Rudolf Oláh
Integral conditions of oscillation of a linear differential equation


Persistent URL: [http://dml.cz/dmlcz/133333](http://dml.cz/dmlcz/133333)

**Terms of use:**

© Mathematical Institute of the Slovak Academy of Sciences, 1989

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.
INTEGRAL CONDITIONS OF OSCILLATION OF A LINEAR DIFFERENTIAL EQUATION

RUDOLF OLÁH

Recently some papers have been devoted to the study of the oscillatory behaviour of second and higher order ordinary or retarded linear differential equations. We shall continue in this course. To be more precise, we consider the linear differential equation

\[ y^{(n)}(t) + p(t)y(t) = 0, \quad n \geq 2, \]  

(1)

where \( p(t) > 0 \) is a continuous function on \([a, \infty), a > 0\).

Our aim is to give new sufficient conditions that guarantee the oscillatory character of equation (1).

We shall use the following definitions.

A solution \( y(t) \) of equation (1) is called oscillatory if it has arbitrarily large zeros, and it is called nonoscillatory otherwise.

Equation (1) is said to have the property A if every solution of this equation is oscillatory if \( n \) is even, and every solution is either oscillatory or \( \lim_{t \to \infty} y^{(i)}(t) = 0, i = 0, \ldots, n - 1 \) if \( n \) is odd.

We introduce the notation:

\( M_n \) is a maximum of the function \( P_n(x) = x(1 - x) \cdots (n - 1 - x) \) on \((0, 1)\).

\( a_1, a_2 \) are the fixed points of the function

\[ f_n(x) = \frac{a}{(1 - x) \cdots (n - 1 - x)} \quad \text{on} \quad (0, 1), \quad 0 < a \leq M_n. \]

In the present paper an attempt is made to establish new conditions under which equation (1) has property A if \( 0 < a \leq M_n \). The case when \( a > M_n \) is solved in [1] for the equation (1), and the problem when \( a > M_2 \) is solved in [6] for the second order retarded differential equation. We can find the other conditions related to the one in Theorem 2 in [4, 5].

We will need the following lemma.

Lemma ([1]). Equation (1) has the property A if and only if the equation

\[ y^{(n)}(t) + (-1)^n p(t)y(t) = 0 \]

(2)
has not the solution $y(t)$ such that

$$y(t) > 0, \quad t \geq t_0 \geq a,$$

$$( -1)^i y^{(i)}(t) > 0, \quad t \geq t_0 \geq a, \quad i = 1, \ldots, n - 1.$$

We can now establish the main results in the next theorems.

**Theorem 1.** Let the following conditions be fulfilled

$$\liminf_{t \to \infty} t^{n-1} \int_t^\infty p(s) \, ds = \frac{a}{n-1}, \quad 0 < a \leq M_n,$$  \hspace{1cm} (3)

$$\liminf_{t \to \infty} t^{n-1-a_1 + \varepsilon} \int_t^\infty s^{a_1-\varepsilon} p(s) \, ds > a_2 (n-2)! \varepsilon, \quad 0 < \varepsilon < a_1.$$  \hspace{1cm} (4)

Then equation (1) has the property A.

**Proof.** Assume the contrary, i.e. equation (1) has not the property A. By the Lemma equation (2) has a solution $v(t)$ such that

$$v(t) > 0, \quad t \geq t_0 \geq a,$$

$$( -1)^i y^{(i)}(t) > 0, \quad t \geq t_0 \geq a, \quad i = 1, \ldots, n - 1.$$  \hspace{1cm} (5)

We remind that

$$\int_t^\infty p(s) \, ds < \infty,$$

which follows from the condition (3).

Integrating equation (2) and according to the fact that $v(t)$ is increasing we obtain

$$( -1)^n v^{(n-1)}(t) \geq \int_t^\infty p(s) v(s) \, ds \geq v(t) \int_t^\infty p(s) \, ds, \quad t \geq t_0.$$  \hspace{1cm} (6)

The condition (3) implies

$$t^{n-1} \int_t^\infty p(s) \, ds > \frac{v}{n-1},$$

where $0 < v < a$, $t \geq t_1 \geq t_0$.

Hence it is obvious that

$$( -1)^n v^{(n-1)}(t) > \frac{v}{n-1} \cdot \frac{v(t)}{t^{n-1}}, \quad t \geq t_1.$$  \hspace{1cm} (7)

Integrating the above inequality we get

$$(-1)^n v^{(n-2)}(t) > \frac{v}{n-1} \cdot v(t) \int_t^\infty \frac{ds}{s^{n-1}} = \frac{v}{(n-1)(n-2)} \cdot \frac{v(t)}{t^{n-2}}.$$
Repeating this argument we find that
\[ v'(t) > \frac{v(t)}{(n-1)!} \cdot \frac{v(t)}{t}, \quad t \geq t_1, \]
which means that the function
\[ \frac{v(t)}{t'}, \quad \text{where} \quad r = \frac{v}{(n-1)!}, \]
is increasing on \([t_1, \infty)\).

