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Mathematica Slovaca, Vol. 54 (2004), No. 5, 487--502

Persistent URL: http://dml.cz/dmlcz/133367

Math. Slovaca, **54** (2004), No. 5, 487–502

Mathematica Slovaca

LOWER BOUNDS FOR THE DISCREPANCY OF SOME SEQUENCES

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(*Communicated by Stanislav Jakubec*)

ABSTRACT. Let $f(x)$ be a function of a class that contains the functions $\beta(\log x)^s$ $(\beta > 0,\ s \geq 1)$ and βn^σ $(\beta > 0,\ 0 < \sigma < 1)$. We obtain lower bounds for the discrepancy of the sequence $(\alpha n + f(n))$, where α is the irrational number with bounded partial quotients. In order to show this result, we estimate exponential sums by the saddle-point method.

1. Introduction

Let $\{x\} = x - [x]$ denote the fractional part of the real number x. We write $e(x) = \mathrm{e}^{2\pi\mathrm{i}x}$ for the real number x. We use $f(x) \asymp g(x)$ to mean that both relations $f(x) \ll g(x)$ and $g(x) \ll f(x)$ hold.

Let (x_n) $(n = 1, 2, \dots)$ be a sequence of real numbers. The *discrepancy* of (x_n) is defined by

$$D_N(x_n) = \sup_{0 \leq a < b \leq 1} \left| \frac{1}{N} \sum_{n=1}^{N} \chi_{[a,\,b)}(x_n) - (b-a) \right|,$$

where N is a positive integer and $\chi_{[a,\,b)}(x)$ is the characteristic function mod 1 of $[a,b)$, that is, $\chi_{[a,b)}(x) = 1$ for $\{x\} \in [a,b)$ and $\chi_{[a,b)}(x) = 0$ otherwise (see [6]).

Ohkubo [7] obtained an upper estimate of $D_N(\alpha n + \beta \log n)$ as follows:

2000 Mathematics Subject Classification: Primary 11K38; Secondary 11L07.
Keywords: discrepancy, exponential sum, saddle-point method, regularly varying function.

If α is of constant type and β is non-zero real, then, for all positive integer N,

$$D_N(\alpha n + \beta \log n) \le C(\beta) N^{-\frac{2}{3}} \log N .$$

See [2] for a multi-dimensional case and also see [3] for an extended result.

Let $r(x)$ be a positive measurable function on $[X, \infty)$ for some $X > 0$. If there exists a real number ρ such that

$$\lim_{x\to\infty} \frac{r(\lambda x)}{r(x)} = \lambda^{\rho} \qquad \text{for each} \quad \lambda > 0 ,$$

then r is said to be *regularly varying function of index* ρ and we write $r \in R_\rho$ (see [1]).

The purpose of this paper is to obtain the lower bounds for the discrepancy of the sequences $\big(\alpha n + f(n)\big)$, where $f(x)$ is some regularly varying function.

We shall need several assumptions on f and α. For convenience, we list the eight assumptions:

(F1) $f(x)$ is real for $x > 1$.
(F2) $f'(x) \to 0$ $(x \to \infty)$.
(F3) $f''(x) \to 0$ $(x \to \infty)$.
(F4) $f''' \in R_\rho$ for some $\rho < -2$.
(F5) $f'''(x)$ is ultimately non-increasing.
(F6) $xf'(x)$ is ultimately non-decreasing.
(A1) α is the irrational number with continued fraction expansion $\alpha = [a_0, a_1, a_2, \dots]$. Let $p_n/q_n = [a_0, a_1, \dots, a_n]$ $(n \ge 0)$ be the sequence of principal convergents to α, and let $h_m = p_{2m+1}$ and $k_m = q_{2m+1}$ for $m \ge 0$.
(A2) $\alpha = [a_0, a_1, a_2, \dots]$ has bounded partial quotients, say $a_i \le c$ for $i \ge 1$.

From (F2), (F3), (F4), and L'Hospital's rule, it follows that $-f'' \in R_{\rho+1}$ and $f' \in R_{\rho+2}$. Since $f'(x) \downarrow 0$ $(x \uparrow \infty)$ and $0 < h_m/k_m - \alpha < k_m^{-2} \to 0$ $(m \to \infty)$, there exists c_m such that $f'(c_m) = \frac{h_m}{k_m} - \alpha$ for sufficiently large m $(m \ge \exists M_0)$. Since $h_m/k_m - \alpha > h_{m+1}/k_{m+1} - \alpha$ and f' is decreasing, we have $c_m < c_{m+1}$ for $m \ge M_0$. Since $f'(c_m) \le k_m^{-2} \to 0$ as $m \to \infty$ and $f'(x) \downarrow 0$ as $x \uparrow \infty$, we have $c_m \uparrow \infty$ as $m \uparrow \infty$.

First, we show the detailed estimate of the exponential sum.

Theorem 1.1. *Suppose that* (F1), (F2), (F3), (F4), (F5), *and* (A1) *are satisfied. Then*

$$\begin{aligned}
&\sum_{n=1}^{N} e\big(k_m(\alpha n + f(n))\big) \\
&= \frac{1}{k_m^{1/2}|f''(c_m)|^{1/2}} \mathrm{e}\big((\alpha k_m - h_m)c_m + k_m f(c_m) - 1/8\big) \\
&\quad + O\left(\frac{1}{k_m^{2/3} f'(c_m)^{1/3}|f''(c_m)|^{1/3}}\right) \\
&\quad + O\left(\frac{1}{k_m f'(c_m)}\right) + O\left(k_m^{1/2}\left(\sup_{1\le x\le c_m/2} xf'(x)\right)^{1/2} \log\big(f'(1)k_m + 2\big)\right)
\end{aligned}$$

for sufficiently large positive integer N, *where* m *is defined by* $2c_m < N \le 2c_{m+1}$.

