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On Tensor Fields Semiconjugated with Torse-forming Vector Fields

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Abstract
The paper deals with tensor fields which are semiconjugated with torse-forming vector fields. The existence results for semitorse-forming vector fields and for convergent vector fields are proved.

Key words: Torse-forming vector fields, Riemannian space, semisymmetric space, $T$-semisymmetric space.

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1 Introduction
Torse-forming vector fields were introduced by K. Yano [8] in 1944 and their properties in Riemannian spaces have been studied by various mathematicians. For example some properties in Ricci semisymmetric Riemannian spaces have been proved by J. Kowolik in [1]. In $T$-semisymmetric Riemannian spaces they are studied by the authors in [4] and [5].

This paper is devoted to the study of tensor fields which are semiconjugated with torse-forming vector fields. We are motivated by the work of J. Kowolik [1].

First we give some definitions and notations. $V_n$ denotes an $n$-dimensional Riemannian space with a metric $g$ and an affine connection $\nabla$. The metric $g$

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need not be positive definite. $TV_n$ is a space of all tangent vector fields on $V_n$. In the whole paper we will assume that $n > 2$ and that all functions, vectors and tensor fields are sufficiently smooth. Further $\xi$ will be a non-zero vector field, i.e. $\xi(x) \neq 0$ for each $x \in V_n$.

We denote the Riemannian tensor in $V_n$ by $R$. This tensor is called harmonic, if $R^\alpha_{ij,k,\alpha} = 0$, where "\cdot" denotes the covariant derivative. This condition can be written in the form $R_{ij,k} = R_{ik,j}$ where $R_{ij} \equiv R^\alpha_{ij,\alpha}$ is the Ricci tensor of $V_n$.

**Definition 1** Vector field $\xi$ is called *torse-forming*, if $\nabla_X \xi = \varrho \cdot X + a(X) \cdot \xi$ for all $X \in TV_n$, where $\varrho$ is some function on $V_n$, $a$ is a linear form on $V_n$. In the local transcription this formula has the form $\xi^h_i = \varrho \delta^h_i + a^i \xi^h$, where $\delta^h_i$ is the Kronecker delta, $a^i$ are components of the form $a$, which is a covector on $V_n$.

**Definition 2** A torse-forming vector field $\xi$ is called:

- recurrent, if $\varrho = 0$,
- concircular, if the form $a$ is gradient (or locally gradient), i.e. there exists (locally) a function $\varphi(x)$ such that $a = \partial_i \varphi(x) dx^i$,
- convergent, if $\xi$ is concircular and $\varrho = \text{const} \cdot \exp(\varphi(x))$,
- semitorse-forming, if $R(X, \xi) \xi = 0$ for each $X \in TV_n$.

Properties of torse-forming vector fields in the Einsteinian spaces are proved by the authors in [5]. In [2] and [3] J. Mikeš proved that in non-Einsteinian Ricci-symmetric and Ricci-two-symmetric ($R_{ij,kl} = 0$) spaces there are no concircular vector fields which are not recurrent.

In what follows we will need a definition of an operator $R(X,Y) \circ T$ for tensors of the type $(0,q)$ or $(1,q)$.

Let $T$ be a tensor of the type $(0,q)$, which is defined as a $q$-linear form $T(X_1, X_2, \ldots, X_q)$, where $X_1, X_2, \ldots, X_q \in TV_n$.

In the space $V_n$ we introduce an operator $R(X,Y) \circ T$ in the following way:

$$R(X,Y) \circ T(X_1, X_2, \ldots, X_q) \overset{df}{=} \sum_{s=1}^{q} T(X_1, \ldots, X_{s-1}, R(X,Y)X_s, X_{s+1}, \ldots, X_q).$$

In the local transcription the tensor $R(X,Y) \circ T$ has a form

$$\sum_{s=1}^{q} T^{i_1\ldots i_{s-1}\alpha i_{s+1}\ldots i_q} R^\alpha_{i_1\ldots i_s}.$$

By the Ricci identity we have

$$T^{i_1\ldots i_q[jk]} = \sum_{s=1}^{q} T^{i_1\ldots i_{s-1}\alpha i_{s+1}\ldots i_q} R^\alpha_{i_1\ldots i_s}.$$

where $[jk]$ denotes the alternation of the tensor with respect to $j$ and $k$. 
If $T$ is a tensor of the type $(0,0)$ (i.e., an invariant, which is a function or a scalar on $V_n$), then we put $R(X,Y) \circ T = 0$, or locally $T_{[jk]} = 0$.

