tag{2}$$

for all positive integers q and k.

Proof. By (1), we have

$$f(q^{2k})+f(q^{2k-2}) = f(q^{2k}+q^{2k-2}) = f(q^{2k-2})f(q^2+1) = f(q^{2k-2})\big[f(q^2)+f(1)\big]$$

from which the lemma $f(q^{2k}) = f(q^{2k-2})f(q^2)$ follows for all positive integers q and k. So, the proof of Lemma 2 is completed, and, moreover, by induction, it gives

$$f(q^{2k}) = \big(f(q^2)\big)^k . \tag{3}$$

□

Lemma 3. *f satisfies* (1), *then $f(4) = 4$ or $f(4) = 0$.*

Proof. By using (3), we have $f(16) = \big(f(4)\big)^2$. On the other hand, by (1), we obtain $f(16) = f(25) - f(9)$. From Lemma 1, it follows that $f(16) = 4f(4)$. Thus we have $\big(f(4)\big)^2 = 4f(4)$, from which $f(4) = 4$ or $f(4) = 0$.

So Lemma 3 is proved. □

Lemma 4. *If f satisfies* (1), *then*

$$f(2^k) = 2^{k-2}f(2^2) \tag{4}$$

for all integers $k \geq 2$.

Proof. We argue by induction on k. When $k = 2$ or 3, equality (4) is obvious.

Assume that n is an integer with $n \geq 3$, and that $f(2^k) = 2^{k-2}f(2^2)$ for all integers k, $2 \leq k \leq n$. We will show that $f(2^{n+1}) = 2^{n-1}f(2^2)$. If $n+1$ is even, then $n+1 = 2k$, where $2k-2 < n$ and $k \geq 2$. By (2) and the induction hypothesis, we have

$$f(2^{n+1}) = f(2^{2k-2})f(2^2) = 2^{2k-4}f(2^2)f(2^2) = 2^{2k-4}\big(f(2^2)\big)^2 .$$

Equality $\big(f(2^2)\big)^2 = 4f(2^2)$ implies that $f(2^{n+1}) = 2^{2k-2}f(2^2) = 2^{n-1}f(2^2)$. It remains to show that $f(2^{n+1}) = 2^{n-1}f(2^2)$ when $n+1$ is odd. If $n+1$ is odd, then $n+1 = 2k+1$, and so $2k = n$. Thus (1) gives that

$$f(2^{n+1}) = f(2^{2k}+2^{2k}) = f(2^{2k}) + f(2^{2k}) = 2f(2^{2k}) = 2\cdot 2^{2k-2}f(2^2) .$$

So $f(2^{n+1}) = 2^{n-1}f(2^2)$, which proves the lemma. □

Lemma 5. *If f satisfies* (1) *and* $f(4) = 4$, *then*

$$f(m^2) = m^2 \tag{5}$$

for all positive integers m.

P r o o f . We shall prove the lemma by induction on m. The lemma is clear for the cases $m = 1, 2, 3$. Assume that M is an integer with $M \geq 3$, and that $f(m^2) = m^2$ for all $m \leq M$. We will show $f\left[(M+1)^2\right] = (M+1)^2$. If $M+1$ is even, then $M+1 = 2^k m$, where $m < M$ and m is odd. By the multiplicativity of f, Lemma 4, $f(4) = 4$, and the induction hypothesis, we have

$$f\left[(M+1)^2\right] = f(2^{2k}m^2) = f(2^{2k})f(m^2) = 2^{2k-2}f(2^2)m^2 = (M+1)^2 .$$

If $M+1 = q$ is odd, then we can write

$$q^2 + 1 = 2\left[\left(\frac{q+1}{2}\right)^2 + \left(\frac{q-1}{2}\right)^2\right],$$

where $\dfrac{q \pm 1}{2}$ are integers.

Since $\dfrac{q \pm 1}{2} \leq M$ and $\left(2, \dfrac{q^2+1}{2}\right) = 1$, we obtain that

$$\begin{aligned} f(q^2) + 1 = f(2)f\left[\left(\frac{q+1}{2}\right)^2 + \left(\frac{q-1}{2}\right)^2\right] &= 2\left[f\left(\left(\frac{q+1}{2}\right)^2\right) + f\left(\left(\frac{q-1}{2}\right)^2\right)\right] \\ = 2\left[\left(\frac{q+1}{2}\right)^2 + \left(\frac{q-1}{2}\right)^2\right] &= q^2 + 1 , \end{aligned}$$

from which $f(q^2) = q^2$, i.e., $f\left((M+1)^2\right) = (M+1)^2$ follows, which completes the proof of the lemma. □

Lemma 6. *If f satisfies* (1) *and* $f(4) = 4$, *then*

$$f(p^k) = p^k \tag{6}$$

for all primes $p \equiv 1 \pmod 4$ and all positive integers k.

