On the integers \(x_i\) for which \(x_i \mid x_1 \ldots x_{i-1}x_i \ldots x_n + 1\) holds
ON THE INTEGERS \( x_i \) FOR WHICH \( x_i \mid x_1 \cdots x_{i-1}x_{i+1} \cdots x_n + 1 \) HOLDS

JAROSLAV JANÁK (PRAHA) AND LADISLAV SKULA (BRNO)

The present paper deals with the question of the existence of positive integers \( x_1, \ldots, x_n \) \((n > 1)\) such that the integer \( x_i \) divides the integer \( x_1 \cdots x_{i-1}x_{i+1} \cdots x_n + 1 \) for each \( 1 \leq i \leq n \). This question was motivated by a problem of Š. Znám (1972, cf. [1]). where the integers \( x_i \) are required to be greater than 1 and the considered divisibility to be proper.

In his paper [1] L. Skula solved the given question for \( 2 \leq n \leq 4 \). From this it follows that for these \( n \)'s there are no integers \( x_i \) required in the problem of Znám. He showed that J. Janák had found by means of the computer that the integers 2, 3, 11, 23, 31 satisfy the conditions of Znám’s problem for \( n = 5 \).

Clearly, we can restrict this question to the positive integers greater than 1.

In this paper the question of the existence of positive integers \( x_1, \ldots, x_n \) \((n > 1)\) greater than 1 with the property \( x_i \mid x_1 \cdots x_{i-1}x_{i+1} \cdots x_n + 1 \) \((1 \leq i \leq n)\) is fully solved for \( n = 5 \) and \( n = 6 \) by Theorems 2 and 3. Theorem 1 gives bounds of the integers \( x_i \) dependent on the integer \( n \). Hence for a fixed \( n \) the question is reduced to a finite number of possibilities and a computer can be used. In this way Theorem 3 was proved. The proof of Theorem 2 is shown directly and it is the development of the method used in [1]. Theorem 2 as well as the Theorem from [1] can be easily obtained by means of a computer and Theorem 1 (as well as Theorem 3); here, however, we have given a direct proof without using a computer. Some of the values \( x_i \) for \( n = 7 \) are given in the table 3.

Throughout this paper \( n \) will denote an integer greater than 1 and \( 1 < x_1 \leq x_2 \leq \ldots \leq x_n \) will be integers such that for each \( 1 \leq i \leq n \)
\[
(1) \quad x_i \mid x_1 \cdots x_{i-1}x_{i+1} \cdots x_n + 1
\]
holds. We shall denote the integer \( \frac{x_1 \cdots x_{i-1}x_{i+1} \cdots x_n + 1}{x_i} \) by \( y_i \).

Clearly, for \( 1 \leq i \neq j \leq n \) we have \((x_i, x_j) = 1\). hence
\[
(2) \quad 1 < x_1 < \ldots < x_n
\]
and
1 ≤ y_n < y_{n-1} < \ldots < y_1

hold.

By multiplying the right and the left-hand sides of the relation (1) we have

\[ x_1 \ldots x_n / \sum_{i=1}^{n} \frac{x_1 \ldots x_n}{x_i} + 1 \]

Hence there exists a positive integer \( m \) such that

(3)

\[ m = \sum_{i=1}^{n} \frac{1}{x_i} + \frac{1}{x_1 \ldots x_n} \]

**Lemma 1.** \( x_1 \leq n \).

Proof. Since \( x_1 \geq 1 \) there holds \( 1 \leq m \leq \frac{n}{x_1} + \frac{n+1}{x_1} \) where \( x_1 < n + 1 \) follows.

