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On Weakly and Pseudo Concircular Symmetric Structures on a Riemannian Manifold

FÜSUN ÖZEN ZENGİN¹, SEZGIN ALTAY DEMIRBAĞ²

Istanbul Technical University, Faculty of Sciences and Letters
Department of Mathematics, Maslak-Istanbul, Turkey
¹ e-mail: fozen@itu.edu.tr
² e-mail: saltay@itu.edu.tr

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Abstract

In this paper, we examine the properties of hypersurfaces of weakly and pseudo concircular symmetric manifolds and we give an example for these manifolds.

Key words: Weakly symmetric manifold, pseudo symmetric manifold, weakly and pseudo symmetric concircular manifold, totally umbilical, totally geodesic, mean curvature, scalar curvature.

2000 Mathematics Subject Classification: 53B20, 53B15

1 Introduction

Firstly, Tamassy and Binh introduced weakly symmetric manifolds, [1].

A non-flat Riemannian manifold \((M_n, g)\), \((n > 2)\) whose the curvature tensor satisfies the following relation is called weakly symmetric

\[
\nabla_l R_{hijk} = A_l R_{hijk} + B_h R_{lijk} + D_i R_{hljk} + E_j R_{hilk} + F_k R_{hijl} \tag{1.1}
\]

where \(A, B, D, E, F\) are non-zero 1-forms and \(\nabla\) denotes the covariant differentiation with respect to the metric tensor of the manifold. These 1-forms are called the associated 1-forms of the manifold and an \(n\)-dimensional manifold of this kind is denoted by \((WS)_n\). It may be mentioned in this connection that
although the definition of a \((WS)_n\) is similar to that of a generalized pseudo-symmetric space studied by Chaki and Mondal, [2], the defining condition of a \((WS)_n\) is weaker than that of a generalized pseudo-symmetric manifold. De and Bandyopadhyay, [3], proved that 1-forms of \((WS)_n\) cannot be all different. Then the equation (1.1) reduces to the form
\[
\nabla_l R_{hijk} = A_l R_{hijk} + B_h R_{lijk} + B_i R_{hljk} + D_j R_{hilk} + D_k R_{hijl} \quad (1.2)
\]
Let us consider a subspace \(V_m\) immersed in a Riemannian manifold \(V_n\) whose parametric representation is \(u^\lambda = u^\lambda(u^1, u^2, \ldots, u^m)\) where \((u^\lambda)\) and \((u^i)\) \((i, j, k, \ldots = 1, 2, \ldots, m)\) denote the coordinate systems of \(V_n\) and \(V_m\), respectively. A conformal transformation \(\bar{g}_{ij} = \rho^2 g_{ij}\) of the fundamental tensor of \(V_n\), being a concircular one with the function \(\rho\) satisfying the equations
\[
\rho_{ij} = \nabla_j \rho_i - \rho_i \rho_j + \frac{1}{2} g^{\alpha\beta} \rho_\alpha \rho_\beta g_{ij} = \phi g_{ij}, \quad \rho_j = \frac{\partial}{\partial u^i} \ln \rho \quad (1.3)
\]
this transformation is called concircular transformation where \(\phi\) is a function of \(u^i\).

The present paper deals with non-concircular flat Riemannian manifold \((M_n, g)\) whose concircular curvature tensor \(Z_{hijk}\) satisfies the condition \((n > 2)\)
\[
\nabla_l Z_{hijk} = A_l Z_{hijk} + B_h Z_{lijk} + D_i Z_{hljk} + E_j Z_{hilk} + F_k Z_{hijl} \quad (1.4)
\]
where
\[
Z_{hijk} = R_{hijk} - \frac{R}{n(n-1)} (g_{hk} g_{ij} - g_{hj} g_{ik})
\]
\(R_{hijk}\) is the curvature tensor and \(R\) is the scalar curvature. Such a manifold will be called a weakly concircular symmetric manifold and denoted by \((WZS)_n\), [4]. It was shown that, in [5], \(Z_{hijk}^b\) is invariant under a concircular transformation.