Applying Theorem 1.1, we can prove the main theorem.

Theorem 1.2. *Suppose that* (F1), (F2), (F3), (F4), (F5), (F6), (A1), *and* (A2) *are satisfied. Then there exists a constant* C *such that*

$$D_N(\alpha n + f(n)) \ge C\frac{f'(N)^{1/4}}{N^{1/2}}$$

for all positive integer N.

The following two corollaries are immediate from Theorem 1.2.

Corollary 1.1. *Let* α *be an irrational number with bounded partial quotients, let* $\beta > 0$ *and* $s \ge 1$. *Then, there exists a constant* $C' > 0$ *such that*

$$D_N\big(\alpha n + \beta(\log n)^s\big) \ge C'\frac{(\log N)^{(s-1)/4}}{N^{3/4}}$$

for all positive integer N.

Corollary 1.2. *Let* α *be an irrational number with bounded partial quotients and let* $\beta > 0$, $0 < \sigma < 1$. *Then there exists a constant* $C'' > 0$ *such that*

$$D_N(\alpha n + \beta n^\sigma) \ge C'' N^{(\sigma-3)/4}$$

for all positive integers N.

2. Preliminary lemmas

Lemma 2.1. ([4; p. 56]) *Let $F(x)$ be a real differentiable function such that $F'(x)$ is monotonic and $F'(x) \ge m > 0$ or $F'(x) \le -m < 0$ for $a \le x \le b$. Let $G(x)$ be a positive, monotonic function for $a \le x \le b$ such that $|G(x)| \le G$. Then*

$$\left| \int_a^b G(x)\, \mathrm{e}^{\mathrm{i}\, F(x)}\, \mathrm{d}x \right| \le 4Gm^{-1} .$$

Lemma 2.2. ([9; Lemma 10.5]) *Let $\varphi(x)$ be a positive decreasing and differentiable function defined on the interval $[a,b]$. If $g''(x)$ is of constant sign and $\varphi'(x)/g''(x)$ is monotone on $[a,b]$, then*

$$\left| \int_a^b \varphi(x) e\big(g(x)\big)\, \mathrm{d}x \right| \le 8 \max_{a\le x\le b} \left(\frac{\varphi(x)}{|g''(x)|^{1/2}} \right) + \max_{a\le x\le b} \left(\left| \frac{\varphi'(x)}{g''(x)} \right| \right) .$$

Lemma 2.3. ([8; Lemma 4.6 and Notes]) *Let $g(x)$ be a real-valued function on the interval $[a,b]$ and suppose that $g(x)$ satisfies the following conditions:*

(i) *$g'''(x)$ is continuous on $[a,b]$;*
(ii) *either $g''(x) > 0$ on $[a,b]$ or $g''(x) < 0$ on $[a,b]$;*
(iii) *there exists positive number A_2 such that $g''(x) \asymp A_2$ on $[a,b]$;*
(iv) *there exists positive number A_3 such that $g'''(x) \ll A_3$ on $[a,b]$;*
(v) *$g'(c) = 0$ for some $c \in [a,b]$.*

Then

$$\int_a^b e\big(g(x)\big)\, \mathrm{d}x = |g''(c)|^{-1/2}\, e\Big(g(c) + \tfrac{1}{8}\operatorname{sgn}\big(g''(c)\big)\Big) + O\big(A_2^{-1} A_3^{1/3}\big)$$

$$+ O\left(\min\left\{ \frac{1}{|g'(a)|}, A_2^{-1/2} \right\} \right) + O\left(\min\left\{ \frac{1}{|g'(b)|}, A_2^{-1/2} \right\} \right) .$$

Lemma 2.4. ([9; Lemma 4.4]) *Let $f(x)$ be a real-valued function and $f'(x)$ be monotone on the interval $[a,b]$ such that $|f'(x)| \le \lambda$ on $[a,b]$ for some $0 < \lambda < 1$. Then*

$$\left| \int_a^b e\big(f(x)\big)\, \mathrm{d}x - \sum_{a<n\le b} e\big(f(n)\big) \right| = O\Big(\frac{1}{1-\lambda} \Big) .$$

The following lemma is a modified saddle-point theorem.

Lemma 2.5. *Suppose that $g(x)$ and $\varphi(x)$ are real-valued functions on the interval $[a,b]$ which satisfy the following conditions:*

(i) *$g'''(x)$ is continuous and $\varphi''(x)$ exists on $[a,b]$;*
(ii) *either $g''(x) > 0$ on $[a,b]$ or $g''(x) < 0$ on $[a,b]$;*
(iii) *there exists a number $A_2 > 0$ such that*
$$g''(x) \asymp A_2 \qquad \text{on} \quad [a,b];$$
(iv) *there exists a number A_3 such that $g'''(x) \ll A_3$;*
(v) *there exist numbers H_0, H_1, and H_2 such that*
$$\varphi(x) \ll H_0, \qquad \varphi'(x) \ll H_1, \qquad \varphi''(x) \ll H_2;$$
(vi) *$g'(c) = 0$ for some $c \in [a,b]$.*

Then
$$\begin{aligned}\int_a^b \varphi(x)e\big(g(x)\big)\,\mathrm{d}x &= \varphi(c)|g''(c)|^{-1/2}e\Big(g(c) + \frac{1}{8}\operatorname{sgn}\big(g''(c)\big)\Big)\\ &\quad + O\big(H_0A_2^{-1}A_3^{1/3}\big) + O\big(H_1A_2^{-1}\big) + O\big(H_2A_2^{-1}(b-a)\big)\\ &\quad + O\big(H_1A_2^{-2}A_3(b-a)\big) + O\big(H_2A_2^{-2}A_3(b-a)^2\big)\\ &\quad + O\big(H_0\min\{A_2^{-1/2}, |g'(a)|^{-1}\}\big) + O\big(H_0\min\{A_2^{-1/2}, |g'(b)|^{-1}\}\big).\end{aligned} \tag{2.1}$$

