Mathematica Slovaca

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*Mathematica Slovaca*, Vol. 44 (1994), No. 1, 117--122

Persistent URL: http://dml.cz/dmlcz/133307

Math. Slovaca, **44** (1994), No. 1, 117–122

Mathematica Slovaca
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Slovak Academy of Sciences

# NOTE ON LINEAR ARBORICITY

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(*Communicated by Martin Škoviera*)

ABSTRACT. The conjecture of linear arboricity requires to decompose any $n$-regular graph into $\lceil \frac{n+1}{2} \rceil$ linear forests. Here, a new approach to this conjecture is developed. We bound the degrees in forests by $\lfloor \frac{n+1}{2} \rfloor$.

## Introduction

In this note, a graph will always mean a finite undirected graph without loops and multiple edges. A graph $\Gamma$ is $n$-regular if the degree of each vertex in $\Gamma$ is $n$. We emphasize that the letter $n$ will always be used only in this meaning.

A letter $T$ will indicate a forest. A linear forest is a forest with all vertex degrees less than or equal to $2$. For any graph $\Gamma$ the arboricity $\Upsilon(\Gamma)$ of $\Gamma$ (the linear arboricity $\Xi(\Gamma)$ of $\Gamma$) is the minimum number of edge disjoint forests (linear forests) whose union is $\Gamma$.

Symbols $V(\Gamma)$ and $E(\Gamma)$ denote the vertex set and the edge set of a graph $\Gamma$, respectively. An edge joining two vertices $x$ and $y$ we denote by $xy$.

The degree of vertex $x$ in a graph $\Gamma$ (a forest $T$) is denoted as $\deg_\Gamma(x)$ ($\deg_T(x)$). The greatest degree in a graph $\Gamma$ is denoted as $\Delta(\Gamma)$.

For a real number $v$, $\lfloor v \rfloor$ denotes the lower integer part of $v$ and $\lceil v \rceil = -\lfloor -v \rfloor$.

In 1961, C. St. J. A. Nash-Williams [9] and W. T. Tutte [12] have determined the arboricity of arbitrary graph. In particular,

$$\Upsilon(\Gamma) = \left\lceil \frac{n+1}{2} \right\rceil$$

for an $n$-regular graph $\Gamma$.

The following conjecture on linear arboricity is due to J. Akiyama, G. Exoo and F. Harrary [3].

AMS Subject Classification (1991): Primary 65C70.
Key words: Graph, Decomposition, Linear arboricity.

**CONJECTURE 1.** *For an arbitrary n-regular graph* $\Gamma$

$$\Xi(\Gamma) = \left\lceil \frac{n+1}{2} \right\rceil .$$

The inequality $\Xi(\Gamma) \geq \lceil \frac{n+1}{2} \rceil$ follows from $\Xi(\Gamma) \geq \Upsilon(\Gamma)$. The converse is not known. However, the conjecture has been proved in some special cases.

For $n = 3,4$ it was proved by J. Akiyama, G. Exoo and F. Harrary in [3] and [4]. For $n = 5,6,8$ it was proved by H. Enomoto and B. Peroche in [6], for $n = 6$ by P. Tomasta in [11], and for $n = 10$ by F. Guldan in [7].

In general, as we mentioned above, the linear arboricity is at least $\lceil \frac{n+1}{2} \rceil$. Already in 1981 it was shown in [4] that $\Xi(\Gamma) \leq \lceil \frac{3}{2} \lceil \frac{n}{2} \rceil \rceil$ for any $n$-regular graph $\Gamma$. In 1987 N. Alon [5] proved by probabilistic methods that for arbitrary $\varepsilon > 0$ and $n$ sufficiently large the linear arboricity of an $n$-regular graph is less than $\left(\frac{1}{2} + \varepsilon\right) \cdot n$.

The problem of linear arboricity in multigraphs was studied by H. Aït-Djafer [1], [2].