Desa and Amur studied the concircular recurrent Riemannian manifold, [6]. The authors proved that the defining condition of a \((WZS)_n\) can always be expressed in the following form, [4]
\[
\nabla_l Z_{hijk} = A_l Z_{hijk} + B_h Z_{lijk} + B_i Z_{hljk} + D_j Z_{hilk} + D_k Z_{hijl} \quad (1.4)
\]
where \(A, B, D\) 1-forms (non-zero simultaneously).

From the first Bianchi identity, we get
\[
R_{hijk} + R_{hjki} + R_{hki} = 0 \quad (1.5)
\]

The second Bianchi identity for a Riemannian manifold is
\[

abla_s R_{hijk} + \nabla_j R_{hiks} + \nabla_k R_{hisj} = 0 \quad (1.6)
\]

Let \((\bar{M}, \bar{g})\) be an \((n + 1)\)-dimensional Riemannian manifold covered by a system of coordinate neighborhoods \(\{U, y^\alpha\}\). Let \((M, g)\) be a hypersurface of \((\bar{M}, \bar{g})\) defined via a system of parametric equation \(y^\alpha = y^\alpha(x^i)\), where Greek
indices take the values 1, 2, . . . , n + 1 and Latin indices take the values 1, 2, . . . , n
a locally coordinate system. Then, we have

\[ g_{ij} = \bar{g}_{\alpha\beta} y_i^\alpha y_j^\beta \] (1.7)

Let \( n^\alpha \) be a local unit normal to \((M, g)\). Thus, we obtain \( \bar{g}_{\alpha\beta} n^\alpha y_i^\beta = 0, \)
\( g_{\alpha\beta} n^\alpha n^\beta = 1 \) and it is easily seen that there are the following conditions between
the contrary metric tensors of the hypersurface \((M, g)\) and \((\bar{M}, \bar{g})\)

\[ g^{\alpha\beta} = g^{ij} y_i^\alpha y_j^\beta + n^\alpha n^\beta, \quad y_i^\alpha = \partial y^\alpha / \partial x^i, \quad (i, j = 1, 2, \ldots, n; \alpha = \beta = 1, 2, \ldots, n + 1) \] (1.8)

A point of a hypersurface, at which the principal directions of the curvature
are indeterminate, is called an umbilical point. In order that the lines of cur-
vature may be indeterminate at every point of the hypersurface, it is necessary
and sufficient that \( \Omega_{ij} = \omega g_{ij} \), where \( \omega \) is an invariant. According to [7],

\[ M = \Omega_{ij} g^{ij} = n\omega \] (1.9)

where the scalar \( M \) is called the mean curvature of such a hypersurface, so that
the conditions for indeterminate lines of curvature are expressible as

\[ \Omega_{ij} = \frac{M}{n} g_{ij} \] (1.10)

If all the geodesics of a hypersurface \((M, g)\) are also geodesics of \((\bar{M}, \bar{g})\), the
former is called a totally geodesic hypersurface of the latter. Such hypersurfaces
are generalizations of planes in ordinary space. A necessary and sufficient condi-
tion that \((M, g)\) be a totally geodesic hypersurface is that the normal curvature
should vanish for all directions in \((M, g)\), and at every point. This requires

\[ \Omega_{ij} = 0 \] (1.11)

Consequently,

\[ M = 0 \] (1.12)

and (1.10) is satisfied.

The structure equations of Gauss and Mainardi–Codazzi, [8]

\[ R_{ijkl} = \bar{R}_{\alpha\beta\gamma\theta} B_{ijkl}^{\alpha\beta\gamma\theta} + \Omega_{ijkl} \]

and

\[ \nabla_k \Omega_{ij} - \nabla_j \Omega_{ik} + \bar{R}_{\beta\gamma\delta\theta} n^\beta B_{ijkl}^{\gamma\delta\theta} = 0 \]

where \( \Omega_{ijkl} = \Omega_{ij} \Omega_{ik} - \Omega_{i} \Omega_{jk} \).