Similarly we can define an operator $R(X,Y) \circ T$ for a tensor $T$ of the type $(1,q)$:

$$R(X,Y) \circ T(X_1, X_2, \ldots, X_q) \equiv \sum_{s=1}^{q} T(X_1, \ldots, X_{s-1}, R(X,Y)X_s, X_{s+1}, \ldots, X_q) - R(X,Y)(T(X_1, \ldots, X_q)).$$

The tensor $R(X,Y) \circ T$ has a local expression

$$\sum_{s=1}^{q} T_{i_1 \ldots i_{s-1} ij i_{s+1} \ldots i_{q}} R_{i_s jk}^\alpha - T_{i_1 \ldots i_{q}}^\alpha \cdot R_{\alpha jk}^h.$$

By the Ricci identity we have

$$T_{i_1 \ldots i_{q},[jk]}^h = \sum_{s=1}^{q} T_{i_1 \ldots i_{s-1} ij i_{s+1} \ldots i_{q}} R_{i_s jk}^\alpha - T_{i_1 \ldots i_{q}}^\alpha \cdot R_{\alpha jk}^h.$$

Now we present Kowolik’s theorems of [1] in a modified form which is more convenient for us. These theorems will be generalized in the next parts of our paper. First, recall notions used in the theorems.

**Definition 3** A Riemannian space $V_n$ is called *semisymmetric*, if

$$R(X,Y) \circ R = 0 \quad \forall X,Y \in TV_n. \quad (1)$$

We write (1) locally in the form $R_{ijkl,[lm]}^h = 0$ or

$$R_{ij}^h R_{klm}^\alpha + R_{ia}^h R_{jklm}^\alpha + R_{ij}^h R_{klm}^\alpha - R_{ijkl}^h R_{\alpha lm}^h = 0.$$

**Definition 4** A Riemannian space $V_n$ is called *Ricci semisymmetric*, if

$$R(X,Y) \circ Ric = 0 \quad \forall X,Y \in TV_n. \quad (2)$$

We write (2) locally

$$R_{ij}^\alpha R_{ijkl}^\alpha + R_{ija}^\alpha R_{jkl}^\alpha = 0 \quad \text{or} \quad R_{ij,[kl]}^\alpha = 0.$$

**Simply conformally recurrent spaces** (*s.c.r. spaces*) were defined by W. Roter [7]. These spaces are characterized by the following conditions:

The Riemannian space $V_n$ is a *s.c.r.* space, if and only if:

1. $C_{hiijk} \neq 0$, where $C_{hiijk}$ is a Weyl tensor of conformal curvature,
2. $C_{hiijk,l} = \varphi_l C_{hiijk}$,
3. a vector $\varphi_k$ is locally gradient,
4. the Ricci tensor is a Codazzi tensor.
Remark 1 It holds that each s.c.r. space is semisymmetric.

Theorem 1 ([1]) Let $V_n$ ($n \geq 4$) be a Ricci semisymmetric space with a harmonic Riemannian tensor. If there is a torse-forming vector field $\xi$ in $V_n$, then $\xi$ is either concircular or recurrent.

Theorem 2 ([1]) If there is a torse-forming vector field $\xi$ in a s.c.r. space $V_n$ ($n \neq 4$), then $\xi$ is recurrent.