P r o o f. The cases $g'' > 0$ and $g'' < 0$ are analogous, so only the former is considered. We have
$$\begin{aligned}\int_a^b \varphi(x)e\big(g(x)\big)\,\mathrm{d}x &= \varphi(c)\int_a^b e\big(g(x)\big)\,\mathrm{d}x + \int_a^b \big(\varphi(x)-\varphi(c)\big)e\big(g(x)\big)\,\mathrm{d}x\\ &= I_1 + I_2, \qquad \text{say.}\end{aligned} \tag{2.2}$$
We split the integral I_2 into two parts:
$$\begin{aligned}I_2 &= \int_a^c \big(\varphi(x)-\varphi(c)\big)e\big(g(x)\big)\,\mathrm{d}x + \int_c^b \big(\varphi(x)-\varphi(c)\big)e\big(g(x)\big)\,\mathrm{d}x\\ &= I_2^{(1)} + I_2^{(2)}, \qquad \text{say.}\end{aligned}$$
By change of variable and integration by parts, we obtain
$$\begin{aligned}I_2^{(1)} &= \left[\frac{\varphi(x+c)-\varphi(c)}{2\pi\mathrm{i}\,g'(x+c)}e\big(g(x+c)\big)\right]_{a-c}^{0}\\ &\quad - \frac{1}{2\pi\mathrm{i}}\int_{a-c}^{0} \frac{\varphi'(x+c)g'(x+c) - g''(x+c)\big(\varphi(x+c)-\varphi(c)\big)}{\big(g'(x+c)\big)^2}e\big(g(x+c)\big)\,\mathrm{d}x.\end{aligned} \tag{2.3}$$

From the conditions concerning $g(x)$ and $\varphi(x)$, it follows that for $x < 0$

$$\varphi'(x+c) = \varphi'(c) + O(H_2 x)\,, \tag{2.4}$$

$$\varphi(x+c) - \varphi(c) = \varphi'(c)x + O(H_2 x^2)\,, \tag{2.5}$$

$$g'(x+c) = g''(c)x + O(A_3 x^2)\,, \tag{2.6}$$

$$g'(x+c) = g'(c) + g''(c+\theta_1 x)x = g''(c+\theta_1 x)x \asymp A_2 x \tag{2.7}$$

for some $0 < \theta_1 < 1$, and

$$g''(x+c) = g''(c) + g'''(c+\theta_2 x)x = g''(c) + O(A_3 x) \tag{2.8}$$

for some $0 < \theta_2 < 1$. By (2.5), (2.7), and $\varphi'(x) \ll H_1$, we obtain

$$\begin{aligned}\frac{\varphi(a)-\varphi(c)}{g'(a)} &= \frac{\varphi'(c)(a-c) + O\big(H_2(a-c)^2\big)}{g'(a)}\\ &= O\big(H_1 A_2^{-1}\big) + O\big(H_2 A_2^{-1}(b-a)\big)\,.\end{aligned} \tag{2.9}$$

By (2.5) and (2.6), we obtain

$$\frac{\varphi(x+c)-\varphi(c)}{g'(x+c)} = \frac{\varphi'(c)x + O(H_2x^2)}{g''(c)x + O(A_3x^2)} = \frac{\varphi'(c) + O(H_2 x)}{g''(c) + O(A_3 x)}\,.$$

From this and $g''(x) \asymp A_2$ and $\varphi'(x) \ll H_1$, it follows that

$$\lim_{x\uparrow 0} \frac{\varphi(x+c)-\varphi(c)}{g'(x+c)} e\big(g(x+c)\big) = \frac{\varphi'(c)}{g''(c)} e\big(g(c)\big) \ll H_1 A_2^{-1}\,. \tag{2.10}$$

Using (2.4), (2.5), (2.6), (2.8), and $\varphi'(x) \ll H_1$, $g''(x) \ll A_2$, we have for $a - c \le x \le 0$,

$$\varphi'(x+c)g'(x+c) - g''(x+c)\big(\varphi(x+c)-\varphi(c)\big) \ll H_1 A_3 x^2 + H_2 A_2 x^2 + H_2 A_3 (b-a) x^2\,,$$

and so

$$\begin{aligned}&\left| \int_{a-c}^{0} \frac{\varphi'(x+c)g'(x+c) - g''(x+c)\big(\varphi(x+c)-\varphi(c)\big)}{\big(g'(x+c)\big)^2} e\big(g(x+c)\big)\,\mathrm{d}x \right|\\ &\ll H_1 A_2^{-2} A_3 (b-a) + H_2 A_2^{-1}(b-a) + H_2 A_2^{-2} A_3 (b-a)^2\,,\end{aligned} \tag{2.11}$$

where in the second step, we also use (2.7).

