Milan Jasem

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ON ISOMETRIES IN PARTIALLY ORDERED GROUPS

MILAN JASEM

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ABSTRACT. In this note congruences in partially ordered groups are studied. A necessary and sufficient condition for any congruence in a Riesz group and a distributive multilattice group to be extendable to an isometry is given. Further, it is shown that all congruences in an abelian lattice ordered group G can be extended to isometries if and only if G is strongly projectable.

Congruences and isometries in an abelian lattice ordered group (l-group) have been introduced and investigated by S w a m y [18], [19]. Isometries in non-abelian l-groups were studied by J a k u b í k [7], [8] and H o l l a n d [6]. R a c h ů n e k [17] generalized the notion of the isometry for any partially ordered group (po-group). Isometries in Riesz groups and multilattice groups were investigated in [10], [11], [12], [13], [14], [17]. In [16] P o w e l l studied conditions under which congruences in abelian l-groups can be extended to isometries.

In this note we study congruences in partially ordered groups. We give a necessary and sufficient condition for any congruence in a Riesz group and a distributive multilattice group to be extended to an isometry. Further, it is proved that if each congruence in an abelian weak polar group G can be extended to an isometry in G, then G is strongly projectable (for definitions see below). It is also shown that all congruences in an abelian l-group G can be extended to isometries if and only if G is strongly projectable. These results correct some of P o w e l 's results on congruences in abelian l-groups from [16].

First we recall some notions and notations used in the paper.

Let G be a po-group. The group operation will be written additively (though it is not assumed that the group is abelian). If $S \subseteq G$, we denote $S^+ = \{x \in S,\ x \geqq 0\}$, $S^- = \{x \in S,\ x \leqq 0\}$. For $a_1, \dots, a_n \in G$, we

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denote by $U(a_1,\dots,a_n)$ and $L(a_1,\dots,a_n)$ the set of all upper bounds and the set of all lower bounds of the set $\{a_1,\dots,a_n\}$ in G, respectively. If for $a,\ b\in G$ there exists the least upper bound (greatest lower bound) of the set $\{a,\ b\}$ in G, then it will be denoted by $a\vee b$ $(a\wedge b)$. For $x\in G$, $|x|=U(x,-x)$. In the case, when the considered *po*-group is an *l*-group, $|x|=x\vee(-x)$ for each element x of the considered *l*-group. If $G=P\times Q$ is a direct decomposition of G, then for $x\in G$ we denote by x_P and x_Q the components of x in the direct factors P and Q, respectively.

A *Riesz group* is any *po*-group H which is directed and satisfies the Riesz interpolation property, i.e., for each a_i, $b_j\in H$ $(i,\ j=1,2)$ such that $a_i\geqq b_j$ $(i,\ j=1,2)$ there exists $c\in H$ such that $a_i\geqq c\geqq b_j$ $(i,\ j=1,2)$. See [3] or [5].

A partially ordered set P is a *multilattice* if for each pair of elements $a,b\in P$, every upper bound of the set $\{a,\ b\}$ in P is over a minimal upper bound of the set $\{a,\ b\}$, and dually. A directed *po*-group H is said to be a *multilattice group* if it is a multilattice under $\geqq$. If x and y are elements of a multilattice group H, then we denote by $x\bigvee_m y$ the set of all minimal elements of the set $U(x,\ y)$ in H. The meaning of $x\bigwedge_m y$ will be analogous. A multilattice group H is said to be *distributive* if for $a,\ b,\ c\in H$ the relations $\left(a\bigvee_m b\right)\cap\left(a\bigvee_m c\right)\neq\emptyset$, $\left(a\bigwedge_m b\right)\cap\left(a\bigwedge_m c\right)\neq\emptyset$ together imply $b=c$. See [1], [15].

Note that every *l*-group is a Riesz group and a distributive multilattice group. But a Riesz group need not be a multilattice group and conversely, a multilattice group need not be a Riesz group.

If S is a subset of a *po*-group G, then a mapping $f\colon S\to G$ is called a *congruence on* S if $|x-y|=|f(x)-f(y)|$ for each $x,\ y\in S$. If $0\in S$ and $f(0)=0$, then a congruence f on S is said to be a *0-congruence.* A congruence (0-congruence) f on S is called an *isometry* (*0-isometry*) if $S=G$ and f is a bijection.