P r o o f . Since $p \equiv 1 \pmod 4$, there exist positive integers x and y such that

$$p^k = x^2 + y^2$$

(see [1; p. 298]). So, from Lemma 5, we get

$$f(p^k) = f(x^2 + y^2) = f(x^2) + f(y^2) = x^2 + y^2 = p^k .$$

□

Lemma 7. *If* (1) *holds and* $f(4) = 0$, *then*

$$f(m^2) = 1 \tag{7}$$

for all odd positive integers m.

Proof. First we note that, using Lemma 4, $f(4) = 0$ implies $f(2^k) = 0$ for all $k \geq 2$, which, with the multiplicativity of f, implies that $f(x^2) = 0$ if x is even.

Equality (7) is true for $m = 1$. Let m be an odd integer $m \geq 3$. Assume that $f(n^2) = 1$ for all odd integers n, $1 \leq n < m$. We have

$$f(m^2) = 2\left[f\left(\left(\frac{m+1}{2}\right)^2\right) + f\left(\left(\frac{m-1}{2}\right)^2\right)\right] - 1, \qquad \text{where } \frac{m \pm 1}{2} < m,$$

and so

$$f(m^2) = \begin{cases} 2f\left(\left(\frac{m+1}{2}\right)^2\right) - 1 & \text{if } m \equiv 1 \pmod 4, \\ 2f\left(\left(\frac{m-1}{2}\right)^2\right) - 1 & \text{if } m \equiv 3 \pmod 4. \end{cases}$$

Using the induction hypothesis, one easily completes the proof of Lemma 7. □

Lemma 8. *If* (1) *holds and* $f(4) = 0$, *then*

$$f(p^k) = 1 \tag{8}$$

for all primes $p \equiv 1 \pmod 4$ *and for all positive integers* k.

Proof. Since $p \equiv 1 \pmod 4$, there exist positive integers x and y such that $p^k = x^2 + y^2$, where x is even and y is odd, from which by (1)

$$f(p^k) = f(x^2) + f(y^2)$$

follows. By (7), we have $f(y^2) = 1$. On the other hand, we have shown in the proof of Lemma 7 that $f(x^2) = 0$.

So $f(p^k) = 1$ and Lemma 8 is proved. □

3. Proof of the Theorem

First, we verify the necessity of the conditions.

If f fulfills the conditions of Theorem, then, by Lemma 3, $f(4)$ may take only the values 4 or 0. If $f(4) = 4$, then, by Lemmas 4, 6 and 5, the conditions (E-1), (E-2) and (E-3) are satisfied. If $f(4) = 0$, then, in Lemmas 1, 4, 7 and 8, we have proved the conditions (E'-1), (E'-2) and (E'-3). So we have proved the necessity of the conditions.

Conversely, suppose that either the conditions (E-1), (E-2), (E-3) or (E'-1), (E'-2), (E'-3) are satisfied for a multiplicative function f.

It is well known that, if $M = m^2 + n^2$, then we can write

$$M = 2^k p_1^{\alpha_1} \dots p_l^{\alpha_l} q_1^{2\beta_1} \dots q_s^{2\beta_s}, \tag{9}$$

where p_i and q_j are primes, $p_i \equiv 1 \pmod 4$ and $q_j \equiv 3 \pmod 4$ for $i = 1, 2, \dots, l$ and $j = 1, 2, \dots, s$ and $k \geq 0$. Suppose that (E-1), (E-2) and (E-3) are fulfilled. Then, by the multiplicativity of f, we have

$$\begin{aligned} f(m^2 + n^2) &= f(2^k) f(p_1^{\alpha_1}) \dots f(p_l^{\alpha_l}) f(q_1^{2\beta_1}) \dots f(q_s^{2\beta_s}) \\ &= 2^k p_1^{\alpha_1} \dots p_l^{\alpha_l} q_1^{2\beta_1} \dots q_s^{2\beta_s} \\ &= m^2 + n^2 = f(m^2) + f(n^2). \end{aligned}$$

So we have shown that f satisfies (E).

Finally, suppose that (E'-1), (E'-2) and (E'-3) hold for the multiplicative function f. Now we consider the values of $f(m^2 + n^2)$.

By (9), the multiplicativity of f, and (E'-2), (E'-3), we have

$$f(m^2 + n^2) = f(2^k).$$

If $k = 0$, then exactly one of the two integers m and n is odd. We may assume m is even and n is odd. So, as above $f(m^2) = 0$, and, by (E'-3), we get $f(n^2) = 1$.

Thus $f(m^2 + n^2) = f(m^2) + f(n^2) = 1$.

If $k = 1$, then both m and n are odd. By (E'-3), $f(m^2) = f(n^2) = 1$, from which we obtain

$$f(m^2 + n^2) = f(m^2) + f(n^2) = 2.$$

If $k \geq 2$, then both m and n are even. (E'-1) and the multiplicativity of f imply $f(m^2) = f(n^2) = 0$, which gives the equality

$$f(m^2 + n^2) = f(m^2) + f(n^2) = 0.$$

This completes the proof of the theorem.

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