From (2.3), (2.9), (2.10), and (2.11), it follows that

$$I_2^{(1)} \ll H_1 A_2^{-1} + H_2 A_2^{-1}(b-a) + H_1 A_2^{-2} A_3 (b-a) + H_2 A_2^{-2} A_3 (b-a)^2\,.$$

Similarly, we can obtain the same estimate for $I_2^{(2)}$. Hence, we have

$$I_2 \ll H_1 A_2^{-1} + H_2 A_2^{-1}(b-a) + H_1 A_2^{-2} A_3 (b-a) + H_2 A_2^{-2} A_3 (b-a)^2\,. \tag{2.12}$$

Applying Lemma 2.3 to the integral I_1, by the assumption $\varphi(x) \ll H_0$, we have

$$\begin{aligned}I_1 =& \varphi(c)|g''(c)|^{-1/2} e\Big(g(c) + \tfrac{1}{8}\operatorname{sgn}\big(g''(c)\big)\Big) + O\big(H_0 A_2^{-1} A_3^{1/3}\big)\\ &+ O\big(H_0 \min\{A_2^{-1/2}, |g'(a)|^{-1}\}\big) + O\big(H_0 \min\{A_2^{-1/2}, |g'(b)|^{-1}\}\big)\,.\end{aligned} \tag{2.13}$$

Combining (2.2), (2.12), and (2.13) completes the proof. □

Lemma 2.6. ([8; Lemma 4.7]) *Let $f(x)$ be a real function with continuous and steadily decreasing derivative $f'(x)$ in (a,b), and let $f'(b) = A$, $f'(a) = B$. Then*

$$\sum_{a<n\le b} e\big(f(n)\big) = \sum_{A-\eta<\nu<B+\eta} \int_a^b e\big(f(x)-\nu x\big)\,\mathrm{d}x + O\big(\log(B-A+2)\big)\,,$$

where η is any positive constant less than 1.

Lemma 2.7. ([1; Monotone Density Theorem]) *Let $U(x) = \int_0^x u(y)\,\mathrm{d}y$.*

If $U(x) \sim cx^{\rho}\ell(x)$ $(x \to \infty)$, where $c \in \mathbb{R}$, $\rho \in \mathbb{R}$, $\ell \in R_0$, and if u is ultimately monotone, then

$$u(x) \sim c\rho x^{\rho-1}\ell(x) \qquad (x \to \infty).$$

Lemma 2.8. *Let $f \in R_\rho$ with $\rho \in \mathbb{R}$. Suppose that $f'(x)$ is ultimately monotone. Then*

$$\lim_{x\to\infty} \frac{xf'(x)}{f(x)} = \rho\,.$$

Proof. Let $f(x)$ is defined on the interval $[X,\infty)$ for some $X > 0$. Since $f \in R_\rho$, we have $f(x) = x^{\rho}\ell(x)$ for some $\ell \in R_0$. We set

$$u(y) = \begin{cases} \frac{f(X)}{X} & \text{if } 0 \le y < X\,,\\ f'(y) & \text{if } X \le y\,. \end{cases}$$

We have

$$\int_0^x u(y)\,\mathrm{d}y = f(x)$$

for $x \ge X$. Lemma 2.7 yields that

$$u(x) \sim \rho x^{\rho-1}\ell(x) \qquad (x \to \infty),$$

and so

$$f'(x) \sim \rho x^{-1} f(x) \qquad (x \to \infty),$$

which completes the proof. □

Lemma 2.9. ([1; Uniform Convergence Theorem]) *If $f \in R_\rho$, then*

$$f(\lambda x)/f(x) \to \lambda^{\rho} \qquad (x \to \infty) \text{ uniformly in } \lambda \text{ on each } [a,b] \quad (0 < a < b < \infty).$$

Lemma 2.10. ([1; Theorem 1.2.12]) *If $f \in R_\rho$ with $\rho > 0$, there exists $g \in R_{1/\rho}$ with*

$$f(g(x)) \sim g(f(x)) \sim x \qquad (x \to \infty).$$

Applying Koksma's inequality (see [6; p. 143, Theorem 5.1]), we obtain the following lemma by the same reasoning as in the proof of [6; p. 143, Corollary 5.1].

Lemma 2.11. ([6]) *Let (x_n) be a sequence of real numbers. Then for any positive integer h,*

$$\frac{1}{4h}\left|\frac{1}{N}\sum_{n=1}^{N} e(hx_n)\right| \le D_N(x_n).$$

3. Proof of Theorem 1.1

Let $m \ge M_0$, and let $a_m = c_m/2$, $b_m = 2c_m$ and $x_n = \alpha n + f(n)$. We have

$$\sum_{a_m<n\le b_m} e(k_m x_n) = \sum_{a_m<n\le b_m} e(k_m x_n - h_m n) = \sum_{a_m<n\le b_m} e(g(n)), \tag{3.1}$$

where $g(x) = (\alpha k_m - h_m)x + k_m f(x)$. Since $f'(x)$ is decreasing, we have

$$g'(x) \le (h_m - \alpha k_m)\left(\frac{f'(c_m/2)}{f'(c_m)} - 1\right) \qquad \text{for} \quad a_m \le x \le b_m. \tag{3.2}$$

Since $h_m - \alpha k_m \to 0$, we have $h_m - \alpha k_m < 2^{\rho+1}$ for sufficiently large m ($m \ge \exists M_1 \ge M_0$). Since $f' \in R_{\rho+2}$ and $c_m \to \infty$ ($m \to \infty$), we have

$$\lim_{m\to\infty} \frac{f'(c_m/2)}{f'(c_m)} = \frac{1}{2^{\rho+2}}.$$

Therefore, we obtain

$$\frac{f'(c_m/2)}{f'(c_m)} < \frac{1}{2^{\rho+2}} + 1 \tag{3.3}$$

for sufficiently large m ($m \ge \exists M_2 \ge M_1$). By (3.2) and (3.3), we have $g'(x) < 1/2$ for $a_m \le x \le b_m$ ($m \ge M_2$). Since $f' > 0$, $k_m > 0$, and $\rho < -2$, we have

$$g'(x) > -(h_m - \alpha k_m) > -2^{\rho+1} \ge -1/2 \qquad \text{for} \quad a_m \le x \le b_m \quad (m \ge M_1).$$

Thus $|g'(x)| < 1/2$ for $a_m \le x \le b_m$ ($m \ge M_2$). Therefore, from (3.1) and Lemma 2.4 with $\lambda = 1/2$, it follows that

$$\sum_{a_m<n\le b_m} e(k_m x_n) = \int_{a_m}^{b_m} e(g(x))\, \mathrm{d}x + O(1) \qquad \text{for} \quad m \ge M_2. \tag{3.4}$$

