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ON ISOMETRIES IN PARTIALLY ORDERED GROUPS

MILAN JASEM

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ABSTRACT. In this note congruences in partially ordered groups are studied. A necessary and sufficient condition for any congruence in a Riesz group and a distributive multilattice group to be extendable to an isometry is given. Further, it is shown that all congruences in an abelian lattice ordered group $G$ can be extended to isometries if and only if $G$ is strongly projectable.

Congruences and isometries in an abelian lattice ordered group ($l$-group) have been introduced and investigated by Swamy [18], [19]. Isometries in non-abelian $l$-groups were studied by Jakubík [7], [8] and Holland [6]. Rachunek [17] generalized the notion of the isometry for any partially ordered group ($po$-group). Isometries in Riesz groups and multilattice groups were investigated in [10], [11], [12], [13], [14], [17]. In [16] Powell studied conditions under which congruences in abelian $l$-groups can be extended to isometries.

In this note we study congruences in partially ordered groups. We give a necessary and sufficient condition for any congruence in a Riesz group and a distributive multilattice group to be extendable to an isometry. Further, it is proved that if each congruence in an abelian weak polar group $G$ can be extended to an isometry in $G$, then $G$ is strongly projectable (for definitions see below). It is also shown that all congruences in an abelian $l$-group $G$ can be extended to isometries if and only if $G$ is strongly projectable. These results correct some of Powell’s results on congruences in abelian $l$-groups from [16].

First we recall some notions and notations used in the paper.

Let $G$ be a $po$-group. The group operation will be written additively (though it is not assumed that the group is abelian). If $S \subseteq G$, we denote $S^+ = \{x \in S, x \geq 0\}$, $S^- = \{x \in S, x \leq 0\}$. For $a_1, \ldots, a_n \in G$, we

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denote by $U(a_1, \ldots, a_n)$ and $L(a_1, \ldots, a_n)$ the set of all upper bounds and the set of all lower bounds of the set $\{a_1, \ldots, a_n\}$ in $G$, respectively. If for $a, b \in G$ there exists the least upper bound (greatest lower bound) of the set $\{a, b\}$ in $G$, then it will be denoted by $a \lor b$ ($a \land b$). For $x \in G$, $|x| = U(x, -x)$. In the case, when the considered po-group is an $l$-group, $|x| = x \lor (-x)$ for each element $x$ of the considered $l$-group. If $G = P \times Q$ is a direct decomposition of $G$, then for $x \in G$ we denote by $x_P$ and $x_Q$ the components of $x$ in the direct factors $P$ and $Q$, respectively.

A Riesz group is any po-group $H$ which is directed and satisfies the Riesz interpolation property, i.e., for each $a_i, b_j \in H$ ($i, j = 1, 2$) such that $a_i \geq b_j$ ($i, j = 1, 2$) there exists $c \in H$ such that $a_i \geq c \geq b_j$ ($i, j = 1, 2$). See [3] or [5].

A partially ordered set $P$ is a multilattice if for each pair of elements $a, b \in P$, every upper bound of the set $\{a, b\}$ in $P$ is over a minimal upper bound of the set $\{a, b\}$, and dually. A directed po-group $H$ is said to be a multilattice group if it is a multilattice under $\lor$. If $x$ and $y$ are elements of a multilattice group $H$, then we denote by $x \lor_m y$ the set of all minimal elements of the set $U(x, y)$ in $H$. The meaning of $x \land_m y$ will be analogous. A multilattice group $H$ is said to be distributive if for $a, b, c \in H$ the relations $(a \lor_m b) \land (a \lor_m c) \neq 0$, $(a \land_m b) \land (a \land_m c) \neq 0$ together imply $b = c$. See [1], [15].

Note that every $l$-group is a Riesz group and a distributive multilattice group. But a Riesz group need not be a multilattice group and conversely, a multilattice group need not be a Riesz group.

If $S$ is a subset of a po-group $G$, then a mapping $f : S \to G$ is called a congruence on $S$ if $|x - y| = |f(x) - f(y)|$ for each $x, y \in S$. If $0 \in S$ and $f(0) = 0$, then a congruence $f$ on $S$ is said to be a 0-congruence. A congruence (0-congruence) $f$ on $S$ is called an isometry (0-isometry) if $S = G$ and $f$ is a bijection.