Then from the equation (2) it follows that
\[ (-1)^n v^{(n-1)}(t) \geq \int_t^\infty p(s) v(s) \, ds \geq \frac{v(t)}{t'} \int_t^\infty s^r p(s) \, ds, \quad t \geq t_1. \]

Using (6) in the next integral we have
\[
\int_t^\infty s^r p(s) \, ds > \left( \frac{v}{n-1} \right)^{n-1} \int_t^\infty p(s) \left( \int_s^\infty p(u) \, du \right) \frac{n-1-r}{n-1} \, ds = \\
= \left( \frac{v}{n-1} \right)^{n-1} \cdot \frac{n-1}{n-1-r} \left( \int_t^\infty p(s) \, ds \right)^{n-1-r},
\]
since
\[ \int_t^\infty p(s) \, ds < \infty. \]

By virtue of (6) the last inequality implies
\[ \int_t^\infty s^r p(s) \, ds > \frac{v}{n-1-r} \cdot \frac{1}{t^{n-1-r}}, \]
and so we obtain
\[ (-1)^n v^{(n-1)}(t) > \frac{v(t)}{n-1-r} \cdot \frac{v(t)}{t^{n-1}}, \quad t \geq t_1. \]

Integrating this inequality and because of the fact that the function
\[ \frac{v(t)}{t'} \]
is increasing we get
\[ (-1)^{n-1} v^{(n-2)}(t) > \frac{v(t)}{n-1-r} \cdot \frac{v(t)}{t'} \int_t^\infty \frac{ds}{s^{n-1-r}} = \]
325
Repeating this argument we find that
\[ v'(t) > \frac{v}{(n - 1 - r) \ldots (1 - r)} \frac{v(t)}{t}, \quad t \geq t_1. \]

Thus the function
\[ \frac{v(t)}{t^k}, \quad \text{where} \quad k = \frac{v}{(n - 1 - r) \ldots (1 - r)}, \]

is increasing on \([t_1, \infty)\). Repeating this procedure we find that
\[ v'(t) > x_i \frac{v(t)}{t}, \quad t \geq t_1, \]

where \(x_i = g_n(x_{i-1})\), \(i = 1, 2, \ldots, x_0 = 0\), and
\[ g_n(x) = \frac{v}{(n - 1 - x) \ldots (1 - x)}. \]

So the functions
\[ \frac{v(t)}{t^{x_i}}, \quad i = 1, 2, \ldots \]

are increasing on \([t_1, \infty)\).

According to the properties of the function \(g_n(x)\), where \(0 < v < M_n\), this function has two fixed points \(v_1, v_2 (v_1 < v_2)\) on \((0, 1)\) and the sequence \(\{x_i\}\) converges to \(v_1\).

The condition (4) implies
\[ t^{n-1 - \alpha_1 + \varepsilon} \int_t^{\infty} s^{\alpha_1 - \varepsilon} p(s) ds > (\alpha_2 + \varepsilon)(n - 2)! = \beta(n - 2)!, \quad (7) \]

where \(\varepsilon > 0, \ t \geq t_2 \geq t_1\).

We can always choose \(v\) such that \(\alpha_1 - \varepsilon < v_1\) and \(v_2 < \beta = \alpha_2 + \varepsilon\). Then the function
\[ \frac{v(t)}{t^{\alpha_1 - \varepsilon}} \]

is increasing on \([t_2, \infty)\) and we obtain from (2) and (7) that
\[ (-1)^n v^{(n-1)}(t) \geq \int_t^{\infty} p(s) v(s) ds \geq \frac{v(t)}{t^{\alpha_1 - \varepsilon}} \int_t^{\infty} s^{\alpha_1 - \varepsilon} p(s) ds, \]

326
\((-1)^n v^{(n-1)}(t) > \beta(n - 2)! \frac{v(t)}{t^{n-1}}, \quad t \geq t_2.\)

Now integrating as above we get that

\[ v'(t) > \beta \cdot \frac{v(t)}{t}, \]

i.e. the function

\[ \frac{v(t)}{t^\beta} \]

is increasing on \([t_2, \infty).\) Repeating the procedure as above we find

\[ v'(t) > x_k \cdot \frac{v(t)}{t}, \quad t \geq t_2, \]

where \(x_k = g_n(x_{k-1}),\) \(k = 1, 2, \ldots, p,\) \(x_0 = \beta\) and \(x_p > 1,\) which follows from the properties of the function \(g_n(x).\) Thus

\[ v'(t) > x_p \cdot \frac{v(t)}{t}, \quad t \geq t_2. \tag{8} \]