Let $m \geq M_2$. Using integration by parts, we have

$$\int_{a_m}^{b_m} e(g(x))\, \mathrm{d}x = \frac{1}{f'(c_m)} \int_{a_m}^{b_m} \varphi(x) e(g(x))\, \mathrm{d}x + O\left(\frac{1}{k_m f'(c_m)}\right), \tag{3.5}$$

where $\varphi(x) = f'(x)$. In order to apply Lemma 2.5 to the integral in the right-hand side of (3.5), we consider the conditions that the functions φ and g satisfy. We observe that $g'(x)$ is decreasing and

$$g'(c_m) = 0 \qquad \text{and} \qquad a_m < c_m < b_m\,.$$

Since $g'' = k_m f'' < 0$, f'' is increasing, and $-f'' \in R_{\rho+1}$, we find

$$-g''(x) \asymp -k_m f''(c_m) \qquad \text{for} \quad a_m \leq x \leq b_m\,.$$

Since $f'''(x)$ is ultimately non-increasing and $f''' \in R_\rho$, we have

$$|g'''(x)| = k_m f'''(x) \ll k_m f'''(c_m/2) \ll k_m f'''(c_m)$$

for $a_m \leq x \leq b_m$ and for sufficiently large m ($m \geq \exists M_3 \geq M_2$). We note that $f' \in R_{\rho+2}$, f' is positive and decreasing. It follows that

$$|\varphi(x)| = f'(x) \leq f'(c_m/2) \ll f'(c_m) \qquad \text{for} \quad a_m \leq x \leq b_m\,.$$

We also note that $-f'' \in R_{\rho+1}$ and $-f''$ is positive and decreasing. It follows that

$$|\varphi'(x)| = -f''(x) \leq -f''(c_m/2) \ll -f''(c_m) \qquad \text{for} \quad a_m \leq x \leq b_m\,.$$

Since $f'''(x)$ is ultimately non-increasing and $f''' \in R_\rho$, we have

$$|\varphi''(x)| = f'''(x) \ll f'''(c_m) \qquad (a_m \leq x \leq b_m)$$

for sufficiently large m ($m \geq M_3$). Hence, we can apply Lemma 2.5 with $A_2 = -k_m f''(c_m)$, $A_3 = k_m f'''(c_m)$, $H_0 = f'(c_m)$, $H_1 = -f''(c_m)$, and $H_2 = f'''(c_m)$ to the integral $\int_{a_m}^{b_m} \varphi(x) e(g(x))\, \mathrm{d}x$. We compute each term on the right-hand side of (2.1) in Lemma 2.5. We have

$$H_0 A_2^{-1} A_3^{1/3} = \frac{f'(c_m)^{2/3}}{k_m^{2/3} \left(-f''(c_m)\right)^{1/3}} \left(\frac{f'(c_m)}{-c_m f''(c_m)}\right)^{1/3} \left(\frac{c_m f'''(c_m)}{-f''(c_m)}\right)^{1/3}. \tag{3.6}$$

We note that $c_m \to \infty$ ($m \to \infty$), $-f'' \in R_{\rho+1}$, and $f'''(x)$ is ultimately non-increasing. Applying Lemma 2.8, we have

$$\lim_{m\to\infty} \frac{c_m f'''(c_m)}{-f''(c_m)} = -(\rho + 1)\,,$$

and so

$$\frac{c_m f'''(c_m)}{-f''(c_m)} = O(1)\,. \tag{3.7}$$

Similarly, we have

$$\lim_{m\to\infty} \frac{-c_m f''(c_m)}{f'(c_m)} = -(\rho+2)\,,$$

and so

$$\frac{f'(c_m)}{-c_m f''(c_m)} = O(1)\,. \tag{3.8}$$

From (3.6), (3.7), and (3.8), it follows that

$$H_0 A_2^{-1} A_3^{1/3} \ll \frac{f'(c_m)^{2/3}}{k_m^{2/3}\big(-f''(c_m)\big)^{1/3}}\,. \tag{3.9}$$

We also obtain

$$H_1 A_2^{-1} = -f''(c_m)\big(-k_m f''(c_m)\big)^{-1} = k_m^{-1}\,. \tag{3.10}$$

From (3.7), it follows that

$$H_2 A_2^{-1}(b_m - a_m) \ll k_m^{-1} \frac{c_m f'''(c_m)}{-f''(c_m)} \ll k_m^{-1}\,, \tag{3.11}$$

$$H_1 A_2^{-2} A_3 (b_m - a_m) \ll k_m^{-1}, \tag{3.12}$$

and

$$H_2 A_2^{-2} A_3 (b_m - a_m)^2 \ll k_m^{-1} \left(\frac{c_m f'''(c_m)}{-f''(c_m)}\right)^2 \ll k_m^{-1}\,. \tag{3.13}$$

Since $f' \in R_{\rho+2}$, we have

$$H_0 \min\{A_2^{-1/2}, |g'(a_m)|^{-1}\} \le H_0 |g'(a_m)|^{-1} = k_m^{-1} \left|\frac{f'(c_m/2)}{f'(c_m)} - 1\right|^{-1} \ll k_m^{-1}\,. \tag{3.14}$$

Similarly, we obtain

$$H_0 \min\{A_2^{-1/2}, |g'(b_m)|^{-1}\} = O(k_m^{-1})\,. \tag{3.15}$$

From Lemma 2.5, (3.9), (3.10), (3.11), (3.12), (3.13), (3.14), and (3.15), it follows that for $m \ge M_3$,

$$\begin{aligned} \int_{a_m}^{b_m} \varphi(x) e\big(g(x)\big)\, \mathrm{d}x = {} & \frac{f'(c_m)}{k_m^{1/2}\big(-f''(c_m)\big)^{1/2}} e\Big(g(c_m) - \frac{1}{8}\Big) \\ & + O\left(\frac{f'(c_m)^{2/3}}{k_m^{2/3}(-f''(c_m))^{1/3}}\right) + O(k_m^{-1})\,. \end{aligned} \tag{3.16}$$