From (1.9), the above equations reduce to the following forms

\[ R_{ijkl} = \bar{R}_{\alpha\beta\gamma\theta} B_{ijkl}^{\alpha\beta\gamma\theta} + \frac{M^2}{n^2} (g_{ij} g_{ik} - g_{ij} g_{jk}) \] (1.13)
and
\[ \tilde{R}_{\alpha\gamma\delta\theta} n^\alpha B^\gamma^\delta^\theta_{ijk} = \frac{1}{n} (g_{ik} \nabla_j M - g_{ij} \nabla_k M) \] (1.14)
respectively, where \( R_{ijkl} \) and \( \tilde{R}_{\alpha\beta\gamma\theta} \) are the curvature tensors \((M, g)\) and \((\tilde{M}, \tilde{g})\), and
\[ B^\alpha^\beta^\gamma^\theta_{ijkl} = B^\alpha^\beta^\gamma^\theta_B^\gamma_{ijk} B^\alpha_t \text{ for } B_t = y_t^2 \text{.} \]

From the Gauss equation, we get
\[ \tilde{R} = R + 2 \tilde{R}_{\alpha\beta\gamma\theta} n^\alpha n^\beta - \Omega_{ijkl} g^{il} g^{jk} \] (1.15)

The concircular curvature tensors of \((M, g)\) and \((\tilde{M}, \tilde{g})\) can be written in the form
\[ Z_{hijk} = R_{hijk} + \frac{R}{n(n - 1)} G_{hijk} \] (1.16)
and
\[ \bar{Z}_{\alpha\beta\gamma\theta} = \tilde{R}_{\alpha\beta\gamma\theta} + \frac{\tilde{R}}{n(n + 1)} G_{\alpha\beta\gamma\theta} \] (1.17)

where
\[ G_{hijk} = g_{hj} g_{ik} - g_{hk} g_{ij} \text{ and } G_{\alpha\beta\gamma\theta} = \bar{g}_{\alpha\gamma} \bar{g}_{\beta\theta} - \bar{g}_{\alpha\theta} \bar{g}_{\beta\gamma} \text{.} \]

On account of (1.7), (1.13), (1.16) and (1.17), we get
\[ Z_{hijk} = \bar{Z}_{\alpha\beta\gamma\theta} B^\alpha^\beta^\gamma^\theta_{hijk} + \frac{M^2}{n^2} G_{hijk} + \frac{1}{n} \left( \frac{R}{n - 1} - \frac{\tilde{R}}{n + 1} \right) G_{hijk} \] (1.18)

2 Totally umbilical hypersurface of a weakly concircular symmetric manifold

Now, we consider an \((n+1)\)-dimensional weakly concircular symmetric Riemannian manifold and we denote this manifold by \((WZS)_{n+1}\). For a \((WZS)_{n+1}\), we have
\[ \nabla_e \bar{Z}_{abcd} = A_e \bar{Z}_{abcd} + B_a \bar{Z}_{ebcd} + B_b \bar{Z}_{acecd} + D_c \bar{Z}_{abed} + D_d \bar{Z}_{abce} \] (2.1)

Using (1.17), we obtain
\[ \bar{Z}_{abcd} n^a B^b_{ijk} = \tilde{R}_{abcd} n^a B^b_{ij} \] (2.2)

