Raad J. K. al Lami

Metric of special 2F-flat Riemannian spaces

Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 44 (2005), No. 1, 7--11

Persistent URL: http://dml.cz/dmlcz/133378

Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **44** (2005) 7–11

Metric of Special 2F-flat Riemannian Spaces

Raad J. AL LAMY

Dept. of Basic Science, Fac. of Sci. & IT., Al Balqa' Applied Univ., Jordan
e-mail: raad@bau.edu.jo

(Received May 2, 2005)

Abstract

In this paper we find the metric in an explicit shape of special $2F$-flat Riemannian spaces V_n, i.e. spaces, which are $2F$-planar mapped on flat spaces. In this case it is supposed, that F is the cubic structure: $F^3 = I$.

Key words: $2F$-flat (pseudo-)Riemannian spaces, $2F$-planar mapping, cubic structure.

2000 Mathematics Subject Classification: 53B20, 53B30, 53B35

1 Introduction

$2F$- and pF-planar mappings are studied in these papers [4, 5, 17]. The mentioned mappings are the generalization of geodesic, holomorphically projective and F-planar mappings [1, 2, 6, 7, 8, 9, 10, 11, 14, 15, 16, 18].

As it is known, the Riemannian space with the constant curvature, resp. the Kählerian space with the constant holomorphically projective curvature, admits a geodesic, resp. holomorphically projective, mapping onto a flat space, i.e. the space with a vanishing curvature tensor.

The consideration in the present paper is perfomed in the tensor form, locally, in a class of substantial real smooth functions. The dimension n of the spaces under consideration, as a rule, is greater then 3. All the spaces are supposed to be connected.

We consider a (pseudo-) Riemannian space V_n with a metric tensor g and an affinor structure F, i.e. a tensor field of type $\binom{1}{1}$. We supposed, that F is the cubic affinor structure, for which it holds

$$F^3 = I.$$

In our paper we find the metric in an explicit shape of special $2F$-flat Riemannian spaces V_n, i.e. spaces, which are $2F$-planar mapped on flat spaces.

It was proved, that the Riemannian tensor of these spaces has the following form [4]:

$$R^h_{ijk} = \sum_{\sigma=0}^{2}(\overset{\sigma}{F}{}^h_i \overset{\sigma}{S}_{jk} + \overset{\sigma}{F}{}^h_j \overset{\sigma}{T}_{ik} - \overset{\sigma}{F}{}^h_k \overset{\sigma}{T}_{ij}),$$

where $\overset{\sigma}{S}_{jk}$ and $\overset{\sigma}{T}_{ik}$ are tensors. Here and after

$$\overset{0}{F}{}^h_i = \delta^h_i, \quad \overset{1}{F}{}^h_i = F^h_i, \quad \overset{2}{F}{}^h_i = F^h_\alpha F^\alpha_i,$$

where δ^h_i is the Kronecker symbol, R^h_{ijk} and F^h_i are components of the Riemannian tensor and the structure F, respectively.

Among other things it is known, that $2F$-flat Riemannian spaces V_n are symmetric, i.e. their Riemannian tensor is covariantly constant.

2 On special $2F$-flat Rimannian space

As it was mentioned, the aim of our interest was to find the metric tensor of the $2F$-flat Riemannian spaces V_n. This problem is considerably extensive, therefore we narrow it by following assumptions.

In the following we study the $2F$-flat Riemannian spaces V_n, for which the Riemannian tensor has the form:

$$R^h_{ijk} = B\,(G^h_k G_{ij} - G^h_j G_{ik}), \tag{1}$$

where

$$G^h_k = \delta^h_i + F^h_i + F^h_\alpha F^\alpha_i, \qquad G_{ij} = g_{i\alpha} G^\alpha_j, \qquad B - \text{const.}$$

There g_{ij} are components of the metric g and F^h_i are components of the structure F, which satisfies the conditions:

$$F^3 = I, \qquad \operatorname{tr} F = \operatorname{tr} F^2 = 0, \tag{2}$$

as well the following characteristic is joined with the metric tensor:

$$\overset{1}{F}_{ij} = \overset{1}{F}_{ji} \quad \text{and} \quad \overset{2}{F}_{ij} = \overset{2}{F}_{ji}, \tag{3}$$

where $\overset{1}{F}_{ij} = g_{i\alpha} F^\alpha_j$ and $\overset{2}{F}_{ij} = g_{i\alpha} \overset{2}{F}{}^\alpha_j$.