By (3.5) and (3.16), we have

$$\int_{a_m}^{b_m} e(g(x))\,\mathrm{d}x = \frac{1}{k_m^{1/2}(-f''(c_m))^{1/2}} e\Big(g(c_m) - \frac{1}{8}\Big) + O\left(\frac{1}{k_m^{2/3} f'(c_m)^{1/3}(-f''(c_m))^{1/3}}\right) + O\left(\frac{1}{k_m f'(c_m)}\right) \tag{3.17}$$

for $m \geq M_3$. From (3.4) and (3.17), it follows that

$$\begin{aligned}\sum_{a_m<n\leq b_m} & e(k_m x_n) \\ &= \frac{1}{k_m^{1/2}(-f''(c_m))^{1/2}} e\Big(g(c_m) - \frac{1}{8}\Big) \\ &\quad + O\left(\frac{1}{k_m^{2/3} f'(c_m)^{1/3}(-f''(c_m))^{1/3}}\right) + O\left(\frac{1}{k_m f'(c_m)}\right) + O(1)\end{aligned} \tag{3.18}$$

for $m \geq M_3$,

Next, we estimate the sum $\sum_{1<n\leq a_m} e(k_m x_n)$. We put $\phi(x) = k_m(\alpha x + f(x))$. Setting $\eta = \alpha k_m - h_m + k_m f'(c_m/2)$, we obtain $0 < \eta < 1$ for sufficiently large m $(m > \exists M_4 > M_3)$. Applying Lemma 2.6, we have for $m > M_4$,

$$\sum_{1<n\leq a_m} e(k_m x_n) = \sum_{A-\eta<\nu<B+\eta} \int_1^{a_m} e(\phi(x) - \nu x)\,\mathrm{d}x + O(\log(B - A + 2)), \tag{3.19}$$

where $A = \phi'(a_m)$, $B = \phi'(1)$. By integration by parts, we have

$$\int_1^{a_m} e(\phi(x) - \nu x)\,\mathrm{d}x = -\frac{k_m}{\alpha k_m - \nu} \int_1^{a_m} f'(x) e(\phi(x) - \nu x)\,\mathrm{d}x + O\left(\frac{1}{|\alpha k_m - \nu|}\right).$$

Applying Lemma 2.2, we obtain

$$\int_1^{a_m} f'(x) e(\phi(x) - \nu x)\,\mathrm{d}x \ll \max_{1\leq x\leq a_m}\left(\frac{f'(x)}{|\phi''(x)|^{1/2}}\right) + \max_{1\leq x\leq a_m}\left(\left|\frac{f''(x)}{\phi''(x)}\right|\right).$$

We have

$$\frac{f'(x)}{|\phi''(x)|^{1/2}} = k_m^{-1/2}(x f'(x))^{1/2}\left(\frac{f'(x)}{-x f''(x)}\right)^{1/2}.$$

Therefore, since $\lim_{x\to\infty} \frac{f'(x)}{-xf''(x)} = -\frac{1}{\rho+2}$ by Lemma 2.8, we have

$$\frac{f'(x)}{|\phi''(x)|^{1/2}} \ll k_m^{-1/2}\Big(\sup_{1\le x\le a_m} xf'(x)\Big)^{1/2} \qquad \text{for} \quad 1 \le x \le a_m .$$

Hence

$$\int_1^{a_m} f'(x)e\big(\phi(x)-\nu x\big)\,\mathrm{d}x \ll k_m^{-1/2}\Big(\sup_{1\le x\le a_m} xf'(x)\Big)^{1/2} + k_m^{-1} . \tag{3.20}$$

We remark that h_m is not contained in the interval $A-\eta < \nu < B+\eta$. By (3.19) and (3.20), we obtain

$$\sum_{1<n\le a_m} e(k_m x_n) \ll \left(k_m^{1/2}\Big(\sup_{1\le x\le a_m} xf'(x)\Big)^{1/2} + 1\right)\log\big(f'(1)k_m + 2\big) . \tag{3.21}$$

Let $b_m < N \le b_{m+1}$. We have

$$\sum_{b_m<n\le N} e(k_m x_n) = \sum_{b_m<n\le N} e(k_m x_n - h_m n) = \sum_{b_m<n\le N} e\big(g(n)\big) .$$

Since $0 \le -g'(x) \le 1/2$ for $b_m \le x \le N$, Lemma 2.4 yields

$$\sum_{b_m<n\le N} e\big(g(n)\big) = \int_{b_m}^{N} e\big(g(x)\big)\,\mathrm{d}x + O(1) . \tag{3.22}$$

Applying integration by parts, we have

$$\int_{b_m}^{N} e\big(g(x)\big)\,\mathrm{d}x = -\frac{k_m}{h_m-\alpha k_m}\int_{b_m}^{N} f'(x)e\big(g(x)\big)\,\mathrm{d}x + O\left(\frac{1}{h_m-\alpha k_m}\right) . \tag{3.23}$$

Since $f'(x)$ is decreasing, we obtain $0 < f'(x) \le f'(b_m)$ for $b_m \le x \le N$. Let $F(x) = 2\pi g(x)$. We have, by the definition $f'(c_m) = \frac{h_m}{k_m} - \alpha$,

$$F'(x) \le 2\pi k_m f'(c_m)\left(\frac{f'(b_m)}{f'(c_m)} - 1\right) .$$

Since $\lim_{m\to\infty} \frac{f'(b_m)}{f'(c_m)} = 2^{\rho+2} < 1$,

$$\frac{f'(b_m)}{f'(c_m)} < \frac{2^{\rho+1}+1}{2} \qquad \text{for sufficiently large } m \quad (m > \exists M_5 > M_4) .$$