In this note, we attempt to look at the problem from another point of view. As we mentioned above, we have $\Upsilon(\Gamma) = \lceil \frac{n+1}{2} \rceil$ for an arbitrary $n$-regular graph $\Gamma$. Let $\Delta_n[\mathcal{R}]$ denote the maximum degree of vertices over all components in decomposition $\mathcal{R}$ of an $n$-regular graph to $\lceil \frac{n+1}{2} \rceil$ forests. Hence, $\Delta_n[\mathcal{R}] \leq n$ is the best possible inequality which can be derived from [9] and [12] because the authors admit vertices of arbitrary degree. However, Conjecture 1 requires to find a decomposition $\mathcal{R}$ satisfying $\Delta_n[\mathcal{R}] = 2$.

Up to date, no better bounds are known in general. In this note, we show that $\Delta_n[\mathcal{R}] \leq \lfloor \frac{n+1}{2} \rfloor$. A short proof of Conjecture 1 for $n = 3$ using techniques similar to those used in the proof of Theorem 1 can be found in [8].

## Main result

The proof of Theorem 1 is constructive. We decompose a graph $\Gamma$ into forests $T_i$, $i = 1, 2, \dots, h$.

We use elementary operation of inserting $i$-admissible edge $xy$ into forest $T_i$, $i = 1, 2, \dots, h$. Let $k$ be a constant to which we decrease the value of $\Delta_n[\mathcal{R}]$. An edge $xy \notin E(T_i)$ is $i$-admissible if and only if:

(i) $T_i \cup xy$ is a forest,
(ii) $\deg_{(T_i \cup xy)}(x) \leq k$,
(iii) $\deg_{(T_i \cup xy)}(y) \leq k$.

We note that insertion of an $i$-admissible edge into a forest $T_i$ cannot increase the number of vertices of degree greater than $k$ in forests $T_j$, $j = 1, 2, \ldots, h$.

We set $h = \lceil \frac{n+1}{2} \rceil$. The following identity will often be used:

$$n - h + 1 = n + 1 - \left\lceil \frac{n+1}{2} \right\rceil = n + 1 + \left\lfloor \frac{-n-1}{2} \right\rfloor = \left\lfloor \frac{n+1}{2} \right\rfloor .$$

**THEOREM 1.** *Let $\Gamma$ be an $n$-regular graph, $n > 3$. Then there are $h = \lceil \frac{n+1}{2} \rceil$ edge disjoint forests $T_1, T_2, \ldots, T_h$ covering $\Gamma$ such that $\Delta(T_i) \le \lfloor \frac{n+1}{2} \rfloor$, $i = 1, 2, \ldots, h$.*

P r o o f. Assume that there is a graph $\Gamma$ which cannot be decomposed into forests, where $\Delta(T_i) \le \lfloor \frac{n+1}{2} \rfloor$, $i = 1, 2, \ldots, h$.

By C. St. J. A. N a s h - W i l l i a m s [9] and [10], there is a decomposition of $\Gamma$ into $h$ forests. We can assume that the decomposition is chosen so that the number of vertices $z \in V(\Gamma)$ with $\deg_{T_i}(z) > \lfloor \frac{n+1}{2} \rfloor$ for any $i$ is minimum.

Let $x$ be a vertex with $\deg_{T_i}(x) > \lfloor \frac{n+1}{2} \rfloor = n - h + 1$. Without loss of generality, let $i = 1$. In the following, we modify our decomposition of $\Gamma$ to a new one with $\deg_{T_1}(x) = n - h + 2$, and then we determine the degrees of some vertices in $T_i$.