Remark. Swamy [18] defined an isometry in an abelian *l*-group C as a surjection $f\colon C\to C$ such that

$$|x-y|=|f(x)-f(y)|\qquad\text{for each}\quad x,\ y\in C. \tag{1}$$

It is obvious that in a *po*-group C any mapping $f\colon C\to C$ which satisfies (1) is an injection. The fact that in a representable *l*-group C (and so in any abelian *l*-group) any mapping $f\colon C\to C$ satisfying (1) is a surjection is not obvious and was proved by Jakubík [9]. This Jakubík's result was extended to isolated Riesz groups and distributive multilattice groups (and hence to *l*-groups) in [13], [14]. It is clear that in a *po*-group C any mapping $f\colon C\to C$ which satisfies (1) need not be a surjection.

Let G be an abelian l-group and L a sublattice of G containing 0. For a congruence $f\colon L \to G$ let $T_f\colon L \to G$ be defined by $T_f(x) = f(x) - f(0)$. Let $A = \{T_f(x) \vee 0,\ x \in L^+\}$, $B = \{-T_f(x) \vee 0,\ x \in L^+\}$. Under this notation Powell proves the following proposition in [16].

PROPOSITION 5. *A congruence $f\colon L \to G$ can be extended to an isometry $\overline{f}\colon G \to G$ if and only if $G = \overline{A} \times \overline{B}$, where $A \subseteq \overline{A}$ and $B \subseteq \overline{B}$.*

The following example shows that Proposition 5 of Powell [16] on congruences in abelian l-groups is not correct.

Example. It is well known that the set C of all continuous functions on the closed interval $[0,1]$ is an abelian l-group under pointwise addition and order. As was shown in [7] (see also [16]), for every 0-isometry f in an l-group G there exists a uniquely determined direct decomposition $G = P \times Q$ such that $f(x) = x_P - x_Q$ for each $x \in G$. Since C has no nontrivial direct factors, there exist only two 0-isometries f_1 and f_2 in C and these are of the form $f_1(x) = x$ or $f_2(x) = -x$ for all $x \in C$. Let $a(x) = \sin 2\pi x$, $b(x) = -|a(x)|$ for each $x \in [0,1]$. Let $L = \{0, b\}$ (0 is the neutral element of C). So L is a sublattice of C containing 0. Let $g(b) = a$, $g(0) = 0$. Then g is a 0-congruence on L. By Proposition 5 [16], g can be extended to an isometry $\overline{g}$ on C. But $\overline{g} \neq f_1$, $\overline{g} \neq f_2$, a contradiction.

Proposition 5 will not be correct even in the case that we take the set L instead of the set L^+ in the definitions of the sets A and B. Namely, if we take $L = \{0,\ a,\ a \vee 0,\ a \wedge 0\}$ and define $h(z) = -z$ for each $z \in L$, then h is a 0-congruence on L and clearly can be extended to an isometry on C. But by Proposition 5 [16], h cannot be extended to an isometry on C.

The following three theorems correct Proposition 5 [16]. Moreover, instead of the assumption of 5 [16] that G is a lattice ordered group we apply more general assumptions.

1. THEOREM. *Let G be a po-group, $S \subseteq G$ and let f be a congruence on S. Let there exist a direct decomposition $G = P \times Q$ of G with Q abelian such that $A = \{x + f(x) + a_Q,\ x \in S\} \subseteq P$, $B = \{x - a_P - f(x),\ x \in S\} \subseteq Q$ for some $a \in G$. Then the mapping $\overline{f}$ defined by $\overline{f}(z) = z_P - z_Q - a$ for each $z \in G$ is an isometry on G and an extension of f.*