Let $T$ be a tensor field of the type $(0,q)$ or $(1,q)$ and $\xi$ be a vector field on $V_n$. By means of the operator $R(X,\xi) \circ T$ let us define the basic notion of our paper:

Definition 5 The tensor field $T$ is semiconjugated with the vector field $\xi$, if

\[ R(X,\xi) \circ T = 0 \quad \text{for each } X \in TV_n. \]  

(3)

In the local transcription (3) has the form

\[ T_{\ldots;[m]}\xi^m = 0, \]  

(4)

where $\xi^m$ are local components of $\xi$.

2 Vector fields semiconjugated with torse-forming vector fields

In this section we will consider 1-covariant vector fields semiconjugated with a torse-forming vector field $\xi$. Denote by $\xi(X)$ a linear form generated by $\xi$, i.e. $\xi(X) \equiv g(X, \xi)$.

Theorem 3 Let $T$ ($\neq 0$) be a 1-covariant vector field semiconjugated with a non-isotropic torse-forming vector field $\xi$, which is not convergent. Then $\xi$ is semitorse-forming and $T$ is colinear with a form $\xi(X)$.

Proof Assume that there is a non-zero vector field $T$ and a non-isotropic non-convergent torse-forming vector field $\xi$, which satisfy (4), i.e.

\[ T_{\alpha}R_{i\beta j \gamma}^\alpha \xi^\beta = 0, \]  

(5)

where $T_i$ are local components of $T$ and $R_{ij\alpha}^h$ are components of the Riemannian tensor $R$. According to [5] we can assume that $\xi$ is normalized, i.e. $g(\xi, \xi) = e = \pm 1$, and the condition

\[ \xi_{\alpha}R_{i\beta j \gamma}^\alpha = g_{ij}c_k - g_{ik}c_j + \xi_\alpha a_{jk} \]  

(6)

holds, where $a_{jk} \equiv -e\xi_{\{j \rho, k\}}$ and

\[ c_k \equiv \mathcal{g}_{k} + eg^2\xi_k. \]  

(7)
Since $\xi$ is not convergent, we have $c_i \neq 0$.

Contracting (6) with $T^k \overset{\text{def}}{=} T_{\alpha} g^{\alpha k}$ and using (5) and properties of the Riemannian tensor we get

$$g_{ij} c_k T^k - T_i c_j + \xi_i a_{jk} T^k = 0. \tag{8}$$

If $c_k T^k \neq 0$, then (8) gives rank $\|g_{ij}\| \leq 2$. Since $n > 2$, we have $c_k T^k = 0$ and (8) leads to

$$-T_i c_j + \xi_i a_{jk} T^k = 0. \tag{9}$$

Since $c_j \neq 0$, the condition (9) implies

$$T_i = a \xi_i,$$

where $a$ if a non-zero function.

Substituting $T_i = a \xi_i$ in (6) we see, that either $\xi$ is semitorse-forming vector field or $T_i = 0$. This completes the proof of Theorem 3. \hfill \square

\section{Symmetric 2-covariant tensors semiconjugated with a torse-forming vector field}

We will prove the following theorem:

\textbf{Theorem 4} Let $n > 2$ and let $T \neq \gamma g$ be a 2-covariant symmetric tensor field semiconjugated with a non-isotropic torse-forming vector field $\xi$, which is not convergent. Then it holds that $\xi$ is semitorse-forming in $V_n$ and

$$T(X, Y) = \gamma \cdot g(X, Y) + \psi \cdot \xi(X) \cdot \xi(Y) \quad \forall X, Y \in TV_n, \tag{10}$$

where $\gamma, \psi$ are functions on $V_n$.

\textbf{Proof} Assume that there is a 2-covariant symmetric tensor field $T$ on $V_n$, which is semiconjugated with a normalised torse-forming vector field $\xi$, which is not convergent. It means that $\xi$ satisfies (6) and $c_i \neq 0$.

Further we have:

$$R(X, \xi) \circ T = 0 \quad \forall X \in TV_n,$$

i.e. locally

$$T_{\alpha j} R_{i l}^{\alpha} \xi^l + T_{i \alpha} R_{j l}^{\alpha} \xi^l = 0. \tag{11}$$

If we substitute (6) in (11) and use properties of the Riemannian tensor we get after computation

$$g_{il} T_{\alpha j} c^\alpha - T_{ij} c_i + g_{ij} T_{i \alpha} c^\alpha - T_{il} c_j + \xi_i \omega_{ij} = 0, \tag{12}$$

where $\omega$ is some tensor of the type $(0, 2)$ and $c^i \equiv c_\alpha g^{\alpha i}$.