We assume that the scalar curvature of \((WZS)_n\) is not constant and \((WZS)_n\) is a totally umbilical hypersurface. In this case, we find that
\[ \nabla_s Z_{hijk} = A_s \bar{Z}_{abcd} B^a_{hij} + B_h \bar{Z}_{ebcd} B^e_{sijk} + B_i \bar{Z}_{acecd} B_{hsij} + D_j \bar{Z}_{abed} B_{hisk} + D_k \bar{Z}_{aceb} B_{hij} + \frac{1}{n^2} G_{hijk} \nabla_s M^2 \]
\[ + \frac{1}{n} G_{hijk} \nabla_s \left( \frac{R}{n - 1} - \frac{\tilde{R}}{n + 1} \right) + \frac{M}{n} \left( g_{hs} \tilde{R}_{abcd} B^b_{jsk} n^a + g_{is} \tilde{R}_{bade} B^a_{hij} n^b + g_{js} \tilde{R}_{cdab} B^a_{khi} n^c + g_{ks} \tilde{R}_{dcba} B^c_{jih} n^d \right) \] (2.3)
By the aid of the Gauss equation, (2.3) can be written as

\[ \nabla_s Z_{hijk} = A_s \left( Z_{hijk} - \frac{M^2}{n^2} G_{hijk} - \frac{1}{n} \left( \frac{R}{n-1} - \frac{\bar{R}}{n+1} \right) G_{hijk} \right) \]

\[ + B_h \left( Z_{sijk} - \frac{M^2}{n^2} G_{sijk} - \frac{1}{n} \left( \frac{R}{n-1} - \frac{\bar{R}}{n+1} \right) G_{sijk} \right) \]

\[ + B_i \left( Z_{hsijk} - \frac{M^2}{n^2} G_{hsijk} - \frac{1}{n} \left( \frac{R}{n-1} - \frac{\bar{R}}{n+1} \right) G_{hsijk} \right) \]

\[ + D_j \left( Z_{hisk} - \frac{M^2}{n^2} G_{hisk} - \frac{1}{n} \left( \frac{R}{n-1} - \frac{\bar{R}}{n+1} \right) G_{hisk} \right) \]

\[ + D_k \left( Z_{hijs} - \frac{M^2}{n^2} G_{hijs} - \frac{1}{n} \left( \frac{R}{n-1} - \frac{\bar{R}}{n+1} \right) G_{hijs} \right) \]

\[ + \frac{1}{n^2} G_{hijk} \nabla_s M^2 + \frac{1}{n} G_{hijk} \nabla_s \left( \frac{R}{n-1} - \frac{\bar{R}}{n+1} \right) \]

\[ + \frac{M}{n^2} (g_{hjsgik} - g_{ijk} g_{hj}) \nabla_j M + (g_{ihs} g_{jk} - g_{ij} g_{hk}) \nabla_k M \]

\[ + (g_{jsgik} - g_{ijk} g_{sj}) \nabla_h M + (g_{ks} g_{hj} - g_{js} g_{hk}) \nabla_i M \]  

(2.4)

Now, we suppose that \((M, g)\) is \((WZS)\).

By the aid of (1.4) and (2.4), we have

\[ \left[ \frac{M^2}{n^2} + \frac{1}{n} \left( \frac{R}{n-1} - \frac{\bar{R}}{n+1} \right) \right] \left( A_s G_{hijk} + B_h G_{sijk} + B_i G_{hsijk} + D_j G_{hisk} + D_k G_{hijs} \right) \]

\[ - G_{hijk} \nabla_s \left( \frac{M^2}{n^2} + \frac{1}{n} \left( \frac{R}{n-1} - \frac{\bar{R}}{n+1} \right) \right) \]

\[ - \frac{M}{n^2} \left( G_{hisk} \nabla_j M + G_{ihsj} \nabla_k M + G_{sijs} \nabla_h M + G_{kjs} \nabla_i M \right) = 0 \]  

(2.5)

Multiplying (2.5) by \(g^{hk} g^{ij}\), we can obtain

\[ \left( \frac{M^2}{n^2} + \frac{1}{n} \left( \frac{R}{n-1} - \frac{\bar{R}}{n+1} \right) \right) (2B_s + 2D_s + n A_s) \]

\[ - \frac{(n+2)}{n^2} \nabla_s M^2 - \nabla_s \left( \frac{R}{n-1} - \frac{\bar{R}}{n+1} \right) = 0 \]  