Remark. Swamy [18] defined an isometry in an abelian $l$-group $C$ as a surjection $f : C \to C$ such that

$$|x - y| = |f(x) - f(y)| \quad \text{for each} \quad x, y \in C.$$  \hfill (1)

It is obvious that in a po-group $C$ any mapping $f : C \to C$ which satisfies (1) is an injection. The fact that in a representable $l$-group $C$ (and so in any abelian $l$-group) any mapping $f : C \to C$ satisfying (1) is a surjection is not obvious and was proved by Jakubík [9]. This Jakubík’s result was extended to isolated Riesz groups and distributive multilattice groups (and hence to $l$-groups) in [13], [14]. It is clear that in a po-group $C$ any mapping $f : C \to C$ which satisfies (1) need not be a surjection.
Let $G$ be an abelian $l$-group and $L$ a sublattice of $G$ containing 0. For a congruence $f: L \rightarrow G$ let $T_f: L \rightarrow G$ be defined by $T_f(x) = f(x) - f(0)$. Let $A = \{T_f(x) \vee 0, \ x \in L^+\}$, $B = \{-T_f(x) \vee 0, \ x \in L^+\}$. Under this notation Powell proves the following proposition in [16].

**PROPOSITION 5.** A congruence $f: L \rightarrow G$ can be extended to an isometry $\overline{f}: G \rightarrow G$ if and only if $G = A \times B$, where $A \subseteq \overline{A}$ and $B \subseteq \overline{B}$.

The following example shows that Proposition 5 of Powell [16] on congruences in abelian $l$-groups is not correct.

**Example.** It is well known that the set $C$ of all continuous functions on the closed interval $[0,1]$ is an abelian $l$-group under pointwise addition and order. As was shown in [7] (see also [16]), for every 0-isometry $f$ in an $l$-group $G$ there exists a uniquely determined direct decomposition $G = P \times Q$ such that $f(x) = x_P - x_Q$ for each $x \in G$. Since $C$ has no nontrivial direct factors, there exist only two 0-isometries $f_1$ and $f_2$ in $C$ and these are of the form $f_1(x) = x$ or $f_2(x) = -x$ for all $x \in C$. Let $a(x) = \sin 2\pi x$, $b(x) = -|a(x)|$ for each $x \in [0,1]$. Let $L = \{0, b\}$ ($0$ is the neutral element of $C$). So $L$ is a sublattice of $C$ containing 0. Let $g(b) = a$, $g(0) = 0$. Then $g$ is a 0-congruence on $L$. By Proposition 5 [16], $g$ can be extended to an isometry $\overline{g}$ on $C$. But $\overline{g} \neq f_1$, $\overline{g} \neq f_2$, a contradiction.

Proposition 5 will not be correct even in the case that we take the set $L$ instead of the set $L^+$ in the definitions of the sets $A$ and $B$. Namely, if we take $L = \{0, a, a \vee 0, a \wedge 0\}$ and define $h(z) = -z$ for each $z \in L$, then $h$ is a 0-congruence on $L$ and clearly can be extended to an isometry on $C$. But by Proposition 5 [16], $h$ cannot be extended to an isometry on $C$.

The following three theorems correct Proposition 5 [16]. Moreover, instead of the assumption of 5 [16] that $G$ is a lattice ordered group we apply more general assumptions.

**1. THEOREM.** Let $G$ be a po-group, $S \subseteq G$ and let $f$ be a congruence on $S$. Let there exist a direct decomposition $G = P \times Q$ of $G$ with $Q$ abelian such that $A = \{x + f(x) + a_Q, \ x \in S\} \subseteq P$, $B = \{x - a_P - f(x), \ x \in S\} \subseteq Q$ for some $a \in G$. Then the mapping $\overline{f}$ defined by $\overline{f}(z) = z_P - z_Q - a$ for each $z \in G$ is an isometry on $G$ and an extension of $f$.