We will prove that

$$T_{i \alpha} c^\alpha = \gamma c_i. \tag{13}$$
Assume, that (13) does not hold. Then there exists a vector $\varepsilon^i$ such that
\[ c_\alpha \varepsilon^\alpha = 0 \quad \text{and} \quad T_{\alpha\beta} \varepsilon^\alpha \varepsilon^\beta = 1. \quad (14) \]
Contract (12) with $\varepsilon^i \varepsilon^j$. Since $T_{ij} = T_{ji}$ and (14) holds, we get
\[ \varepsilon_l = h \xi_l, \quad (15) \]
where $h \overset{\text{def}}{=} -\frac{1}{2} \omega_{\alpha\beta} \varepsilon^\alpha \varepsilon^\beta$.

If we contract (12) with $\varepsilon^j$, we obtain by means of (14) and (15)
\[ g_{li} - T_{l\alpha} \varepsilon^\alpha c_i + \xi_l (h T_{i\alpha} c^\alpha + \omega_{i\beta} \varepsilon^\beta) = 0. \]
This implies that rank $\|g_{ij}\| \leq 2$, which contradicts the assumption that (13) does not hold.

By (13) we extract the member $T_{\alpha i} c^\alpha$ in (12). After computation we obtain
\[ F_{ij} c_i + F_{il} c_j + \xi_l \omega_{ij} = 0, \quad (16) \]
where
\[ F_{ij} \overset{\text{def}}{=} T_{ij} - \gamma g_{ij}. \quad (17) \]
Since $c_i \neq 0$, then there exists $\varphi^i$ such, that $c_\alpha \varphi^\alpha = 1$.
Contracting (16) with $\varphi^i \varphi^j$ we get $F_{l\alpha} \varphi^\alpha = f \cdot \xi_l$, where $f \overset{\text{def}}{=} -\frac{1}{2} \omega_{\alpha\beta} \varepsilon^\alpha \varepsilon^\beta$.
Similarly, if we contract (16) with $\varphi^j$, we get
\[ F_{il} = \xi_l \chi_i, \quad (18) \]
where $\chi_i \overset{\text{def}}{=} -f c_i - \omega_{i\alpha} \varphi^\alpha$.
Since $F_{ij}$ is a symmetric tensor, the equality (18) implies
\[ F_{ij} = \psi \cdot \xi_i \xi_j. \quad (19) \]
By the assumption $F_{ij} \neq 0$, we have $\psi \neq 0$. Substituting (17) to (19) we see, that (10) is true. It remains to prove that the vector field $\xi$ is semitorse-forming.

Therefore we covariantly derive the equality (19) by indices $l$ and $m$, then we alternate it with respect to $l$ and $m$ and finally we contract it with $\xi^m$. Since
\[ F_{ij,[lm]} \xi^m = 0 \quad \text{and} \quad \psi \neq 0, \]
we reach the formula
\[ \xi_{i,[lm]} \xi^m \cdot \xi_j + \xi_i \cdot \xi_{j,[lm]} \xi^m = 0, \]
wherefrom it follows
\[ \xi_{i,[lm]} \xi^m = 0. \]
This means that the vector field $\xi$ is semitorse-forming. \qed
4 Antisymmetric 2-covariant tensors semiconjugated with a torse-forming vector field

The following theorem deals with antisymmetric tensor fields.

**Theorem 5** In a Riemannian space $V_n$ ($n > 3$) there is no non-zero 2-covariant antisymmetric tensor field $T$ semiconjugated with a non-isotropic torse-forming vector field $\xi$, which is not convergent.