(2.6)

Similarly, multiplying (2.5) by \(g^{ik} g^{hs}\), it is easily obtained that

\[ \left( \frac{M^2}{n^2} + \frac{1}{n} \left( \frac{R}{n-1} - \frac{\bar{R}}{n+1} \right) \right) (B_s + A_s + (n-1) D_s) \]

\[ - \frac{(n+2)}{2n^2} \nabla_s M^2 - \frac{1}{n} \nabla_s \left( \frac{R}{n-1} - \frac{\bar{R}}{n+1} \right) = 0 \]  

(2.7)

Let us suppose that

\[ R = (1 - \frac{2}{n+1}) \bar{R} \]  

(2.8)
where the scalar curvature $R$ is not constant.

From (2.6) and (2.7), we get

$$A_s = 2D_s \quad \text{or} \quad M = 0 \quad (2.9)$$

We assume that $A_s = 2D_s$. Transvecting (1.4) with $g^{lk}$ and $g^{ij}$, we get

$$g^{ks}\nabla_s G_{hk} = (A_k - B_k + D_k)G_{hs}g^{sk} \quad (2.10)$$

where $G_{hk} = R_{hk} - \frac{R}{n}g_{hk}$ ($n > 2$) is the Einstein tensor.

Similarly, transvecting (1.4) with $g^{hk}$ and $g^{ij}$, we have

$$(B_k + D_k)G_{hs}g^{sk} = 0 \quad (2.11)$$

Hence, using the equations (2.9) and (2.10), it can be obtained that

$$(A_k + 2B_k)G_{hs}g^{sk} = 0 \quad (2.12)$$

Now, multiplying the equation (1.4) by $g_{hk}g_{ij}$ and using the result $\nabla_s R_{hk}^k = \frac{1}{2}\nabla_h R$, we obtain $R \equiv \text{const}$. In the beginning, we suppose that $R \neq \text{const}$. Thus, $A_s \neq 2D_s$. From (2.9), we have $M = 0$, i.e., the hypersurface is totally geodesic. Thus, we can state the following theorem:

**Theorem 2.1** In the totally umbilical hypersurface $(WZS)_n$ of $(WZS)_{n+1}$, if the expression $R = (1 - \frac{2}{n+1})\bar{R}$, $(R \neq \text{const.})$ is satisfied then the hypersurface is totally geodesic.

**Theorem 2.2** If the totally umbilical hypersurface $(WZS)_n$ of a $(WZS)_{n+1}$ satisfies the condition $\frac{R}{n-1} - \frac{R}{n+1} = c$ ($c < 0$, const.) then either the mean curvature or the scalar curvature of this hypersurface is constant.

**Proof** We assume that the totally umbilical hypersurface $(WZS)_n$ of $(WZS)_{n+1}$ satisfies the condition

$$-\frac{\bar{R}}{n+1} + \frac{R}{n-1} = c \quad (2.13)$$

From (2.5) and (2.13), we obtain

$$\left(\frac{M^2}{n^2} + \frac{c}{n}\right)(A_sG_{hijk} + B_hG_{sijk} + B_iG_{hsjk} + D_jG_{hisk} + D_kG_{hijs})$$

$$- \frac{1}{n^2}G_{hijk}\nabla_sM^2 - \frac{M}{n^2}(G_{hisk}\nabla_jM$$

$$+ G_{ihsj}\nabla_kM + G_{sijk}\nabla_hM + G_{kjsb}\nabla_iM) = 0 \quad (2.14)$$