**Proof.** Let $g(z) = f(z) + a$ for each $z \in S$. Then $g$ is a congruence on $S$. Let $\overline{g}(z) = z_P - z_Q$ for each $z \in G$. By Theorem 1.22 [13], $\overline{g}$ is a 0-isometry on $G$. Then $\overline{f}$ is an isometry on $G$, too. Let $x \in S$. Since $x + f(x) + a_Q = x_P + x_Q + f(x)_P + f(x)_Q + a_Q = x_P + f(x)_P + x_Q + f(x)_Q + a_Q \in P$, $x - a_P - f(x) = x_P + x_Q - a_P - f(x)_Q - f(x)_P = x_P - a_P - f(x)_P + x_Q - f(x)_Q \in Q$, $23$
we obtain \( x_Q + f(x)_Q + a_Q = 0, \) \( x_P - a_P - f(x)_P = 0. \) Thus \( f(x)_P = x_P - a_P, \)
\( f(x)_Q = -x_Q - a_Q. \) Therefore \( f(x) = f(x)_P + f(x)_Q = (x_P - a_P) + (-x_Q - a_Q) = x_P - x_Q - (a_P + a_Q) = \overline{f}(x). \) Hence \( \overline{f} \) is an extension of \( f. \)

2. **THEOREM.** Let \( G \) be a Riesz group, \( S \subseteq G \) and let \( f \) be a congruence on \( S. \) Let \( f \) be extendable to an isometry \( \overline{f} \) on \( G. \) Then there exists a direct decomposition \( G = P \times Q \) of \( G \) with \( Q \) abelian such that

\[
A = \{ x + f(x) + a_Q, \ x \in S \} \subseteq P, \quad B = \{ x - a_P - f(x), \ x \in S \} \subseteq Q
\]

for some \( a \in G. \)

**Proof.** Define \( \overline{g}(x) = \overline{f}(x) - \overline{f}(0) \) for each \( x \in G. \) Then \( \overline{g} \) is a 0-isometry on \( G. \) By Theorem 3.20 [13], there exists a direct decomposition \( G = P \times Q \) with \( Q \) abelian such that \( \overline{g}(x) = x_P - x_Q \) for each \( x \in G. \) Let \( x \in S, \)
\( a = -\overline{f}(0). \) Then \( x + f(x) + a_Q = x + \overline{f}(x) - \overline{f}(0)_Q = x + \overline{g}(x) + \overline{f}(0) - \overline{f}(0)_Q = x_P + x_Q + x_P - x_Q + \overline{f}(0)_P + \overline{f}(0)_Q - \overline{f}(0)_Q = 2x_P + \overline{f}(0)_P \in P, \ x - a_P - f(x) = x - a_P - \overline{f}(x) = x - a_P - \overline{f}(0) - \overline{g}(x) = x_P + x_Q + \overline{f}(0)_P - \overline{f}(0)_Q - \overline{f}(0)_P + x_Q - x_P = 2x_Q - \overline{f}(0)_Q \in Q. \) Thus \( A \subseteq P, \ B \subseteq Q. \)

3. **THEOREM.** Let \( G \) be a distributive multilattice group, \( S \subseteq G \) and let \( f \) be a congruence on \( S. \) Let \( f \) be extendable to an isometry \( \overline{f} \) on \( G. \) Then there exists a direct decomposition \( G = P \times Q \) of \( G \) with \( Q \) abelian such that

\[
A = \{ x + f(x) + a_Q, \ x \in S \} \subseteq P, \quad B = \{ x - a_P - f(x), \ x \in S \} \subseteq Q
\]

for some \( a \in G. \)

The proof of this theorem is the same as the proof of Theorem 2, only instead of Theorem 3.20 [13] it is needed to use Theorem 17 [12], under which to every 0-isometry \( g \) in a distributive multilattice group \( G \) there exists a direct decomposition \( G = P \times Q \) of \( G \) with \( Q \) abelian such that \( g(x) = x_P - x_Q \) for each \( x \in G. \)

Let \( G \) be a po-group, \( S \subseteq G, \) \( 0 \in S \) and let \( f \) be a congruence on \( S. \) If we put \( g(x) = f(x) - f(0) \) for each \( x \in S, \) then \( g \) is a 0-congruence on \( S. \) It is clear that if \( g \) can be extended to an isometry, then \( f \) can be extended to an isometry, too. Thus it suffices to examine only 0-congruences on subsets containing 0.