Since $2(n - h + 2) \ge n + 2$, the only forest $T_i$ with $\deg_{T_i}(x) > \lfloor \frac{n+1}{2} \rfloor$ is $T_1$. Let $\deg_{T_i}(x) = \lfloor \frac{n+1}{2} \rfloor + j$. Since $n - (n - h + 1 + j) = h - j - 1$, there are $j$ forests, say, $T_2, T_3, \ldots, T_{j+1}$ with $\deg_{T_i}(x) = 0$ for all $i \in \{2, 3, \ldots, j+1\}$. Since $n - h + 2 \ge 2$ if $n > 3$, there are at least two vertices $y$ with $xy \in E(T_1)$. Let $y$ be such that $xy \in E(T_1)$. Since $1 + 2(n - h + 1) \ge n + 1$, we have $\deg_{T_i}(y) < n - h + 1$ for some $i \in \{2, 3, \ldots, j+1\}$ if $j \ge 2$. Assume $\deg_{T_{j+1}}(y) < n - h + 1$. Then $xy$ is $(j+1)$-admissible, and we can insert $xy$ into $T_{j+1}$. We decreased $\deg_{T_1}(x)$ by one.

Now we have $j - 1$ forests $T_2, T_3, \ldots, T_j$ with $\deg_{T_i}(x) = 0$ for all $i \in \{2, 3, \ldots, j\}$. Let $y$ be such that $xy \in E(T_1)$. If $j - 1 \ge 2$, $xy$ is $i$-admissible for some $i \in \{2, 3, \ldots, j\}$, and we can insert $xy$ into $T_i$.

Thus, $j - 1$ neighbours of $x$ in $T_1$ we can insert into $T_i$, $i \in \{2, 3, \ldots, j+1\}$. Then $\deg_{T_1}(x) = n - h + 2$, and there is a forest, say, $T_2$ with $\deg_{T_2}(x) = 0$. But $\deg_{T_2}(y) \ge n - h + 1$ for all $y$ with $xy \in E(T_1)$ since otherwise we get a contradiction with the original choice of $T_1, T_2, \ldots, T_k$ in $\Gamma$.

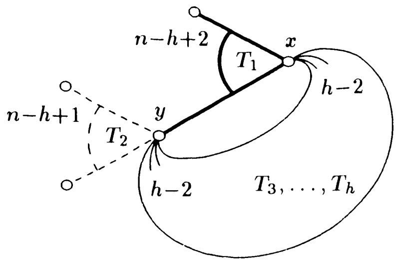


Figure 1.

The edge $xy$ is $i$-admissible if $x$ and $y$ are in distinct components of $T_i$, $i > 2$, because of $(n-h+2)+(n-h+1) \geq n+1$ (see Fig. 1). Thus,

$$\deg_{T_1}(x) = n-h+2 \quad \text{and} \quad \deg_{T_i}(x) = 1, \quad i = 3,4,\ldots,h,$$
$$\deg_{T_2}(y) = n-h+1 \quad \text{and} \quad \deg_{T_i}(y) = 1, \quad i = 1,3,4,\ldots,h$$

for all $y$ with $xy \in E(T_1)$ because $\Gamma$ is $n$-regular (see Fig. 1).

Let $y$ be a fixed vertex of $\Gamma$ with $xy \in E(T_1)$. Then $x$ and $y$ are joined by a path in $T_3$. Let us denote $y = a_0, a_1, \ldots, a_m = x$ the vertices of this path (see Fig. 2).

We claim $\deg_{T_1}(a_1) \geq n-h+1$. Otherwise we can insert $a_1a_0$ into $T_1$ and $xy$ into $T_3$. We get again forests because $xy$ is 3-admissible if $a_1a_0 \notin E(T_3)$, and $a_1a_0$ is 1-admissible if $xy \notin E(T_1)$. But then $\deg_{T_1}(x) = n-h+1$, that is a contradiction with the original choice of $T_1, T_2, \ldots, T_k$ in $\Gamma$.

Vertices $a_0$ and $a_1$ must be in the same component of $T_i$, $i = 4,5,\ldots,h$. Otherwise, we can insert $a_1a_0$ into $T_i$, and $xy$ into $T_3$ because $a_1a_0$ is $i$-admissible. Thus, $\deg_{T_i}(a_1) \geq 1$, $i > 3$.