Multiplying (2.14) by $g^{hk}g^{ij}$, we find that

$$\left(\frac{M^2}{n^2} + \frac{c}{n}\right)(2B_s + 2D_s + nA_s) - \frac{(n+2)}{n^2}\nabla_sM^2 = 0 \quad (2.15)$$
Similarly, multiplying (2.14) by $g^{ik}g^{hs}$, we can easily obtain that
\[
\left(\frac{M^2}{n^2} + \frac{c}{n}\right)(B_s + A_s + (n - 1)D_s) - \frac{(n + 2)}{2n^2} \nabla_s M^2 = 0 \tag{2.16}
\]
Using (2.15) and (2.16), we get
\[
M^2 = -cn \quad \text{or} \quad A_s = 2D_s \tag{2.17}
\]
On the other hand, from (1.4), we have
\[
\nabla_l Z_{hijk} = A_l Z_{hijk} + B_h Z_{lijk} + B_i Z_{hljk} + D_j Z_{hkil} + D_k Z_{hijl} \tag{2.18}
\]
Permutating $j, k, l$ by cyclic in (2.18), adding the three equations and using the expression (1.5) and the first Bianchi Identity, we obtain
\[
(A_l - 2D_l)Z_{hijk} + (A_j - 2D_j)Z_{hikl} + (A_k - 2D_k)Z_{hilj} - \frac{1}{n(n - 1)}(G_{hijk} \nabla_l R + G_{hikl} \nabla_j R + G_{hilj} \nabla_k R) \tag{2.19}
\]
Transvecting (2.19) with $g^{ij}g^{hk}$, we can obtain
\[
2(A_k - 2D_k)g^{hk}G_{hl} = \frac{(n - 2)}{n} \nabla_l R \tag{2.20}
\]
If $A_k = 2D_k$, from (2.20), then we say that the scalar curvature of this hypersurface is constant. If $A_k \neq 2D_k$, from (2.17), the mean curvature of this hypersurface must be constant. If $c = 0$ then it is clear that this hypersurface is totally geodesic. Thus, the proof is completed. \hfill \Box

**Theorem 2.3** If a totally geodesic hypersurface of a $(WZS)_{n+1}$ satisfies the condition $R = (1 - \frac{2}{n+1})\bar{R}$ then this hypersurface is $(WZS)_n$.

**Proof** From (1.4) and (2.4), the proof is easily seen that.

### 3 Totally umbilical hypersurface of a pseudo concircular symmetric manifold

We consider a non-concircular flat Riemannian manifold $(M, g)$ whose concircular curvature tensor $Z_{hijk}$ satisfies the condition
\[
\nabla_l Z_{hijk} = 2\lambda_l Z_{hijk} + \lambda_h Z_{lijk} + \lambda_i Z_{hljk} + \lambda_j Z_{hkil} + \lambda_k Z_{hijl} \tag{3.1}
\]
where $\lambda_l$ is a non-zero covariant vector. Such a manifold will be called a pseudo-concircular symmetric manifold and denoted by $(PZS)_n$. Permutating $j,k,l$ by cyclic in (3.1), we obtain the following equations
\[
\nabla_j Z_{hikl} = 2\lambda_j Z_{hikl} + \lambda_h Z_{jikl} + \lambda_i Z_{hjkl} + \lambda_k Z_{hjil} + \lambda_l Z_{hijl} \tag{3.2}
\]

Adding the equations (3.1), (3.2) and (3.3) and by using the first and the second Bianchi identities, it is obtained that

$$G_{hijk} \nabla_l R + G_{hikl} \nabla_j R + G_{hilj} \nabla_k R = 0 \quad (3.4)$$

Transvecting (3.4) with $g^{hk} g^{ij}$, we get

$$(1 - n)(2 - n) \nabla_l R = 0.$$  

Since $n > 2$, we find that the scalar curvature of the hypersurface is constant.

Now, we can state the following theorem:

**Theorem 3.1** The scalar curvature of a pseudo concircular symmetric manifold is constant.

**Theorem 3.2** Let us suppose that a hypersurface $(PZS)_n$ of a pseudo concircular symmetric manifold $(PZS)_{n+1}$ be totally umbilical. Then the scalar curvature of $(PZS)_{n+1}$ is constant.