**Lemma 2.** Let \( 1 \leq k \leq n - 1 \). Then

\[ x_{k+1} < x_1 \ldots x_k (n - k + 1) \]

Proof. Let \( 1 \leq k \leq n - 1 \). According to (3) we have \( m - \sum_{i=1}^{k} \frac{1}{x_i} > 0 \) and because

\[ mx_1 \ldots x_k - \sum_{i=1}^{k} \frac{x_1 \ldots x_k}{x_i} \]

is an integer, we have

(4)

\[ mx_1 \ldots x_k - \sum_{i=1}^{k} \frac{x_1 \ldots x_k}{x_i} \geq 1 \]

According to (2)

\[ 2 < x_{k+1} < x_{k+2} < \ldots < x_n \]

holds and implies

(5)

\[ \frac{1}{x_{k+1} \ldots x_n} < \frac{1}{2 \cdot 3 \ldots (n - k + 1)} \leq \frac{1}{n - k + 1} \]

From (3), (4) and (5) we obtain

\[ \frac{n - k}{x_{k+1}} \geq \sum_{i=1}^{n} \frac{1}{x_i} - \sum_{i=1}^{k} \frac{1}{x_1 \ldots x_n} = \]

\[ = \frac{1}{x_1 \ldots x_k} \left( mx_1 \ldots x_k - \sum_{i=1}^{k} \frac{x_1 \ldots x_k}{x_i} \right) \geq \]

\[ \geq \frac{1}{x_1 \ldots x_k} \left( 1 - \frac{1}{x_{k+1} \ldots x_n} \right) > \frac{1}{x_1 \ldots x_k} \frac{n - k}{n - k + 1} \]

306
Thus \( x_{k+1} < x_1 \ldots x_k (n - k + 1) \).

*From Lemma 1 and 2 we get easily*

**Theorem 1.** Let \( C_1 = n \) and \( C_{k+1} = C_1 \ldots C_k (n - k + 1) - 1 \) for \( 1 \leq k \leq n - 1 \). Then

\[ x_i \leq C_i \]

for each \( 1 \leq i \leq n \).

**Remark.** For the constants \( C_i \) it obviously holds \( C_i \leq n^{2^{i-1}} \).

The valuations in Lemma 1 and 2 and hence also in Theorem 1 can be improved. B. Novák, e.g., has communicated that for \( n > 3 \) we get \( x_i \leq n - 2 \) and in (5) we can use the relation \( x_{k+1} \ldots x_n \geq (k + 2) \ldots (n + 1) \). But the aim of these assertions is only to show the finiteness of possibilities of integers \( x_i \). Using the computer, the bounds \( n^{2^{i-1}} \) were applied.

<table>
<thead>
<tr>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( y_1 )</th>
<th>( y_2 )</th>
<th>( y_3 )</th>
<th>( y_4 )</th>
<th>( y_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>7</td>
<td>43</td>
<td>1 807</td>
<td>815 861</td>
<td>362 605</td>
<td>66 601</td>
<td>1 765</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>7</td>
<td>47</td>
<td>395</td>
<td>194 933</td>
<td>86 637</td>
<td>15 913</td>
<td>353</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>11</td>
<td>23</td>
<td>31</td>
<td>11 765</td>
<td>5 229</td>
<td>389</td>
<td>89</td>
</tr>
</tbody>
</table>

**Theorem 2.** Let \( n = 5 \). Then the following table gives all the possibilities of the integers \( x_1, \ldots, x_5 \).

**Proof.** By direct calculation we find out that the values given in the table satisfy the given requirements.

For simplicity of notation we put \( x_1 = a, x_2 = b, x_3 = c, x_4 = d, x_5 = e, y_5 = x, y_4 = y \).

By (2) we have

\[ 2 \leq a < b < c < d < e. \]

For the integer defined by the relation (3) there holds

\[ m = \frac{1}{a} + \ldots + \frac{1}{e} + \frac{1}{abcde} \leq \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} = \frac{209}{144} < 2, \]

thus \( m = 1 \) and

(6) \[ \frac{1}{a} + \ldots + \frac{1}{e} + \frac{1}{abcde} = 1. \]
For \( A = ab \) we obtain

(7) \[ Acd + 1 = ex , \]
(8) \[ Ace + 1 = dy \]

Put \( D = xy - A^2c^2 \). Since \( (A, x) = (A, y) = 1 \), we have \( D \neq 0 \) and from (7) and (8) we get

(9) \[ d = \frac{Ac + x}{D} , \quad e = \frac{Ac + y}{D} , \]

where from it follows that \( D \) is a positive integer. Obviously,

(10) \( x < y \).