Hence
$$F'(x) \le -\pi\big(1-2^{\rho+2}\big)k_m f'(c_m) < 0 \qquad \text{for} \quad m \ge M_4 .$$
Using Lemma 2.1 and $f' \in R_{\rho+2}$, we have
$$\left|\int_{b_m}^{N} f'(x)e\big(g(x)\big)\,\mathrm{d}x\right| \le \frac{4f'(b_m)}{\big(1-2^{\rho+2}\big)\pi f'(c_m)k_m} \ll \frac{1}{k_m}.$$
Hence, by (3.23)
$$\int_{b_m}^{N} e\big(g(x)\big)\,\mathrm{d}x = O\left(\frac{1}{h_m-\alpha k_m}\right) = O\left(\frac{1}{k_m f'(c_m)}\right),$$
and so, by (3.22)
$$\sum_{b_m<n\le N} e\big(g(n)\big) = O\left(\frac{1}{k_m f'(c_m)}\right) + O(1) \quad \text{for} \quad m \ge M_5 . \tag{3.24}$$
Combining (3.18), (3.21), and (3.24) completes the proof of Theorem 1.1.

4. Proof of Theorem 1.2

We note that $f' \in R_{\rho+2}$ and f'' is increasing. Applying Lemma 2.8, we obtain
$$\lim_{x\to\infty} \frac{xf''(x)}{f'(x)} = \rho+2 .$$
Hence, we have
$$-f''(x) = -\frac{xf''(x)}{f'(x)}\frac{f'(x)}{x} \asymp \frac{f'(x)}{x} . \tag{4.1}$$
Applying Theorem 1.1, we have for sufficiently large N,
$$\begin{aligned}
&\sum_{n=1}^{N} e\big(k_m(\alpha n + f(n))\big)\\
&= \frac{1}{k_m^{1/2}|f''(c_m)|^{1/2}} e\Big(g(c_m)-\tfrac{1}{8}\Big) + O\left(\frac{1}{k_m^{2/3}|f'(c_m)|^{1/3}|f''(c_m)|^{1/3}}\right)\\
&\quad + O\left(\frac{1}{k_m|f'(c_m)|}\right) + O\bigg(k_m^{1/2}\Big(\sup_{1\le x\le c_m/2} x|f'(x)|\Big)^{1/2}\log\big(f'(1)k_m+2\big)\bigg),
\end{aligned} \tag{4.2}$$

where $2c_m < N \le 2c_{m+1}$ and $g(x) = (\alpha k_m - h_m)x + k_m f(x)$.

We compare the order of magnitude of the main term with that of each error term in (4.2). Because of (4.1), for the main term, we have

$$\frac{1}{k_m^{1/2}\big(-f''(c_m)\big)^{1/2}} \asymp \frac{c_m^{1/2}}{k_m^{1/2} f'(c_m)^{1/2}} = \frac{c_m}{k_m^{1/2}\big(c_m f'(c_m)\big)^{1/2}} . \tag{4.3}$$

Because of (4.1), for the first error term, we have

$$\frac{1}{k_m^{2/3} f'(c_m)^{1/3}\big(-f''(c_m)\big)^{1/3}} \asymp \frac{c_m}{k_m^{2/3}\big(c_m f'(c_m)\big)^{2/3}} .$$

Since the function $xf'(x)$ is ultimately non-decreasing, we have $0 < \lim_{m\to\infty} c_m f'(c_m) \le \infty$. By comparing the right-hand sides of the above two formulas, it follows that the order of magnitude of the main term is properly bigger than that of the first error term. Similarly, it follows that the order of magnitude of the main term is properly bigger than that of the second error term. For the third error term, we have for sufficiently large m

$$k_m^{1/2}\Big(\sup_{1\le x\le c_m/2} xf'(x)\Big)^{1/2} \log\big(f'(1)k_m+2\big) \le k_m^{1/2}\big(c_m f'(c_m)\big)^{1/2} \log\big(f'(1)k_m+2\big) ,$$

because $xf'(x)$ is ultimately non-decreasing. Since $k_m f'(c_m) \le 1/k_m$, we have

$$k_m^{1/2}\big(c_m f'(c_m)\big)^{1/2} \log\big(f'(1)k_m + 2\big) = o\left(\frac{c_m^{1/2}}{k_m^{1/2} f'(c_m)^{1/2}}\right) .$$

Hence, by comparing the second term of (4.3) with the right-hand side of the above formula, it follows that the order of magnitude of the main term is properly bigger than that of the third error term. After all, the order of magnitude of the main term (4.3) is properly bigger than that of each error term. Therefore we obtain

$$\left|\sum_{n=1}^{N} e\big(k_m(\alpha n + f(n))\big)\right| \ge C_1 \frac{c_m^{1/2}}{k_m^{1/2} f'(c_m)^{1/2}} \tag{4.4}$$

for some $C_1 > 0$ and for sufficiently large N.