Proof. Let $g(z) = f(z) + a$ for each $z \in S$. Then g is a congruence on S. Let $\overline{g}(z) = z_P - z_Q$ for each $z \in G$. By Theorem 1.22 [13], $\overline{g}$ is a 0-isometry on G. Then $\overline{f}$ is an isometry on G, too. Let $x \in S$. Since $x + f(x) + a_Q = x_P + x_Q + f(x)_P + f(x)_Q + a_Q = x_P + f(x)_P + x_Q + f(x)_Q + a_Q \in P$, $x - a_P - f(x) = x_P + x_Q - a_P - f(x)_Q - f(x)_P = x_P - a_P - f(x)_P + x_Q - f(x)_Q \in Q$, we obtain $x_Q + f(x)_Q + a_Q = 0$, $x_P - a_P - f(x)_P = 0$. Thus $f(x)_P = x_P - a_P$, $f(x)_Q = -x_Q - a_Q$. Therefore $f(x) = f(x)_P + f(x)_Q = (x_P - a_P) + (-x_Q - a_Q) = x_P - x_Q - (a_P + a_Q) = \overline{f}(x)$. Hence $\overline{f}$ is an extension of f.

2. THEOREM. *Let G be a Riesz group, $S \subseteq G$ and let f be a congruence on S. Let f be extendable to an isometry $\overline{f}$ on G. Then there exists a direct decomposition $G = P \times Q$ of G with Q abelian such that*

$$A = \{x + f(x) + a_Q\,,\ x \in S\} \subseteq P\,, \quad B = \{x - a_P - f(x)\,,\ x \in S\} \subseteq Q$$

for some $a \in G$.

P r o o f. Define $\overline{g}(x) = \overline{f}(x) - \overline{f}(0)$ for each $x \in G$. Then $\overline{g}$ is a 0-isometry on G. By Theorem 3.20 [13], there exists a direct decomposition $G = P \times Q$ with Q abelian such that $\overline{g}(x) = x_P - x_Q$ for each $x \in G$. Let $x \in S$, $a = -\overline{f}(0)$. Then $x + f(x) + a_Q = x + \overline{f}(x) - \overline{f}(0)_Q = x + \overline{g}(x) + \overline{f}(0) - \overline{f}(0)_Q = x_P + x_Q + x_P - x_Q + \overline{f}(0)_P + \overline{f}(0)_Q - \overline{f}(0)_Q = 2x_P + \overline{f}(0)_P \in P$, $x - a_P - f(x) = x - a_P - \overline{f}(x) = x - a_P - \overline{f}(0) - \overline{g}(x) = x_P + x_Q + \overline{f}(0)_P - \overline{f}(0)_Q - \overline{f}(0)_P + x_Q - x_P = 2x_Q - \overline{f}(0)_Q \in Q$. Thus $A \subseteq P$, $B \subseteq Q$.

3. THEOREM. *Let G be a distributive multilattice group, $S \subseteq G$ and let f be a congruence on S. Let f be extendable to an isometry $\overline{f}$ on G. Then there exists a direct decomposition $G = P \times Q$ of G with Q abelian such that*

$$A = \{x + f(x) + a_Q\,,\ x \in S\} \subseteq P\,, \quad B = \{x - a_P - f(x)\,,\ x \in S\} \subseteq Q$$

for some $a \in G$.

The proof of this theorem is the same as the proof of Theorem 2, only instead of Theorem 3.20 [13] it is needed to use Theorem 17 [12], under which to every 0-isometry g in a distributive multilattice group G there exists a direct decomposition $G = P \times Q$ of G with Q abelian such that $g(x) = x_P - x_Q$ for each $x \in G$.

Let G be a *po*-group, $S \subseteq G$, $0 \in S$ and let f be a congruence on S. If we put $g(x) = f(x) - f(0)$ for each $x \in S$, then g is a 0-congruence on S. It is clear that if g can be extended to an isometry, then f can be extended to an isometry, too. Thus it suffices to examine only 0-congruences on subsets containing 0.