By multiplying (7) and (8) we get \( xyde = A^2c^2de + Acd + Ace + 1 \), hence

\[
D = \frac{Ac}{e} + \frac{Ac}{d} + \frac{1}{de} = Ac\left( \frac{1}{e} + \frac{1}{d} + \frac{1}{abcd} \right) ,
\]

whence and from (6) \( D = Ac\left( 1 - \frac{1}{a} - \frac{1}{b} - \frac{1}{c} \right) \), therefore

(11) \[ D = c(A - a - b) - A . \]

The following cases I—V cover all the possibilities of integers \( a, b \).

I. \( a = 2, b = 3 \). Then \( A = 6 \) and according to (11) \( D = c - 6 \).

Since \( a + c, b + c \) and \( D > 0 \), the following cases (a)—(d) cover all the possibilities of the integer \( c \).

(a) \( c = 7 \). Then \( D = 1 \), hence \( xy = D + A^2c^2 = 5.353 \). Since 353 is a prime, we get from (10) \( x = 1, y = 5 \cdot 353 = 1765 \) or \( x = 5, y = 353 \). According to (9) \( d = 43, e = 1807 \) or \( d = 47, e = 395 \), which are values given in the table 1.

(b) \( c = 11 \). Then \( D = 5 \), hence \( xy = 7^2.89 \) and thus it follows that \( x = 1, y = 7^2.89 \) or \( x = 7, y = 7.89 \) or \( x = 49, y = 89 \). From (9) we obtain \( d = 67/5 \) or \( d = 73/5 \) or \( d = 23 \). Hence \( d = 23, e = 31 \), which are values given in the table 1.

(c) \( c = 13 \). Then \( D = 7, xy = 6091 \) and, since 6091 is a prime, it holds \( x = 1, y = 6091 \) and according to (9) \( d = 79/7 \), which is a contradiction.

(d) \( c \geq 17 \). Then according to (6) \( 1 = \frac{1}{a} + \ldots + \frac{1}{e} + \frac{1}{a \ldots e} \leq \frac{1}{2} + \frac{1}{3} + \frac{1}{17} + \ldots + \frac{1}{19} + \frac{1}{23} + \frac{1}{2 \cdot 3 \cdot 17 \cdot 19 \cdot 23} = \frac{7342}{7429} < 1 \), which is a contradiction.

II. \( a = 2, b = 5, c = 7 \). Then \( D = 11, xy = 3 \cdot 1637 \). Since 1637 is a prime, we get from (10) \( x = 1, y = 3 \cdot 1637 \) or \( x = 3, y = 1637 \) whence according to (9) \( d = 71/11 \) or \( d = 73/11 \), which is a contradiction.
### Table 2

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>$y_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>7</td>
<td>43</td>
<td>1807</td>
<td>3263443</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>7</td>
<td>43</td>
<td>1823</td>
<td>193667</td>
<td>17</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>7</td>
<td>47</td>
<td>395</td>
<td>779731</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>7</td>
<td>47</td>
<td>403</td>
<td>19403</td>
<td>41</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>7</td>
<td>47</td>
<td>415</td>
<td>8111</td>
<td>101</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>7</td>
<td>47</td>
<td>583</td>
<td>1223</td>
<td>941</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>7</td>
<td>55</td>
<td>179</td>
<td>24323</td>
<td>17</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>11</td>
<td>23</td>
<td>31</td>
<td>47059</td>
<td>1</td>
</tr>
</tbody>
</table>