**Proof** Taking the relation $A_s^2 = B_s = D_s = \lambda_s$ in (2.3), (2.4) and (2.5) and using the equation (3.1), we get

$$\left( \frac{M^2}{n^2} + \frac{1}{n} \left( \frac{R}{n-1} - \frac{\bar{R}}{n+1} \right) \right) \left( 2\lambda_s G_{hijk} + \lambda_i G_{hjsk} + \lambda_j G_{hisk} + \lambda_k G_{hijs} + \lambda_l G_{shikj} \right)$$

$$- \frac{1}{n^2} G_{hijk} \nabla_s M^2 - \frac{1}{n} G_{hijk} \nabla_s \left( \frac{R}{n-1} - \frac{\bar{R}}{n+1} \right)$$

$$- \frac{M}{n^2} (G_{hisk} \nabla_j M + G_{ihsj} \nabla_k M + G_{sijk} \nabla_h M + G_{kjsi} \nabla_i M) = 0 \quad (3.5)$$

Multiplying (3.5) by $g^{hk} g^{ij}$ and $g^{ik} g^{hs}$, respectively, we obtain

$$\left( \frac{M^2}{n^2} + \frac{1}{n} \left( \frac{R}{n-1} - \frac{\bar{R}}{n+1} \right) \right) 2\lambda_s (2 + n) - \frac{(n+2)}{n^2} \nabla_s M^2$$

$$- \nabla_s \left( \frac{R}{n-1} - \frac{\bar{R}}{n+1} \right) = 0 \quad (3.6)$$

and

$$\left( \frac{M^2}{n^2} + \frac{1}{n} \left( \frac{R}{n-1} - \frac{\bar{R}}{n+1} \right) \right) \lambda_s (2 + n) - \frac{(n+2)}{2n^2} \nabla_s M^2$$

$$- \frac{1}{n} \nabla_s \left( \frac{R}{n-1} - \frac{\bar{R}}{n+1} \right) = 0 \quad (3.7)$$

From (3.6) and (3.7), we obtain

$$\frac{\bar{R}}{n+1} + \frac{R}{n-1} = c \quad (3.8)$$
where \( c \) is a positive constant. By using Theorem 3.1, we can say that
\[
\bar{R} \equiv \text{const.} \quad (3.9)
\]

**Theorem 3.3** If a totally geodesic hypersurface of \((PZS)_{n+1}\) satisfies the condition \( R = (1 - \frac{2}{n+1})\bar{R} \) then the hypersurface is \((PZS)_n\).

**Proof** Let us suppose that a hypersurface of \((PZS)_{n+1}\) be totally geodesic. From the expressions (1.12) and (2.4) and the condition \( \frac{1}{2} A_s = B_s = D_s = \lambda_s \), the proof is clear. \( \square \)

### 4 An example of a \((WZS)_n\)

In this section, we want to construct a \((WZS)_n\) spaces. On the coordinate space \( R^n \) (with coordinates \( x^1, x^2, \ldots, x^n \)), we define a Riemannian space \( V^n \) and calculate the components of the curvature tensor and its covariant derivative.

Let each Latin index run over 1, 2, \ldots, \( n \) and each Greek index over 2, 3, \ldots, \( n-1 \). We define a Riemannian metric on \( R^n \) \((n > 3)\) by the formula
\[
d s^2 = \phi(d x^1)^2 + k_{\alpha\beta} d x^\alpha d x^\beta + 2 d x^1 d x^n \quad (4.1)
\]
where \([k_{\alpha\beta}]\) is a symmetric and non-singular matrix consisting of constants and \( \phi \) is a function of \((x^1, x^2, \ldots, x^{n-1})\) and independent of \( x^n \). In the metric considered, the only non-vanishing components of the curvature tensor, \([9]\]
\[
R_{1\alpha\beta1} = \frac{1}{2} \phi_{,\alpha\beta} \quad (4.2)
\]
where “,” denotes the partial differentiation with respect to the coordinates and \( k_{\alpha\beta} \) are the elements of the matrix inverse to \([k_{\alpha\beta}]\).