It is clear, that V_n with this Riemannian tensor is symmetric. Therefore we use for the computation procedure of the metric tensor the formula by P. A. Shirokov [14], in accordance with this formula the metric tensor of the symmetric space in some point $M(x_0) \in V_n$ is calculate by sequences:

$$g_{ij}(y) = \underset{\circ}{g}_{ij} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^k \, 2^k}{(2k+2)!} \overset{(k)}{m}_{ij}, \tag{4}$$

where

$$\overset{(1)}{m}_{ij} = m_{ij}, \qquad \overset{(k+1)}{m}_{ij} = \overset{(k)}{m}_{i\alpha} m_{j\beta} \, \underset{\circ}{g}^{\alpha\beta}, \qquad m_{ij} = \underset{\circ}{R}_{i\alpha\beta j} y^{\alpha} y^{\beta}, \tag{5}$$

$\underset{\circ}{g}_{ij}$, $\underset{\circ}{g}^{ij}$, $\underset{\circ}{R}_{i\alpha\beta j}$ are values of components of the metric, inverse and Riemannian tensors in a point x_0, $y \equiv (y^1, y^2, \ldots, y^n)$ are Riemannian coordinates in the point x_0.

3 The computation procedure of the metric of the 2F-flat space

We substitute (1) to (5) in some point $M(x_0)$ and obtain:

$$m_{ij} = \overset{(1)}{m}_{ij} = B\,(y_{ij} + \overset{1}{y}_{ij} + \overset{2}{y}_{ij}),$$

where

$$y_{ij} = y_i y_j + \overset{1}{y}_i \overset{2}{y}_j + \overset{2}{y}_i \overset{1}{y}_j - y\, \underset{\circ}{g}_{ij} - \overset{2}{y}\, \underset{\circ}{\overset{1}{F}}_{ij} - \overset{1}{y}\, \underset{\circ}{\overset{2}{F}}_{ij},$$

$$\overset{1}{y}_{ij} = y_{\alpha j}\, \underset{\circ}{F}^{\alpha}_{i}, \qquad \overset{2}{y}_{ij} = \overset{1}{y}_{\alpha j}\, \underset{\circ}{F}^{\alpha}_{i},$$

$$y_i = \underset{\circ}{g}_{i\alpha} y^{\alpha}, \quad \overset{1}{y}_i = y_{\alpha}\, \underset{\circ}{F}^{\alpha}_{i}, \quad \overset{2}{y}_j = \overset{1}{y}_{\alpha}\, \underset{\circ}{F}^{\alpha}_{i},$$

$$y = \underset{\circ}{g}_{\alpha\beta} y^{\alpha} y^{\beta}, \quad \overset{1}{y} = \underset{\circ}{\overset{1}{F}}_{\alpha\beta} y^{\alpha} y^{\beta}, \quad \overset{2}{y} = \underset{\circ}{\overset{2}{F}}_{\alpha\beta} y^{\alpha} y^{\beta},$$

and $\underset{\circ}{F}^{h}_{i}$, $\underset{\circ}{\overset{1}{F}}{}^{h}_{i}$, $\underset{\circ}{\overset{2}{F}}{}^{h}_{i}$ are components of the corresponding tensors in the point x_0.

We notice, that

$$y_{ij} = y_{ji}, \quad \overset{1}{y}_{ij} = \overset{1}{y}_{ji}, \quad \overset{2}{y}_{ij} = \overset{2}{y}_{ji},$$

$$y_{i\alpha}\, \underset{\circ}{g}^{\alpha\beta} y_{\beta j} = -y\, y_{ij} - \overset{1}{y}\, \overset{2}{y}_{ij} - \overset{2}{y}\, \overset{1}{y}_{ij}.$$

Therefore

$$\overset{(2)}{m}_{ij} = -3B^2 (y + \overset{1}{y} + \overset{2}{y})(y_{ij} + \overset{1}{y}_{ij} + \overset{2}{y}_{ij}) = A\, \overset{(1)}{m}_{ij} = A\, m_{ij},$$

where

$$A = -3B\,(y + \overset{1}{y} + \overset{2}{y}).$$

By analogy we obtain

$$\overset{(3)}{m}_{ij} = A\,\overset{(2)}{m}_{ij} = A^2\, m_{ij}, \quad \cdots, \quad \overset{(k)}{m}_{ij} = A^{k-1}\, m_{ij}.$$

Then we substitute this one to (4) and we obtain

$$g_{ij}(y) = \underset{\circ}{g}_{ij} + \frac{1}{2}\, m_{ij} \sum_{k=1}^{\infty} \frac{(-1)^k 2^k A^{k-1}}{(2k+2)!}.$$

We make sure of the convergency of the sequences for an arbitrary value of coordinates y^h.