From the proof of Theorem 2 it follows that if $0 \in S$ and f is a 0-congruence on S, then $a = 0$ in this theorem. Thus from 1 and 2 we obtain:

4. THEOREM. *Let G be a Riesz group, $S \subseteq G$, $0 \in S$. Let f be a 0-congruence on S and let $A = \{x + f(x),\ x \in S\}$, $B = \{x - f(x),\ x \in S\}$. Then 0-congruence f can be extended to an isometry $\overline{f}$ on G if and only if there exists a direct decomposition $G = P \times Q$ of G with Q abelian such that $A \subseteq P$, $B \subseteq Q$.*

Analogously we obtain:

5. THEOREM. *Let G be a distributive multilattice group, $S \subseteq G$, $0 \in S$. Let f be a 0-congruence on S, $A = \{x + f(x),\ x \in S\}$, $B = \{x - f(x),\ x \in S\}$. Then 0-congruence f can be extended to an isometry $\overline{f}$ on G if and only if there exists a direct decomposition $G = P \times Q$ with Q abelian such that $A \subseteq P$, $B \subseteq Q$.*

6. THEOREM. *Let G be a Riesz group, $S \subseteq G$, $0 \in S$ and let f be a 0-congruence on S. Let $x \in S^+$. Then there exist $x_1, x_2 \in G^+$ such that $x = x_1 + x_2$, $f(x) = x_1 - x_2$, $x_1 + x_2 = x_2 + x_1$. Moreover, $x_1 \vee x_2 = x$, $x_1 \wedge x_2 = 0$, $x_1 = f(x) \vee 0$, $x_2 = \big(-f(x)\big) \vee 0$.*

P r o o f. Since $x \geqq 0$, from the relation $U(x) = |x| = |f(x)| = U\big(f(x), -f(x)\big)$ we get $x = \big(-f(x)\big) \vee f(x)$. Since G is a Riesz group, from the relations $x \in U\big(0, -f(x)\big)$, $-f(x) + x \in U\big(0, -f(x)\big)$ we obtain that there exists $x_2 \in G$ such that $0 \leqq x_2 \leqq x$, $-f(x) \leqq x_2 \leqq -f(x) + x$. Let $x_1 = x - x_2$. Then $x_1 \in U\big(0, f(x)\big)$. Thus $x = x_1 + x_2$, where $x_1 \in U\big(0, f(x)\big)$, $x_2 \in U\big(0, -f(x)\big)$. Let $z \in U\big(0, f(x)\big)$, $t \in U\big(0, -f(x)\big)$. Then $z + x_2, x_1 + t \in U\big(f(x), -f(x)\big) = |f(x)| = |x| = U(x_1 + x_2)$. This implies $z \geqq x_1$, $t \geqq x_2$. Therefore $x_1 = f(x) \vee 0$, $x_2 = \big(-f(x)\big) \vee 0$. Clearly $x_1 \vee x_2 = x$ and hence $x_1 \wedge x_2 = 0$. Then it is easy to verify that $x_2 = x_1 - f(x) = -f(x) + x_1$. From this we obtain $f(x) = x_1 - x_2$, $x_1 + x_2 = x_2 + x_1$.

Polars in Riesz group were introduced and investigated by G l a s s in [4] and we shall make use of the theory developed there.

Let H be a Riesz group. If $X \subseteq H$, then $\langle X \rangle$ will denote the *subgroup of H generated by X*.

For $h \in H^+$, let $h^\perp = \{x \in H^+,\ x \wedge h = 0\}$, $p_0(h) = \langle h^\perp \rangle$.

For $h \in H$, the set $p(h) = \bigcup_{g \in U(h,0,-h)} p_0(g)$ is called the *polar of h in H*.

Thus $p(h)$ is a directed convex subgroup of H for each $h \in H$. If H is an l-group, then the definition given for l-groups coincides with the one given here.

For $S \subseteq H$, the *polar of S* is defined to be the set $p(S) = \Big\langle \Big(\bigcap_{s \in S} p(s) \Big)^+ \Big\rangle$.

Thus, polars are directed convex subgroups of H and $p(h) = p(\{h\})$ for each $h \in H$. A definition of *higher polars* is given using induction on the positive integer n.

For $S \subseteq H$, let $p^1(S) = p(S)$ and $p^{n+1}(S) = p(p^n(S))$.

A subset S of H is said to be *weakly positive* if for all $s \in S$ there exist $s_1, s_2 \in S^+ \cup S^-$ such that $s_1 \leqq s \leqq s_2$.

A Riesz group H is said to be a *weak polar group* if $p^3(S) = p(S)$ for all subsets S of G.

Note that every l-group is a weak polar group.