We consider \( V_n \) and
\[
\phi = f(x^1)(V_{\alpha\beta} x^\alpha x^\beta \cos g(x^1) + w_{\alpha\beta} x^\alpha x^\beta \sin g(x^1) + k_{\alpha\beta} x^\alpha x^\beta h(x^1))
\]
where \( f, g, h \) are functions of \( x^1 \) only and the matrices \([w_{\alpha\beta}]\), \([V_{\alpha\beta}]\) and \([k_{\alpha\beta}]\) are the form
\[
w_{\alpha\beta} = -1 \quad \text{for} \quad \alpha = \beta \quad \text{and} \quad w_{\alpha\beta} = 0 \quad \text{for} \quad \alpha \neq \beta \quad (4.3)
\]
\[
V_{\alpha\beta} = 1 \quad \text{for} \quad \alpha = \beta \quad \text{and} \quad V_{\alpha\beta} = 0 \quad \text{for} \quad \alpha \neq \beta \quad (4.4)
\]

and
\[
k_{\alpha\beta} = \begin{cases} 1 & \text{for } \alpha = \beta \\ 0 & \text{otherwise} \end{cases} \quad (4.5)
\]

From (4.2), the only non-vanishing components of the concircular curvature tensor \( Z_{hijk} \) are
\[
Z_{1\alpha\beta1} = \begin{cases} f(\cos g - \sin g + h) & \text{for } \alpha = \beta \\ 0 & \text{for } \alpha \neq \beta \end{cases} \quad (4.6)
\]
Here, we consider

\[ A_i = B_i = D_i = 0 \quad \text{for} \ i \neq 1 \quad \text{and} \quad A_1 + B_1 + D_1 = c_1, \ c_1 \neq 0 \quad \text{and const.} \quad (4.7) \]

Thus, from (1.4), \( V_n \) will be \((WZS)_n\) if and only if the following relations

\[
\nabla_1 Z_{1a1} = A_1 Z_{1a1} + B_1 Z_{1a1} + B_\alpha Z_{1a1} + D_\alpha Z_{1a1} + D_1 Z_{1a1} \quad (4.8)
\]

\[
\nabla_\alpha Z_{1a1} = A_\alpha Z_{1a1} + B_1 Z_{1a1} + B_1 Z_{1a1} + D_\alpha Z_{1a1} + D_1 Z_{1a1} \quad (4.9)
\]

\[
\nabla_\alpha Z_{1a1} = A_\alpha Z_{1a1} + B_1 Z_{1a1} + B_1 Z_{1a1} + D_\alpha Z_{1a1} + D_1 Z_{1a1} \quad (4.10)
\]

Thus, using (4.8), (4.9) and (4.10), we find

\[
f'(x^1)(\cos g - \sin g + h) + f(x^1)(-g' \sin g - g' \cos g + h')
\]

\[
= (A_1 + B_1 + D_1)f(x^1)(\cos g - \sin g + h). \quad (4.11)
\]

By the aid of (4.11), we get

\[
f(\cos g - \sin g + h) = c_2 e^{(A_1 + B_1 + D_1)x^1}, \quad c_2 > 0. \quad (4.12)
\]

So, the \( n \)-dimensional weakly concircular recurrent Riemannian manifold has the metric of the form

\[
ds^2 = \phi(dx^1)^2 + k_{\alpha\beta}dx^\alpha dx^\beta + 2dx^1 dx^n,
\]

\[
\phi = c_2 e^{c_1 x^1} \sum_{k=2}^{n-1} (x^k)^2.
\]

References