Let $f_1(x) = 1/f'(x)$. Since $f' \in R_{\rho+2}$, we have $f_1 \in R_{-(\rho+2)}$. By Lemma 2.10, there exists $F \in R_{-1/(\rho+2)}$ such that $F\big(f_1(x)\big) \sim x$ as $x \to \infty$. Hence, $c_n \sim F\big(f_1(c_m)\big) = F\big(k_m/(h_m - \alpha k_m)\big)$ as $m \to \infty$. Therefore

$$\frac{c_{m+1}}{c_m} \sim \frac{F\big(k_{m+1}/(h_{m+1} - \alpha k_{m+1})\big)}{F\big(k_m/(h_m - \alpha k_m)\big)} \qquad \text{as} \quad m \to \infty . \tag{4.5}$$

Setting $\lambda_m := \frac{k_{m+1}/(h_{m+1}-\alpha k_{m+1})}{k_m/(h_m-\alpha k_m)}$, we obtain $1/2 \le \lambda_m \le 2(c+1)^4 =: b$ for $m \ge 0$, because $1/(2q_{n+1}) \le p_n - \alpha q_n \le 1/q_{n+1}$ and $q_{n+1}/q_n = a_{n+1} + q_{n-1}/q_n \le c+1$ for $n \ge 1$ by the assumptions (A1) and (A2). By Lemma 2.9, $F(\lambda x)/F(x) \to \lambda^{-1/(\rho+2)}$ $(x \to \infty)$ uniformly in $\lambda \in [1/2, b]$. Hence, there exists X_0 such that if $x \ge X_0$, then

$$\left| \frac{F(\lambda x)}{F(x)} - \lambda^{-1/(\rho+2)} \right| < 1$$

for all $\lambda \in [1/2, b]$. We note that

$$\frac{F\big(k_{m+1}/(h_{m+1} - \alpha h_{m+1})\big)}{F\big(k_m/(h_m - \alpha h_m)\big)} = \frac{F(\lambda_m x_m)}{F(x_m)}, \tag{4.6}$$

where $x_m = k_m/(h_m - \alpha h_m)$. Since $x_m \to \infty$ as $m \to \infty$ and $\lambda_m \in [1/2, b]$, if m is sufficiently large, then

$$\frac{F(\lambda_m x_m)}{F(x_m)} \le \lambda_m^{-1/(\rho+2)} + 1 \le b^{-1/(\rho+2)} + 1. \tag{4.7}$$

From (4.5), (4.6), and (4.7), there exists $K > 0$ such that

$$\frac{c_{m+1}}{c_m} \le K \tag{4.8}$$

for sufficiently large m $(m \ge \exists M > 0)$. For $2c_m < N \le 2c_{m+1}$ $(m \ge M)$, we have $1/(2K) \le c_m/N < 1/2$. By Lemma 2.9, $f'(\lambda x)/f'(x) \to \lambda^{\rho+2}$ $(x \to \infty)$ uniformly in $\lambda \in \big[1/(2K), 1/2\big)$. Therefore, there exists $X_1 > 0$ such that if $x \ge X_1$, then

$$\left| \frac{f'(\lambda x)}{f'(x)} - \lambda^{\rho+2} \right| < 1$$

for all $\lambda \in \big[1/(2K), 1/2\big)$. Hence, we have

$$\left| \frac{f'\big((c_m/N)N\big)}{f'(N)} - \left(\frac{c_m}{N}\right)^{\rho+2} \right| < 1$$

for all $N \ge X_1$, and so

$$\frac{f'(c_m)}{f'(N)} < \left(\frac{c_m}{N}\right)^{\rho+2} + 1 \le \left(\frac{1}{2K}\right)^{\rho+2} + 1.$$

Hence, we have

$$f'(c_m) \ll f'(N). \tag{4.9}$$

Hence, from (4.4), (4.8), and (4.9), it follows that

$$\frac{1}{k_m} \left| \frac{1}{N} \sum_{n=1}^{N} e\big(k_m(\alpha n + f(n))\big) \right| \ge \frac{C_2}{k_m^{3/2} f'(N)^{1/2} N^{1/2}}$$

for some $C_2 > 0$ and for sufficiently large m. Since f' is decreasing, $f' \in R_{\rho+2}$, and $f'(c_m) \le k_m^{-2}$, we have

$$f'(N) < f'(2c_m) \ll f'(c_m) \le k_m^{-2},$$

so that

$$k_m \ll f'(N)^{-1/2}.$$

Therefore

$$\frac{1}{k_m}\left|\frac{1}{N}\sum_{n=1}^{N} e\big(k_m(\alpha n + f(n))\big)\right| \ge C\frac{f'(N)^{1/4}}{N^{1/2}} \tag{4.10}$$

for some $C > 0$. By (4.10) and Lemma 2.11, we obtain the desired inequality.

REFERENCES

[1] BINGHAM, H.—GOLDIE, C. M.—TEUGELS, J. L.: *Regular Variation*, Cambridge University Press, Cambridge, 1987.

[2] GOTO, K.—OHKUBO, Y.: *The discrepancy of the sequence* $(n\alpha + (\log n)\beta)$, Acta Math. Hungar. **86** (2000), 39–47.

[3] GOTO, K.—OHKUBO, Y.: *Discrepancy of some special sequences.* In: Analytic Number Theory (C. Jia, K. Matsumoto, eds.), Dev. Math. 6, Kluwer Academic Publishers, Dordrecht, 2002, pp. 143–155.

[4] IVIĆ, A.: *The Riemann Zeta-Function*, John Wiley and Sons, New York, 1985.

[5] KARATSUBA, A. A.—VORONIN, S. M.: *The Riemann Zeta-Function*, Walter de Gruyter, Berlin, 1992.

[6] KUIPERS, L.—NIEDERREITER, H.: *Uniform Distribution of Sequences*, John Wiley and Sons, New York, 1974.

[7] OHKUBO, Y.: *Notes on Erdös-Turán inequality*, J. Aust. Math. Soc. Ser. A **67** (1999), 51–57.

[8] TITCHMARSH, E. C.: *The Theory of the Riemann Zeta-Function* (2nd ed. revised by D. R. Heath-Brown), Clarendon Press, Oxford, 1986.

[9] ZYGMUND, A.: *Trigonometric Series Vol. I*, Cambridge University Press, Cambridge, 1977.

Received October 9, 2003

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