A Riesz group H is said to be *strongly projectable* if to every polar A in H there exists a directed convex subgroup B of H such that $H = A \times B$.

We shall need the following properties of polars in a Riesz group H (Glass [4]).

(A) If S and T are subsets of H and $S \subseteq T$, then $p(T) \subseteq p(S)$.

(B) For every subset S of H, $S^+ \cup S^- \subseteq p^2(S)$. If S is a weakly positive set, then $S \subseteq p^2(S)$.

(C) For each subset S of H, $p(S) \cap p^2(S) = \{0\}$.

(D) If $H = P \times Q$ is a direct decomposition of H, then $p(P) = Q$, $p(Q) = P$.

7. THEOREM. *Let G be an abelian weak polar group. Let any 0-congruence on a subset S of G be extendable to an isometry on G. Then G is strongly projectable.*

Proof. Let $H = p(S) + p^2(S)$. By (C), $p(S) \cap p^2(S) = \{0\}$. Then from Proposition 5.8 [2] it follows that $H = p(S) \times p^2(S)$ is a direct decomposition of H. Let $f(x+y) = x - y$ for each $x \in p(S)$, $y \in p^2(S)$. By 1.22 [13], f is a 0-isometry on H. Since f can be extended to an 0-isometry $\overline{f}$ on G, from 3.20 [13] we have that there exists a direct decomposition $G = P \times Q$ of G such that $\overline{f}(x) = x_P - x_Q$ for each $x \in G$. Let $y, z \in G^+$, $\overline{f}(y) = y$, $\overline{f}(z) = -z$. Then we get $y_P + y_Q = y_P - y_Q$, $-z_Q - z_P = z_P - z_Q$. Thus $2y_Q = 0$, $2z_P = 0$. Therefore $y_Q = 0$, $z_P = 0$. Hence $y \in P^+$, $z \in Q^+$. Let $t \in p(S)^+$, $v \in p^2(S)^+$. Then $\overline{f}(t) = t$, $\overline{f}(v) = -v$. Thus $p(S)^+ \subseteq P^+$, $p^2(S)^+ \subseteq Q^+$ and hence $p(S) \subseteq P$, $p^2(S) \subseteq Q$. Then from (A) and (D) it follows that $p^2(S) \supseteq p(P) = Q$, $p(S) \supseteq p^3(S) \supseteq p(Q) = P$. Therefore $P = p(S)$, $Q = p^2(S)$.

8. THEOREM. *Let G be an abelian Riesz group, S a weakly positive subset of G. Let every congruence on S be extendable to an isometry on G. Then $G = p(S) \times p^2(S)$.*

The proof is the same as the proof of Theorem 7, only instead of assumption that G is a weak polar group, it is needed to make use of (B).

9. THEOREM. *Let G be a representable l-group, L a subset of G containing 0 and let f be a 0-congruence on L. Then $|x+f(x)| \wedge |y-f(y)| = 0$ for each $x, y \in L$.*

P r o o f. Without loss of generality we may suppose that G is a subgroup of the l-group $\prod_{i \in I} G_i$, where

(a) all G_i are linearly ordered,
(b) for each $i \in I$, the natural projection of G into G_i is a surjection.

Let $i \in I$. For $z \in G$ we denote by z_i the i-th component of z and by 0_i the neutral element of G_i. From Lemma 1 [18] it follows that either $f(t)_i = t_i$ for each $t \in L$ or $f(t)_i = -t_i$ for each $t \in L$. Let x, $y \in L$. Then we have that either $x_i + f(x)_i = 0_i$, $y_i + f(y)_i = 0_i$ or $x_i - f(x)_i = 0_i$, $y_i - f(y)_i = 0_i$. Thus $\big(|x+f(x)| \wedge |y-f(y)|\big)_i = |x_i + f(x)_i| \wedge |y_i - f(y)_i| = 0_i$.

Therefore $|x+f(x)| \wedge |y-f(y)| = 0$.

The proof of Theorem 6 of P o w e l l [16], which is the main result of [16], is based on the Proposition 5 [16]. We now show not only that Theorem 6 is valid for abelian l-groups as it was established in [16] but also that in this theorem L can be any subset containing 0.