We have the following identities:

$$\deg_{T_3}(a_1) = 2, \qquad \deg_{T_1}(a_1) = n-h+1, \qquad \deg_{T_i}(a_1) = 1, \quad i > 3$$

because $n = \deg_{\Gamma}(a_1) \geq 2 + (n-h+1) + h - 3 = n$.

Analogously, we have $\deg_{T_2}(a_2) \geq n-h+1$ because otherwise we can insert $a_1a_2$ into $T_2$, and $xy$ into $T_3$. Similarly, $a_2$ and $a_1$ must be in the same component of $T_i$ for each $i > 3$ because, otherwise, we can insert $a_1a_2$ into $T_i$, and $xy$ into $T_3$ (see Fig. 2). It means that:

$$\deg_{T_3}(a_2) = 2, \qquad \deg_{T_2}(a_2) = n-h+1, \qquad \deg_{T_i}(a_2) = 1, \quad i > 3.$$

We can repeat this construction till $a_m$ is reached. Finally, we obtain:

$$\deg_{T_3}(x) = 1\,, \qquad \deg_{T_1}(x) = n - h + 2\,, \qquad \deg_{T_i}(x) = 1\,, \quad i > 3\,.$$

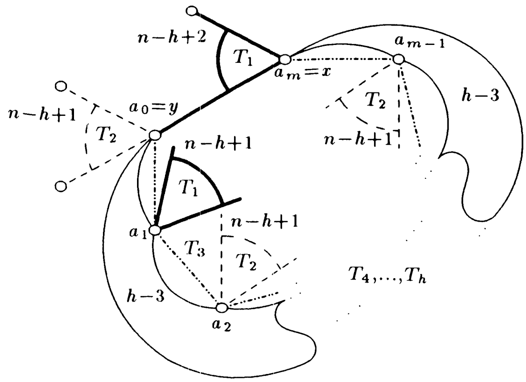


Figure 2.

Hence $\deg_{T_3}(x) = \deg_{T_3}(y) = 1$ and $\deg_{T_3}(a_i) = 2$, $i = 1, 2, \ldots, m-1$. But as we mentioned above, there exists $\overline{y} \neq y$ with $\overline{y}x \in E(T_1)$. We obtain existence of a path in $T_3$ with vertices $\overline{y} = b_0, b_1, \ldots, b_{\overline{m}} = x$ by an analogous process. Here $\deg_{T_3}(\overline{y}) = \deg_{T_3}(x) = 1$ and $\deg_{T_3}(b_i) = 2$, $i = 1, 2, \ldots, \overline{m} - 1$. It means that $x$, $y$ and $\overline{y}$ are three distinct vertices of degree 1 in linear tree, that is a contradiction.

This concludes the proof of Theorem 1. □

We have proved that every $n$-regular graph for which $n > 3$ can be decomposed into $\lceil \frac{n+1}{2} \rceil$ forests with maximum degree $\lfloor \frac{n+1}{2} \rfloor$. Assumption $n > 3$ was used to establish three forests which yield the path $a_0, a_1, \ldots, a_m$ in the proof. Since $\lfloor \frac{n+1}{2} \rfloor = 2$ if $n = 4$, we proved the Conjecture 1 for $n = 4$.

Every graph of degree not greater than $k$ can be completed to $k$-regular graph by adding new vertices and edges. Thus, Theorem 1 implies that each graph $\Gamma$ with $\Delta(\Gamma) = k$ can be decomposed into $\lceil \frac{k+1}{2} \rceil$ forests of degree not greater than $\lfloor \frac{k+1}{2} \rfloor$. The decreasing of degrees in forests to some function asymptotically equal even to $o(n)$ is still open.

### Acknowledgement

I would thank to S. Poljak and P. Tomasta for their interest in this work and helpful comments.

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Received January 8, 1991

Revised December 15, 1992

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