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A Result on Segmenting Jungck–Mann Iterates

MEMUDU OLAPOSI OLATINWO

Department of Mathematics, Obafemi Awolowo University, Ile-Ife, Nigeria
e-mail: polatinwo@oauife.edu.ng

(Received May 5, 2007)

Abstract

In this paper, following the concepts in [5, 7], we shall establish a convergence result in a uniformly convex Banach space using the Jungck–Mann iteration process introduced by Singh et al [13] and a certain general contractive condition. The authors of [13] established various stability results for a pair of nonself-mappings for both Jungck and Jungck–Mann iteration processes. Our result is a generalization and extension of that of [7] and its corollaries. It is also an improvement on the result of [7].

Key words: Jungck–Mann iteration process; uniformly convex Banach space.

2000 Mathematics Subject Classification: 47H06, 47H10

1 Introduction

Suppose that $A = (a_{nk})$ is an infinite, lower triangular, regular row-stochastic matrix, $E$ a closed convex subset of a Banach space and $T$ a continuous mapping of $E$ into itself and $x_1 \in E$. Then, the general Mann iteration process $M(x_1, A, T)$ which was introduced in Mann [9] is defined by

$$v_n = \sum_{k=1}^{n} a_{nk} x_k, \quad x_{n+1} = T v_n, \quad n = 1, 2, \ldots,$$

(1)
If $A$ is the identity matrix, then each sequence of $M(x_1, A, T)$ becomes the sequence of Picard iterates of $T$ at $x_1$. It was established in [9] that if either of the sequences $\{x_n\}$ and $\{v_n\}$ converges, then the other also converges to the same point, and their common limit is a fixed point of $T$.

In [5, 7], it is said that the matrix $A$ is segmenting for the Mann process if $a_{n+1,k} = (1 - a_{n+1,n+1})a_{nk}$ for $k \leq n$. In this case, $v_{n+1}$ lies on the segment joining $v_n$ and $Tv_n$:

$$v_{n+1} = (1 - d_n)v_n + d_nTv_n, \quad n = 1, 2, \ldots,$$

where $d_n = a_{n+1,n+1}$. A segmenting matrix is determined by its sequence of diagonal elements. Some authors including [3, 11, 12] have investigated the case $d_n = \lambda$, $0 < \lambda < 1$, while Mann [9] approximated the fixed points of continuous functions on a closed interval of the real line using the segmenting matrix determined by $d_n = \frac{1}{n}$, $\forall n$. Dotson [6] considered the case when $d_n$ is bounded away from 0 and 1. Groetsch [7] generalized the results of [3, 6, 9, 11, 12] in a uniformly convex Banach space by employing (2) and assuming that $A$ is a segmenting matrix for which $\sum_{n=1}^{\infty} d_n(1 - d_n) = \infty$.

We shall give another definition of a segmenting matrix in the next section with a view to generalizing and extending Groetsch [7] and others mentioned earlier in this paper.

2 Preliminaries

Singh et al [13] introduced the following iteration process: Let $(E, \|\|)$ be a normed linear space, $S, T: Y \to E$ and $T(Y) \subseteq S(Y)$. Then, for $x_0 \in Y$, consider the iteration process

$$Sx_{n+1} = (1 - \alpha_n)Sx_n + \alpha_nTx_n, \quad n = 0, 1, 2, \ldots,$$

where $\{\alpha_n\}_{n=0}^{\infty}$ satisfies

(i) $\alpha_0 = 1$,

(ii) $0 \leq \alpha_n \leq 1$ for $n > 0$,

(iii) $\sum \alpha_n = \infty$, and

(iv) $\sum_{j=0}^{n} \alpha_j \Pi_{i=j+1}^{n}(1 - \alpha_i + a\alpha_i)$ converges.

The iteration process (3) is called the Jungck–Mann iteration.

For $Y = E$, $S = I$ (identity operator) in (3) with $\{\alpha_n\}_{n=0}^{\infty}$ satisfying (i)–(iv), then we have the Mann iteration process introduced by Mann [9]. Also, if in (3), $Y = E$, $S = I$ (identity operator) and $\alpha_n = 1$, then we obtain the Jungck iteration introduced by Jungck [8].