From the proof of Theorem 2 it follows that if \( 0 \in S \) and \( f \) is a 0-congruence on \( S, \) then \( a = 0 \) in this theorem. Thus from 1 and 2 we obtain:
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4. Theorem. Let $G$ be a Riesz group, $S \subseteq G$, $0 \in S$. Let $f$ be a 0-congruence on $S$ and let $A = \{x + f(x), x \in S\}$, $B = \{x - f(x), x \in S\}$. Then 0-congruence $f$ can be extended to an isometry $\bar{f}$ on $G$ if and only if there exists a direct decomposition $G = P \times Q$ of $G$ with $Q$ abelian such that $A \subseteq P$, $B \subseteq Q$.

Analogously we obtain:

5. Theorem. Let $G$ be a distributive multilattice group, $S \subseteq G$, $0 \in S$. Let $f$ be a 0-congruence on $S$, $A = \{x + f(x), x \in S\}$, $B = \{x - f(x), x \in S\}$. Then 0-congruence $f$ can be extended to an isometry $\bar{f}$ on $G$ if and only if there exists a direct decomposition $G = P \times Q$ with $Q$ abelian such that $A \subseteq P$, $B \subseteq Q$.

6. Theorem. Let $G$ be a Riesz group, $S \subseteq G$, $0 \in S$ and let $f$ be a 0-congruence on $S$. Let $x \in S^+$. Then there exist $x_1, x_2 \in G^+$ such that $x = x_1 + x_2$, $f(x) = x_1 - x_2$, $x_1 + x_2 = x_2 + x_1$. Moreover, $x_1 \vee x_2 = x$, $x_1 \wedge x_2 = 0$, $x_1 = f(x) \vee 0$, $x_2 = (-f(x)) \vee 0$.

Proof. Since $x \geq 0$, from the relation $U(x) = |x| = |f(x)| = U(f(x), -f(x))$ we get $x = (-f(x)) \vee f(x)$. Since $G$ is a Riesz group, from the relations $x \in U(0, -f(x))$, $-f(x) + x \in U(0, -f(x))$ we obtain that there exists $x_2 \in G$ such that $0 \leq x_2 \leq x$, $-f(x) \leq x_2 \leq -f(x) + x$. Let $x_1 = x - x_2$. Then $x_1 \in U(0, f(x))$. Thus $x = x_1 + x_2$, where $x_1 \in U(0, f(x))$, $x_2 \in U(0, -f(x))$.

Let $z \in U(0, f(x))$, $t \in U(0, -f(x))$. Then $z + x_2, x_1 + t \in U(f(x), -f(x)) = |f(x)| = |x| = U(x_1 + x_2)$. This implies $z \geq x_1, t \geq x_2$. Therefore $x_1 = f(x) \vee 0$, $x_2 = (-f(x)) \vee 0$. Clearly $x_1 \vee x_2 = x$ and hence $x_1 \wedge x_2 = 0$. Then it is easy to verify that $x_2 = x_1 - f(x) = -f(x) + x_1$. From this we obtain $f(x) = x_1 - x_2$, $x_1 + x_2 = x_2 + x_1$.

Polars in Riesz group were introduced and investigated by Glass in [4] and we shall make use of the theory developed there.

Let $H$ be a Riesz group. If $X \subseteq H$, then $\langle X \rangle$ will denote the subgroup of $H$ generated by $X$.

For $h \in H^+$, let $h^\perp = \{x \in H^+, x \wedge h = 0\}$, $p_0(h) = \langle h^\perp \rangle$.

For $h \in H$, the set $p(h) = \bigcup_{g \in U(h, 0, -h)} p_0(g)$ is called the polar of $h$ in $H$.

Thus $p(h)$ is a directed convex subgroup of $H$ for each $h \in H$. If $H$ is an $l$-group, then the definition given for $l$-groups coincides with the one given here.

For $S \subseteq H$, the polar of $S$ is defined to be the set $p(S) = \left\langle \left( \bigcap_{s \in S} p(s) \right)^+ \right\rangle$. 

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Thus, polars are directed convex subgroups of $H$ and $p(h) = p({h})$ for each $h \in H$. A definition of higher polars is given using induction on the positive integer $n$.

For $S \subseteq H$, let $p^1(S) = p(S)$ and $p^{n+1}(S) = p\left(p^n(S)\right)$.

A subset $S$ of $H$ is said to be weakly positive if for all $s \in S$ there exist $s_1, s_2 \in S^+ \cup S^-$ such that $s_1 \leq s \leq s_2$.

A Riesz group $H$ is said to be a weak polar group if $p^3(S) = p(S)$ for all subsets $S$ of $G$.