10. THEOREM. *Let G be an abelian l-group. Then the following conditions are equivalent:*

(1) *Every congruence f on a subset L of G containing 0 can be extended to an isometry.*
(2) *G is strongly projectable.*

P r o o f. (1) $\Longrightarrow$ (2). Since every l-group is a weak polar group, it is a consequence of Theorem 7.

(2) $\Longrightarrow$ (1). Let G be strongly projectable, $L \subseteq G$, $0 \in L$ and let f be a congruence on L. Let $g(x) = f(x) - f(0)$ for each $x \in L$. Then g is a 0-congruence on L. Let $A = \{x + g(x),\ x \in L\}$, $B = \{x - g(x),\ x \in L\}$. Then $G = p^2(A) \times p(A)$, $A \subseteq p^2(A)$. By 9, $B \subseteq p(A)$. From Theorem 4 it follows that g can be extended to an isometry $\overline{g}$. Let $\overline{f}(x) = \overline{g}(x) + f(0)$ for each $x \in G$. Then $\overline{f}$ is an isometry and an extension of f.

11. THEOREM. *Let H be an l-group, $a, b \in H$. If $|a| = |b|$, then $|a+b| \wedge |a-b| = 0$.*

P r o o f. By the distributivity of H, $\big[(a+b)\vee(-b-a)\big] \wedge \big[(a-b)\vee(b-a)\big] = \big[(a+b)\wedge(a-b)\big] \vee \big[(-b-a)\wedge(a-b)\big] \vee \big[(a+b)\wedge(b-a)\big] \vee \big[(-b-a)\wedge(b-a)\big] =$

$\big[a+(-b\wedge b)\big]\vee\big[(-b\wedge b)-a\big]\vee\big\{\big[(-b-a)\wedge(a-b)\big]\vee\big[(a+b)\wedge(b-a)\big]\big\}=$ $\big[a+(-a\wedge a)\big]\vee\big[(-a\wedge a)-a\big]\vee\big\{\big[(-b-a)\vee(a+b)\big]\wedge\big[(a-b)\vee(a+b)\big]\wedge$ $\big[(-b-a)\vee(b-a)\big]\wedge\big[(a-b)\vee(b-a)\big]\big\}=\big[(2a\wedge 0)\vee(-2a\wedge 0)\big]\vee\big\{\big[a+(-b\vee b)\big]\wedge$ $\big[(-b\vee b)-a\big]\wedge\big[(-b-a)\vee(a+b)\big]\wedge\big[(a-b)\vee(b-a)\big]\big\}=\big[(-2a\vee 2a)\wedge$ $(2a\vee 0)\wedge(-2a\vee 0)\wedge 0\big]\vee\big\{\big[a+(-a\vee a)\big]\wedge\big[(-a\vee a)-a\big]\wedge|a+b|\wedge|a-b|\big\}=$ $0\vee\big[(0\vee 2a)\wedge(-2a\vee 0)\wedge|a+b|\wedge|a-b|\big]=0\vee\big[0\wedge|a+b|\wedge|a-b|\big]=0\,.$

12. THEOREM. *Let H be an l-group, $S\subseteq H$, $0\in S$ and let f be a 0-congruence on S. Then $|x+f(x)|\wedge|x-f(x)|=0$ for each $x\in S$.*

Proof. This is an immediate consequence of 11.

13. THEOREM. *Let G be a Riesz group, $S\subseteq G$, $0\in S$ and let f be a 0-congruence on S. If $x\in S^+$, then $\big(x+f(x)\big)\wedge\big(x-f(x)\big)=0$.*

Proof. By 6, $x+f(x)=2x_1$, $x-f(x)=2x_2$, $x_1\wedge x_2=0$, where $x_1, x_2\in G^+$. From the proposition (b) [2, p. 10] we obtain $2x_1\wedge 2x_2=0$. Thus $\big(x+f(x)\big)\wedge\big(x-f(x)\big)=0$.

The question whether any 0-congruence in an abelian weak polar group which is strongly projectable can be extended to an isometry (see Theorem 7 above) remains open.

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Department of Mathematics
Faculty of Chemical Technology
Slovak Technical University
Radlinského 9
812 37 Bratislava
Slovakia