Following (3), we shall generalize and extend Groetsch [7] and others mentioned earlier in this paper by assuming that $A$ is a segmenting matrix for which

$$Sv_{n+1} = (1 - d_n)Sv_n + d_nTv_n, \quad n = 1, 2, \ldots,$$
such that $\sum_{n=1}^{\infty} d_n (1 - d_n) = \infty$ and $S, T: C \rightarrow C$ are selfmappings on a nonempty convex subset $C$ of a uniformly convex Banach space $E$. The operators $S$ and $T$ are assumed to have a common fixed point and satisfy in addition the contractive condition

$$\|Tx - Ty\| \leq \|Sx - Sy\|, \quad \forall x, y \in C. \quad (**)$$

If $S = I$ (identity operator) in (*), then we obtain (2) and if $S = I$ in (**) then we have $\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C$ (that is, $T$ becomes a nonexpansive mapping).

We shall establish our main result in the next section. However, the following lemma is required in the sequel.

**Lemma 2.1** (Groetsch [7]) Let $X$ be a uniformly convex Banach space and let $x, y \in X$. If $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| \geq \epsilon > 0$, then

$$\|\lambda x + (1 - \lambda)y\| \leq 1 - 2\lambda(1 - \lambda)\delta(\epsilon)$$

for $0 \leq \lambda < 1$ and $\delta(\epsilon) > 0$.

The proof of this Lemma is contained in [4, 7].

### 3 The Main Result

**Theorem 3.1** Let $C$ be a convex subset of a uniformly convex Banach space $E$ and $S, T: C \rightarrow C$ selfmappings satisfying condition (***) and $T(C) \subseteq S(C)$. Suppose that $S$ and $T$ have at least a common fixed point. Let $\{Sv_n\}_{n=1}^{\infty}$ be the sequence defined by (*). Then, the sequence $\{(S - T)v_n\}_{n=1}^{\infty}$ converges strongly to 0 for each $x_1 \in C$ such that $\sum_{n=1}^{\infty} d_n (1 - d_n) = \infty$.

**Proof** If $p$ is a common fixed point of $S$ and $T$ (i.e. $Sp = Tp = p$), then

$$\|Sv_{n+1} - p\| = \|(1 - d_n)Sv_n + d_nTv_n - (1 - d_n + d_n)p\|$$

$$= \|(1 - d_n)(Sv_n - p) + d_n(Tv_n - p)\|$$

$$\leq (1 - d_n)\|Sv_n - p\| + d_n\|Tv_n - p\|$$

$$= (1 - d_n)\|Sv_n - p\| + d_n\|Tv_n - Tp\|$$

$$\leq (1 - d_n)\|Sv_n - p\| + d_n\|Sv_n - Sp\|$$

$$= (1 - d_n)\|Sv_n - p\| + d_n\|Sv_n - p\|$$

$$= \|Sv_n - p\| \leq \|Sv_{n-1} - p\| \leq \cdots \leq \|Sv_1 - p\|, \quad (4)$$

from which we have that the sequence $\{Sv_n - p\}_{n=1}^{\infty}$ is decreasing.

Now,

$$\|(S - T)v_n\| = \|Sv_n - Tv_n\| \leq \|Sv_n - p\| + \|p - Tv_n\|$$

$$= \|Sv_n - p\| + \|Tp - Tv_n\| \leq \|Sv_n - p\| + \|Sp - Sv_n\| = 2\|Sv_n - p\|.$$
Suppose on the contrary that \( \{(S - T)v_n\}_{n=1}^{\infty} \) does not converge to 0. Since \( \|Sv_n - Tv_n\| \leq 2\|Sv_n - p\| \), we may assume that there is an \( a > 0, a \in (0, 1) \) such that \( \|Sv_n - p\| \geq a \) for any \( n \). If \( \{(S - T)v_n\}_{n=1}^{\infty} \) does not converge to 0, then there is an \( \epsilon > 0 \) such that \( \|Sv_n - T v_n\| \geq \epsilon \) for any \( n \).