Note that every $l$-group is a weak polar group.

A Riesz group $H$ is said to be strongly projectable if to every polar $A$ in $H$ there exists a directed convex subgroup $B$ of $H$ such that $H = A \times B$.

We shall need the following properties of polars in a Riesz group $H$ (Glass [4]).

(A) If $S$ and $T$ are subsets of $H$ and $S \subseteq T$, then $p(T) \subseteq p(S)$.

(B) For every subset $S$ of $H$, $S^+ \cup S^- \subseteq p^2(S)$. If $S$ is a weakly positive set, then $S \subseteq p^2(S)$.

(C) For each subset $S$ of $H$, $p(S) \cap p^2(S) = \{0\}$.

(D) If $H = P \times Q$ is a direct decomposition of $H$, then $p(P) = Q$, $p(Q) = P$.

7. Theorem. Let $G$ be an abelian weak polar group. Let any $0$-congruence on a subset $S$ of $G$ be extendable to an isometry on $G$. Then $G$ is strongly projectable.

Proof. Let $H = p(S) + p^2(S)$. By (C), $p(S) \cap p^2(S) = \{0\}$. Then from Proposition 5.8 [2] it follows that $H = p(S) \times p^2(S)$ is a direct decomposition of $H$. Let $f(x + y) = x - y$ for each $x \in p(S)$, $y \in p^2(S)$. By 1.22 [13], $f$ is a $0$-isometry on $H$. Since $f$ can be extended to an $0$-isometry $\overline{f}$ on $G$, from 3.20 [13] we have that there exists a direct decomposition $G = P \times Q$ of $G$ such that $\overline{f}(x) = x_P - x_Q$ for each $x \in G$. Let $y, z \in G^+$, $\overline{f}(y) = y$, $\overline{f}(z) = -z$. Then we get $y_P + y_Q = y_P - y_Q$, $-z_Q - z_P = z_P - z_Q$. Thus $2y_Q = 0$, $2z_P = 0$. Therefore $y_Q = 0$, $z_P = 0$. Hence $y \in P^+$, $z \in Q^+$. Let $t \in p(S)^+$, $v \in p^2(S)^+$. Then $\overline{f}(t) = t$, $\overline{f}(v) = -v$. Thus $p(S)^+ \subseteq P^+$, $p^2(S)^+ \subseteq Q^+$ and hence $p(S) \subseteq P$, $p^2(S) \subseteq Q$. Then from (A) and (D) it follows that $p^2(S) \supseteq p(P) = Q$, $p(S) \supseteq p^3(S) \supseteq p(Q) = P$. Therefore $P = p(S)$, $Q = p^2(S)$.

8. Theorem. Let $G$ be an abelian Riesz group, $S$ a weakly positive subset of $G$. Let every congruence on $S$ be extendable to an isometry on $G$. Then $G = p(S) \times p^2(S)$. 

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The proof is the same as the proof of Theorem 7, only instead of assumption that $G$ is a weak polar group, it is needed to make use of (B).

9. **Theorem.** Let $G$ be a representable $l$-group, $L$ a subset of $G$ containing 0 and let $f$ be a 0-congruence on $L$. Then $|x + f(x)| \land |y - f(y)| = 0$ for each $x, y \in L$.

**Proof.** Without loss of generality we may suppose that $G$ is a subgroup of the $l$-group $\prod G_i$, where

(a) all $G_i$ are linearly ordered,
(b) for each $i \in I$, the natural projection of $G$ into $G_i$ is a surjection.

Let $i \in I$. For $z \in G$ we denote by $z_i$ the $i$-th component of $z$ and by $0_i$ the neutral element of $G_i$. From Lemma 1 [18] it follows that either $f(t)_i = t_i$ for each $t \in L$ or $f(t)_i = -t_i$ for each $t \in L$. Let $x, y \in L$. Then we have that either $x_i + f(x)_i = 0_i$, $y_i + f(y)_i = 0_i$ or $x_i - f(x)_i = 0_i$, $y_i - f(y)_i = 0_i$. Thus

$$ (|x + f(x)| \land |y - f(y)|)_i = |x_i + f(x)_i| \land |y_i - f(y)_i| = 0_i. $$

Therefore $|x + f(x)| \land |y - f(y)| = 0$.