These sequences can be introduced in the following form

$$g_{ij}(y) = \underset{\circ}{g}_{ij} + \frac{1}{4A^2}\, m_{ij} \left(1 - A - \sum_{k=0}^{\infty} \frac{(-2A)^k}{(2k)!}\right),$$

which is easy to express such as

$$g_{ij}(y) = \underset{\circ}{g}_{ij} + \frac{1}{4A^2}\, m_{ij} \left(1 - A - \left\{ \begin{matrix} \cos\sqrt{2A}, & A > 0, \\ \mathrm{ch}\sqrt{2|A|}, & A < 0, \end{matrix} \right\}\right). \qquad (6)$$

We can easily see that

$$\lim_{y \to 0} g_{ij}(y) = \underset{\circ}{g}_{ij}$$

and above functions $g_{ij}(y)$ are analytical onto domain.

Theorem 1 *Let V_n be a 2F-flat Riemannian space and y its Riemannian coordinates. Suppose that the conditions (1), (2) and (3) hold. Then the metric V_n is expressed by the formula (6).*

References

[1] Beklemishev, D. V.: *Differential geometry of spaces with almost complex structure.* Geometria. Itogi Nauki i Tekhn., All-Union Inst. for Sci. and Techn. Information (VINITI), Akad. Nauk SSSR, Moscow, 1965, 165–212.

[2] Eisenhart, L. P.: Riemannian Geometry. *Princenton Univ. Press*, 1926.

[3] Kurbatova, I. N.: *HP-mappings of H-spaces.* Ukr. Geom. Sb., Kharkov, **27** (1984), 75–82.

[4] Al Lamy, R. J.: *About 2F-plane mappings of affine connection spaces.* Coll. on Diff. Geom., Eger (Hungary), 1989, 20–25.

[5] Al Lamy, R. J., Kurbatova, I. N.: *Invariant geometric objects of 2F-planar mappings of affine connection spaces and Riemannian spaces with affine structure of III order.* Dep. of UkrNIINTI (Kiev), 1990, No. 1004Uk90, 51p.

[6] Al Lamy, R. J., Mikeš, J., Škodová, M.: *On linearly pF-planar mappings.* Diff. Geom. and its Appl. Proc. Conf. Prague, 2004, Charles Univ., Prague (Czech Rep.), 2005, 347–353.

[7] J. Mikeš, *On special F-planar mappings of affine-connected spaces.* Vestn. Mosk. Univ., 1994, 3, 18–24.

[8] Mikeš, J.: *Geodesic mappings of affine-connected and Riemannian spaces.* J. Math. Sci., New York, **78**, 3 (1996), 311–333.

[9] Mikeš, J.: *Holomorphically projective mappings and their generalizations.* J. Math. Sci., New York, **89**, 3 (1998), 1334–1353.

[10] Mikeš, J., Pokorná, O.: *On holomorphically projective mappings onto Kählerian spaces.* Suppl. Rend. Circ. Mat. Palermo, II. Ser. 69, (2002), 181–186.

[11] Mikeš, J., Sinyukov, N. S.: *On quasiplanar mappings of spaces of affine connection.* Sov. Math. **27**, 1 (1983), 63–70; translation from Izv. Vyssh. Uchebn. Zaved., Mat., **248**, 1 (1983), 55–61.

[12] Petrov, A. Z.: New Method in General Relativity Theory. *Nauka, Moscow*, 1966.

[13] Petrov, A. Z.: *Simulation of physical fields.* In: Gravitation and the Theory of Relativity, Vol. 4–5, Kazan' State Univ., Kazan, 1968, 7–21.

[14] Shirokov, P. A.: Selected Work in Geometry. *Kazan State Univ. Press, Kazan*, 1966.

[15] Sinyukov, N. S.: Geodesic Mappings of Riemannian Spaces. *Nauka, Moscow*, 1979.

[16] Sinyukov, N. S.: *Almost geodesic mappings of affinely connected and Riemannian spaces.* J. Sov. Math. **25** (1984), 1235–1249.

[17] Škodová, M., Mikeš, J., Pokorná, O.: *On holomorphically projective mappings from equiaffine symmetric and recurrent spaces onto Kählerian spaces.* Circ. Mat. di Palermo, Suppl. Rend. Circ. Mat. Palermo, II. Ser. 75, (2005), 309–316.

[18] Yano, K.: Differential Geometry on Complex and Almost Complex Spaces. *Pergamon Press, Oxford–London–New York–Paris–Frankfurt*, XII, 1965, 323p.