Let

\[
\begin{align*}
b &= 2\delta \left( \frac{\epsilon}{\|Sv_1 - p\|} \right), \\
x_n &= \frac{Sv_n - p}{\|Sv_n - p\|} \quad \text{and} \quad y_n = \frac{Tv_n - p}{\|Sv_n - p\|}.
\end{align*}
\]

Then, we have

\[
\|x_n\| = \left\| \left( \frac{Sv_n - p}{\|Sv_n - p\|} \right) \right\| \leq \frac{\|Sv_n - p\|}{\|Sv_n - p\|} = 1
\]

and

\[
\|y_n\| = \left\| \left( \frac{Tv_n - p}{\|Sv_n - p\|} \right) \right\| \leq \frac{\|Tv_n - Tp\|}{\|Sv_n - p\|} \leq \frac{\|Sv_n - Sp\|}{\|Sv_n - p\|} = \frac{\|Sv_n - p\|}{\|Sv_n - p\|} = 1.
\]

Hence, we have by (\( \ast \)) that

\[
\begin{align*}
\|Sv_{n+1} - p\| &= \|(1 - d_n)Sv_n + d_nTv_n - (1 - d_n + d_n)p\| \\
&= \|(1 - d_n)(Sv_n - p) + d_n(Tv_n - p)\| \\
&= \left\| (\|Sv_n - p\|) \left[ (1 - d_n)\frac{Sv_n - p}{\|Sv_n - p\|} + d_n\frac{Tv_n - p}{\|Sv_n - p\|} \right] \right\| \\
&= \|(\|Sv_n - p\|)(1 - d_n)x_n + d_ny_n\| \\
&\leq \|Sv_n - p\| \|(1 - d_n)x_n + d_ny_n\|. \quad (5)
\end{align*}
\]

Using (4) and Lemma 2.1 in (5) yield

\[
\begin{align*}
\|Sv_{n+1} - p\| &\leq \|Sv_n - p\| - b d_n (1 - d_n) \|Sv_n - p\| \\
&\leq \|Sv_{n-1} - p\| - b d_{n-1} (1 - d_{n-1}) \|Sv_{n-1} - p\| - b d_n (1 - d_n) \|Sv_n - p\| \\
&\leq \|Sv_{n-1} - p\| - b d_{n-1} (1 - d_{n-1}) \|Sv_{n-1} - p\| - b d_n (1 - d_n) \|Sv_n - p\| \\
&\leq \|Sv_{n-1} - p\| - b \left[ d_{n-1} (1 - d_{n-1}) + d_n (1 - d_n) \right] \|Sv_n - p\|.
\end{align*}
\]

Repeating this process inductively leads to

\[
a \leq \|Sv_{n+1} - p\| \leq \|Sv_1 - p\| - b \left[ d_1 (1 - d_1) \|Sv_n - p\| + d_2 (1 - d_2) \|Sv_n - p\| + \cdots + d_n (1 - d_n) \|Sv_n - p\| \right]
\]

\[
= \|Sv_1 - p\| - b \sum_{j=1}^{n} d_j (1 - d_j) \|Sv_n - p\| \leq \|Sv_1 - p\| - ab \sum_{j=1}^{n} d_j (1 - d_j).
\]
Therefore, we obtain
\[ a \left[ 1 + b \sum_{j=1}^{n} d_j (1 - d_j) \right] \leq \| Sv_1 - p \|, \]
from which it follows that
\[ a \leq \frac{\| Sv_1 - p \|}{1 + b \sum_{j=1}^{n} d_j (1 - d_j)} \to 0 \quad \text{as } n \to \infty, \]
leading to a contradiction. Therefore, we have \( a = 0 \). Hence,
\[ \lim_{n \to \infty} \| Sv_n - Tv_n \| = 0. \]

**Remark 3.1** Theorem 3.1 is also a generalization of the results of [3, 6, 7, 9, 11, 12].

**References**