The proof of Theorem 6 of Powell [16], which is the main result of [16], is based on the Proposition 5 [16]. We now show not only that Theorem 6 is valid for abelian $l$-groups as it was established in [16] but also that in this theorem $L$ can be any subset containing 0.

10. **Theorem.** Let $G$ be an abelian $l$-group. Then the following conditions are equivalent:

(1) Every congruence $f$ on a subset $L$ of $G$ containing 0 can be extended to an isometry.

(2) $G$ is strongly protectable.

**Proof.** (1) $\implies$ (2). Since every $l$-group is a weak polar group, it is a consequence of Theorem 7.

(2) $\implies$ (1). Let $G$ be strongly protectable, $L \subseteq G$, $0 \in L$ and let $f$ be a congruence on $L$. Let $g(x) = f(x) - f(0)$ for each $x \in L$. Then $g$ is a 0-congruence on $L$. Let $A = \{x + g(x), x \in L\}$, $B = \{x - g(x), x \in L\}$. Then $G = p^2(A) \times p(A)$, $A \subseteq p^2(A)$. By 9, $B \subseteq p(A)$. From Theorem 4 it follows that $g$ can be extended to an isometry $\bar{g}$. Let $\bar{f}(x) = \bar{g}(x) + f(0)$ for each $x \in G$. Then $\bar{f}$ is an isometry and an extension of $f$.

11. **Theorem.** Let $H$ be an $l$-group, $a, b \in H$. If $|a| = |b|$, then $|a + b| \land |a - b| = 0$.

**Proof.** By the distributivity of $H$, $[(a+b)\lor (-b-a)] \land [(a-b)\lor (b-a)] = [(a+b)\land (a-b)] \lor [(-b-a)\land (a-b)] \lor [(a+b)\land (b-a)] \lor [(-b-a)\land (b-a)] = 27$
[a + (−b ∧ b)] ∨ [−(b ∧ b) − a] ∨ \{[(−b − a) ∧ (a − b)] ∨ [(a + b) ∧ (b − a)]\} =
[a + (−a ∧ a)] ∨ [(−a ∧ a) − a] ∨ \{[(−b − a) ∨ (a + b)] ∧ [(a − b) ∨ (a + b)] ∧
[(−b − a) ∧ (a − b) ∨ (b − a)]\} = [(2a ∧ 0) ∨ (−2a ∧ 0)] ∨ \{[a + (−b ∨ b)] ∧
[(−b ∨ b) − a] ∧ [(−b − a) ∨ (a + b)] ∧ [(a − b) ∨ (b − a)]\} = [(−2a ∨ 2a) ∧
(2a ∨ 0) ∧ (−2a ∨ 0) ∧ 0] ∨ \{[(a + (−a ∨ a)] ∧ [(−a ∨ a) − a] ∧ [a + b] ∧ [a − b]\} =
0 ∨ [(0 ∨ 2a) ∧ (−2a ∨ 0) ∧ [a + b] ∧ [a − b]] = 0 ∨ [0 ∧ [a + b] ∧ [a − b]] = 0.

12. **Theorem.** Let \( H \) be an l-group, \( S \subseteq H \), \( 0 \in S \) and let \( f \) be a
0-congruence on \( S \). Then \( |x + f(x)| ∧ |x − f(x)| = 0 \) for each \( x \in S \).

**Proof.** This is an immediate consequence of 11.

13. **Theorem.** Let \( G \) be a Riesz group, \( S \subseteq G \), \( 0 \in S \) and let \( f \) be a
0-congruence on \( S \). If \( x \in S^+ \), then \((x + f(x)) ∧ (x − f(x)) = 0 \).

**Proof.** By 6, \( x + f(x) = 2x_1 \), \( x − f(x) = 2x_2 \), \( x_1 ∧ x_2 = 0 \), where
\( x_1, x_2 \in G^+ \). From the proposition (b) [2, p. 10] we obtain \( 2x_1 ∧ 2x_2 = 0 \). Thus
\((x + f(x)) ∧ (x − f(x)) = 0 \).

The question whether any 0-congruence in an abelian weak polar group which
is strongly projectable can be extended to an isometry (see Theorem 7 above)
remains open.

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Department of Mathematics
Faculty of Chemical Technology
Slovak Technical University
Radlinského 9
812 37 Bratislava
Slovakia