### Table 3

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>$x_7$</th>
<th>$y_7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>7</td>
<td>43</td>
<td>1807</td>
<td>3263443</td>
<td>10650056950807</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>7</td>
<td>43</td>
<td>1807</td>
<td>3263447</td>
<td>2130014000915</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>7</td>
<td>43</td>
<td>1807</td>
<td>3263591</td>
<td>71480133827</td>
<td>149</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>7</td>
<td>43</td>
<td>1807</td>
<td>3264187</td>
<td>14298637519</td>
<td>745</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>7</td>
<td>43</td>
<td>1823</td>
<td>193667</td>
<td>637617223447</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>7</td>
<td>47</td>
<td>395</td>
<td>779731</td>
<td>607979652631</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>7</td>
<td>47</td>
<td>395</td>
<td>779831</td>
<td>6020372531</td>
<td>101</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>7</td>
<td>47</td>
<td>403</td>
<td>19403</td>
<td>15435513367</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>7</td>
<td>47</td>
<td>415</td>
<td>8111</td>
<td>6644612311</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>7</td>
<td>47</td>
<td>583</td>
<td>1223</td>
<td>1404749767</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>7</td>
<td>55</td>
<td>179</td>
<td>24323</td>
<td>10057317271</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>11</td>
<td>23</td>
<td>31</td>
<td>47059</td>
<td>2214502423</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>11</td>
<td>23</td>
<td>31</td>
<td>47063</td>
<td>442938131</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>11</td>
<td>23</td>
<td>31</td>
<td>47095</td>
<td>59897203</td>
<td>37</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>11</td>
<td>23</td>
<td>31</td>
<td>47131</td>
<td>30382063</td>
<td>73</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>11</td>
<td>23</td>
<td>31</td>
<td>47243</td>
<td>12017087</td>
<td>185</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>11</td>
<td>23</td>
<td>31</td>
<td>47423</td>
<td>6114059</td>
<td>365</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>11</td>
<td>31</td>
<td>35</td>
<td>67</td>
<td>369067</td>
<td>13</td>
</tr>
</tbody>
</table>
III. $a = 2, b \geq 5, c \geq 9$. Then according to (6) we have
$$1 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{1}{2} + \frac{1}{5} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{2 \cdot 5 \cdot 9 \cdot 11 \cdot 13} = \frac{140}{143} < 1,$$
which is a contradiction.

IV. $a = 3, b = 4, c = 5$. Then $D = 13, xy = 3613$. Since 3613 is a prime, we have, according to (10), $x = 1, y = 3613$ and by (9) there holds $d = \frac{61}{13}$, which is a contradiction.

V. $a \geq 3, \{a, b, c\} \neq \{3, 4, 5\}$. Then $b \geq 4, c \geq 7, d \geq 8, e \geq 11$ and according to
$$1 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{1}{3} + \frac{1}{4} + \frac{1}{7} + \frac{1}{8} + \frac{1}{11} + \frac{1}{3 \cdot 4 \cdot 7 \cdot 8 \cdot 11} = \frac{995}{1056} < 1,$$
which is a contradiction.

The proof of Theorem 2 is now complete.

**Theorem 3.** Let $n = 6$. Then the following table 2 gives all the possibilities of the integers $x_1, \ldots, x_6$.

**Proof.** Theorem 1 and the computer.

**Remark.** Concluding we introduce table 3 for $n = 7$ giving some values of integers $x_1, \ldots, x_7$, obtained by means of a computer. But we do not know if they are complete.

**REFERENCES**


Received February 3, 1977

Výzkumní ústav matematických strojů
Stodůleká 16
150 00 Praha-Jinonice

Katedra algebra a geometrie
Přírodovědecké fakulty UJEP
Janáčkovo nám. 2a
662 95 Brno

О ЦЕЛЫХ ЧИСЛАХ $x_i$, ДЛЯ КОТОРЫХ СПРАВЕДЛИВО
$$x_i | x_1 \ldots x_{i-1} x_{i+1} \ldots x_n + 1$$

Ярослав Янак и Ладислав Скула

Резюме

В работе рассматривается вопрос существования натуральных чисел $x_1, \ldots, x_n (n > 1)$, которые удовлетворяют отношению $x_i | x_1 \ldots x_{i-1} x_{i+1} \ldots x_n + 1$ для всех $1 \leq i \leq n$. Для $2 \leq n \leq 4$ этот вопрос решал Л. Цула. В нашей статье даны все решения для $n = 5$ и $n = 6$. Для натурального $n > 1$ найдена верхняя граница чисел $x_i$ и в этом случае можно использовать вычислительную машину. Таким образом было получено решение для $n = 6$. Некоторые величины $x_i$ для $n = 7$ тоже даны.