**Proof** Assume that there is a 2-covariant anti-symmetric tensor field $T$ on $V_n$, which is semiconjugated with a non-isotropic torse-forming vector field $\xi$, which is not convergent. It means, that $\xi$ satisfies (6) and $c_i \neq 0$. Similarly as in the proof of Theorem 4 we get, that (11), (12) and (13) are true. Substituting (13) in (12) and using the antisymmetric property of $T$ (i.e. $T_{ij} = -T_{ji}$), we get after computation

$$(T_{ii} - \mu g_{ii})c_j - (T_{ij} - \mu g_{ij})c_i - \xi_i \omega_{ij} = 0. \quad (20)$$

Since $c_j \neq 0$, then there exists $\varphi^i$, for which $\varphi^\alpha c_\alpha = 1$. Contracting (20) with $\varphi^j$ we find

$$T_{ii} - \mu g_{ii} = \xi_i \eta_i + \chi_i c_i, \quad (21)$$

where $\eta_i$ and $\chi_i$ are some covectors.

Symmetrising (21) we obtain

$$-2\mu g_{ii} = \xi_i \eta_i + \xi_i \eta_i + \chi_i c_i + \chi_i c_i. \quad (22)$$

If $n > 4$, we deduce that $\mu = 0$.

Assume that $n = 4$ and $\mu \neq 0$. Then covectors $\xi_i$, $c_i$, $\eta_i$, $\chi_i$ must be linearly independent. Hence their coordinates in a given point $x$ can be chosen in the following way:

$$\xi_i = \delta^1_i, \quad \eta_i = \delta^2_i, \quad c_i = \delta^3_i, \quad \chi_i = \delta^4_i.$$ 

Then

$$g_{ij} = \frac{1}{2\mu} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$ 

The inverse matrix $g^{ij}$ has the form

$$g^{ij} = -2\mu \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$ 

We can check that

$$g^{ij} \xi_i \xi_j = 0$$

holds, i.e. $\xi$ is isotropic, a contradiction.
Thus for $n > 3$ the formula (22) implies, that $\mu = 0$. Therefore we can simplify (21) and (22) as follows:

$$T_{ij} = \xi_i \eta_j + \chi_i c_j$$

and

$$\xi_l \eta_i + \xi_i \eta_l + \chi_i c_i + \chi_l c_i = 0.$$  \hspace{1cm} (23)

Vectors $\xi_i$ and $\chi_i$ are not colinear. Otherwise it should be $T_{ij} = 0$. Therefore there is $\varphi^i$ such that

$$\xi_\alpha \varphi^\alpha = 1 \quad \text{and} \quad \chi_\alpha \varphi^\alpha = 0.$$  

Contracting (23) with $\varphi^i \varphi^l$ we find $\eta_\alpha \varphi^\alpha = 0$ and contracting (23) with $\varphi^l$ we get $\eta_i = -c_\alpha \varphi^\alpha \cdot \chi_i$. Then (23) has a form

$$(c_i - c_\alpha \varphi^\alpha \xi_i) \chi_l + (c_l - c_\alpha \varphi^\alpha \xi_l) \chi_i = 0.$$  

Since $\chi_l \neq 0$, we obtain

$$c_i = c_\alpha \varphi^\alpha \xi_i.$$  \hspace{1cm} (24)

Using (7) and (24) we derive

$$\varphi_{,k} = (c_\alpha \varphi^\alpha - e \varphi^2) \xi_k.$$  

Hence we have $\varphi = \varphi(\xi)$, where $\xi$ is a scalar field satisfying $\xi_k = \partial_k \xi$. It means that $\xi$ is concircular and, by [3], is convergent.

\section{Main results}

By means of Theorem 4 (for symmetric tensors) and Theorem 5 (for antisymmetric tensors) we will prove the following assertion for arbitrary 2-covariant tensors.

\textbf{Theorem 6} Let $n > 3$ and let $T$ (≠ $\gamma g$) be a 2-covariant tensor field semiconjugated with a non-isotropic torse-forming vector field $\xi$, which is not convergent. Then it holds that $\xi$ is semitorse-forming in $V_n$ and

$$T(X, Y) = \gamma \cdot g(X, Y) + \psi \cdot \xi(X) \cdot \xi(Y) \quad \forall X, Y \in TV_n,$$

where $\gamma, \psi$ are functions on $V_n$.