On the other hand with regard to (5) we obtain

\[ v(t) \geq (t - t_0) v'(t), \quad t \geq t_0, \]

so

\[ x_p v(t) - (x_p - 1) v(t) \geq tv'(t) - t_0 v'(t), \]

\[ x_p v(t) + t_0 v'(t) - (x_p - 1) v(t) \geq tv'(t). \tag{9} \]

Since (3) implies (see e.g. [5])

\[ \int_0^\infty t^{n-1} p(t) \, dt = \infty, \]

and this condition and (5) imply that

\[ \lim_{t \to \infty} v(t) = \infty; \]

then for sufficiently large \(t\) we have

\[ t_0 (v'(t) - (x_p - 1) v(t) < 0. \]

Thus from (9) for sufficiently large \(t\) we get

\[ x_p v(t) > tv'(t), \]

which is a contradiction with (8). This completes the proof.
Theorem 2. Assume that condition (3) and
\[\lim_{t \to -\infty} \left( \frac{1}{t} \int_{t}^{t'} s^{n} p(s) ds + t^{1-\varepsilon} \int_{t}^{\infty} s^{n+\varepsilon-2} p(s) ds \right) > (n-1)!,\]
\[\varepsilon \in (0, a),\] are fulfilled.

Then equation (1) has the property A.

Proof. Suppose that equation (1) has not the property A. Then equation (2) has a solution \( v(t) \) which satisfies (5). Now using the same argument as in the proof of Theorem 1 we find that the function
\[v(t), \quad \varepsilon \in (0, a),\] is increasing on \([t_{1}, \infty), t_{1} \geq t_{0} \geq a,\) since \( (v(t) t^{-\varepsilon})' > 0. \)

On the other hand from the identity
\[v^{(k)}(t) = \sum_{i=j}^{k-1} (-1)^{i-j} \frac{(s-t)^{i-j}}{(i-j)!} v^{(i)}(s) + \frac{(-1)^{k-j}}{(k-j-1)!} \int_{t}^{s} (u-t)^{k-j-1} v^{(k)}(u) du,\]
\[s \geq t \geq t_{1},\] where \( 1 \leq k \leq n, 0 \leq j \leq n-1, \) for \( k = n, j = 1, \) we obtain
\[v'(t) = \sum_{i=1}^{n-1} (-1)^{i-1} \frac{(s-t)^{i-1}}{(i-1)!} v^{(i)}(s) + \frac{(-1)^{n-1}}{(n-2)!} \int_{t}^{s} (u-t)^{n-2} v^{(n)}(u) du.\]

According to (5) and (12) we get
\[v'(t) \geq \frac{(-1)^{n-1}}{(n-2)!} \int_{t}^{\infty} (u-t)^{n-2} v^{(n)}(u) du.\]
Since \( v(t) \) satisfies (2) from the last inequality we have
\[v'(t) \geq \frac{1}{(n-2)!} \int_{t}^{\infty} (u-t)^{n-2} p(u) v(u) du.\]

Integrating this inequality from \( T \) to \( t, t > T \geq t_{1}, \) we get the following one.
\[v(t) \geq \frac{1}{(n-1)!} \int_{T}^{t'} (u-T)^{n-1} p(u) v(u) du + \]
\[+ \frac{1}{(n-2)!} \int_{T}^{\infty} p(u) v(u) \int_{T}^{s} (u-s)^{n-2} ds du,\]
and using the inequality \( a^{n} - b^{n} \geq (a-b) a^{n-1}, a \geq b \geq 0, \) we obtain
\[v(t) \geq \frac{1}{(n-1)!} \left[ \int_{T}^{t'} (u-T)^{n-1} p(u) v(u) du + (t-T) \int_{T}^{\infty} (u-T)^{n-2} p(u) v(u) du \right].\]
According to (5) we have
\[ v(t) \geq (t - T) v'(t) + v(T) \geq (t - T) v'(t) \]
and the inequality \( v(t) \geq (t - T) v'(t) \) implies that the function
\[ \frac{v(t)}{t - T}, \quad t > T, \]
is nonincreasing and with regard to (11) from (13) we have
\[ (n - 1)! \geq \frac{1}{t - T} \int_{t}^{t'} (u - T)^n p(u) \, du + \frac{t - T}{t^e} \int_{t}^{\infty} u^e (u - T)^{n - 2} p(u) \, du, \]
which contradicts condition (10). This completes the proof.

REFERENCES


Received December 1, 1986

Katedra matematiky
Vysokej školy dopravy a spojov
Marx—Engelsa 25
010 88 Žilina

ИНТЕГРАЛЬНЫЕ ПРИЗНАКИ КОЛЕБАНИЯ ЛИНЕЙНОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ

Rudolf Oláh

Резюме

В работе приведены достаточные условия для того, чтобы каждое решение уравнения (1) при четном \( n \) являлось колеблющимся, а при нечетном \( n \), либо колеблющимся, либо удовлетворяло условию \( \lim_{t \to \infty} y^{(i)}(t) = 0, i = 0, ..., n - 1. \)