\textbf{Proof} Assume that there is a 2-covariant tensor field $T$ on $V_n$, which is semiconjugated with a normalised torse-forming vector field $\xi$, which is not convergent.

Tensor $T$ can be uniquely expressed in the form $T = U + V$, where $U$ is a symmetric part and $V$ is an antisymmetric part of $T$. It holds

$$U(X, Y) = \frac{1}{2}(T(X, Y) + T(Y, X))$$
and
\[ V(X, Y) = \frac{1}{2} (T(X, Y) - T(Y, X)) \]
for any vector fields \( X, Y \in TV_n \). Therefore \( U \) and \( V \) are also semiconjugated with \( \xi \). Theorem 5 implies, that \( V = 0 \). Hence \( T \equiv U \) and so \( T \) is symmetric and the assertion of Theorem 6 follows from Theorem 4. \( \square \)

Now we will prove theorems for Riemannian spaces having Riemannian and Ricci tensors semiconjugated with a torse-forming vector field. These theorems generalize Kowolik’s results in [1].

**Theorem 7** Let \( n > 2 \) and let \( V_n \) be a non-Einsteinian Riemannian space, where the Ricci tensor is semiconjugated with a non-isotropic torse-forming vector field \( \xi \). Then \( \xi \) is convergent.

**Proof** Assume that the Ricci tensor \( Ric \) is semiconjugated with a torse-forming vector field \( \xi \).

Since \( Ric \) is a symmetric tensor, we get by Theorem 4
\[
Ric(X, Y) = \gamma g(X, Y) + \psi \cdot \xi(X) \cdot \xi(Y) \quad \forall X, Y \in TV_n,
\]
(25)
where \( \xi(X) \equiv g(X, \xi) \) and \( \psi \) is a function on \( V_n \).

Semitorse-forming fields fulfil \( R^h_{\alpha j \beta} \xi^\alpha \xi^\beta = 0 \). Contracting it with respect to \( h \) and \( j \) we obtain \( R_{\alpha \beta} \xi^\alpha \xi^\beta = 0 \), which can be written in the form
\[ Ric(\xi, \xi) = 0. \]

Let us put \( X = \xi \) a \( Y = \xi \) in (25). Since we can assume that \( \xi \) is normalized, i.e. \( g(\xi, \xi) \equiv \xi(\xi) = e = \pm 1 \), we get \( \psi = -e \gamma \) and so the formula (25) has the form
\[
Ric(X, Y) = \gamma \cdot \left( g(X, Y) - e \xi(X) \cdot \xi(Y) \right) \quad \forall X, Y \in TV_n.
\]
(26)
Substituting \( Y = \xi \) in (26) we obtain
\[ Ric(X, \xi) = 0 \quad \forall X \in TV_n. \]
It means that \( \xi \) is an eigenvector of the Ricci tensor corresponding to the zero eigenvalue. Therefore \( \xi \) is convergent. \( \square \)

**Theorem 8** Let \( n > 2 \) and let \( V_n \) be a Riemannian space with a non-constant curvature, where the Riemannian tensor is semiconjugated with a non-isotropic torse-forming vector field \( \xi \). Then \( \xi \) is convergent.

**Proof** Assume that a Riemannian space \( V_n \) with a non-constant curvature has the Riemannian tensor which is semiconjugated with a torse-forming vector field \( \xi \) which is not convergent. Then \( V_n \) has the Ricci tensor which is also semiconjugated with \( \xi \). Therefore by Theorem 7 the space \( V_n \) has to be an Einsteinian space. We can easily see that \( \xi \) is concircular.
Then, according to the result of [4] the Riemannian tensor has the form
\[ R_{hijk} = K(g_{hj}g_{ik} - g_{hk}g_{ij}), \]
which means that \( V_n \) has a constant curvature, a contradiction. We have proved that \( \xi \) has to be convergent. \( \square \)

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