

Igor V. Protasov

Topologies on groups determined by right cancellable ultrafilters

*Commentationes Mathematicae Universitatis Carolinae*, Vol. 50 (2009), No. 1, 135--139

Persistent URL: <http://dml.cz/dmlcz/133421>

## Terms of use:

© Charles University in Prague, Faculty of Mathematics and Physics, 2009

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://project.dml.cz>

## Topologies on groups determined by right cancellable ultrafilters

I.V. PROTASOV

*Abstract.* For every discrete group  $G$ , the Stone-Čech compactification  $\beta G$  of  $G$  has a natural structure of a compact right topological semigroup. An ultrafilter  $p \in G^*$ , where  $G^* = \beta G \setminus G$ , is called right cancellable if, given any  $q, r \in G^*$ ,  $qp = rp$  implies  $q = r$ . For every right cancellable ultrafilter  $p \in G^*$ , we denote by  $G(p)$  the group  $G$  endowed with the strongest left invariant topology in which  $p$  converges to the identity of  $G$ . For any countable group  $G$  and any right cancellable ultrafilters  $p, q \in G^*$ , we show that  $G(p)$  is homeomorphic to  $G(q)$  if and only if  $p$  and  $q$  are of the same type.

*Keywords:* Stone-Čech compactification, right cancellable ultrafilters, left invariant topologies

*Classification:* Primary 54H11; Secondary 54C05, 54G15

A topology  $\tau$  on a group  $G$  is called *left invariant* if, for every element  $g \in G$ , the left shift  $x \mapsto gx$  is continuous in  $\tau$ . Given an infinite group  $G$ , we denote by  $G(p)$  the group  $G$  provided with the strongest left invariant topology in which  $p$  converges to the identity of  $G$ . By [4, Theorem 4.12], the space  $G(p)$  is *strongly extremally disconnected* in the sense that, for every open non-closed subset  $U$  of  $G(p)$ , there exists  $g \in \text{cl } U \setminus U$  such that  $\{g\} \cup U$  is a neighbourhood of  $g$ . To distinguish the spaces  $G(p)$  for different ultrafilters  $p$  on  $G$ , we need some algebra in the Stone-Čech compactification of a discrete group.

Given a discrete space  $X$ , we take the points of  $\beta X$ , the Stone-Čech compactification of  $X$ , to be the ultrafilters on  $X$ , with the points of  $X$  identified with the principal ultrafilters, and denote by  $X^* = \beta X \setminus X$  the set of all free ultrafilters on  $X$ . The topology of  $\beta X$  can be defined by stating that the sets of the form  $\bar{A} = \{p \in \beta X : A \in p\}$ , where  $A$  is a subset of  $X$ , are a base for the open sets. We shall also use the universal property of  $\beta X$  stating that every mapping  $f : X \rightarrow Y$ , where  $Y$  is a compact Hausdorff space, can be extended to the continuous mapping  $f^\beta : \beta X \rightarrow Y$ .

Let  $G$  be a discrete group. Using the universal property of the space  $\beta G$ , we extend the group multiplication from  $G$  to  $\beta G$  in two steps. Given  $g \in G$ , the mapping

$$x \mapsto gx : G \rightarrow \beta G$$

extends to the continuous mapping

$$q \mapsto gq : \beta G \rightarrow \beta G.$$

Then, for each  $q \in \beta G$ , we extend the mapping  $g \mapsto gq$ , defined from  $G$  into  $\beta G$ , to the continuous mapping

$$p \mapsto pq : \beta G \rightarrow \beta G.$$

The product  $pq$  of ultrafilters  $p, q$  can also be defined by the rule: given a subset  $A \subseteq G$ ,

$$A \in pq \Leftrightarrow \{g \in G : g^{-1}A \in q\} \in p.$$

It is easy to verify that the binary operation  $(p, q) \mapsto pq$  is associative, so  $\beta G$  is a semigroup, and  $G^*$  is a subsemigroup of  $\beta G$ . It follows from the second step of the extension that, for every  $q \in \beta G$ , the mapping  $p \mapsto pq$  is continuous, so the semigroup  $\beta G$  is right topological. For the structure of compact right topological semigroup  $\beta G$  and its combinatorial applications see [1].

An ultrafilter  $p \in \beta G$  is called an *idempotent* if  $pp = p$ . By [1, Corollary 6.43], for every infinite group  $G$ , there are  $2^{2^{|G|}}$  idempotents in  $G^*$ . Given an idempotent  $p \in G^*$ , the space  $G(p)$  is Hausdorff and *maximal*, i.e.  $G(p)$  has no isolated points but  $G(p)$  has an isolated point in any stronger topology. The existence of maximal topological groups is consistent with ZFC [3]. For every infinite group  $G$ , in ZFC there exists an idempotent  $p$  such that  $G(p)$  is regular. To my knowledge, these are the only ZFC-examples of homogeneous regular maximal spaces. For these and other results concerning the topologies on a group  $G$  determined by idempotents from  $\beta G$  see [3], [4], [5]. For topologies on a semigroup  $S$  determined by idempotents from  $\beta S$  see [2].

An ultrafilter  $p \in G^*$  is called *right cancellable* if, for any  $q, r \in G^*$ ,  $qp = rp$  implies  $q = r$ . For every countable group  $G$ , there exists an open and dense in  $G^*$  subset consisting of right cancellable ultrafilters [1, Theorem 8.10]. For characterizations and properties of right cancellable ultrafilters see [1, Chapter 8].

In this paper, given a countable group  $G$ , we classify up to homeomorphisms the topologies on  $G$  determined by right cancellable ultrafilters. To this end, we use the spaces  $\text{Seq}(q), q \in \omega^*$  defined in [6].

We denote by  $\text{Seq}$  the set of all words in the alphabet  $\omega = \{0, 1, \dots\}$ . Every ultrafilter  $q \in \omega^*$  determines a topology on  $\text{Seq}$  in the following way: a subset  $U \subseteq \text{Seq}$  is open if and only if

$$(\forall t \in U)\{n \in \omega : tn \in U\} \in q.$$

The set  $\text{Seq}$  endowed with this topology is denoted by  $\text{Seq}(q)$ .

**Lemma 1.** *Let  $p, q \in \omega^*$ . The spaces  $\text{Seq}(p)$  and  $\text{Seq}(q)$  are homeomorphic if and only if  $p$  and  $q$  are of the same type, i.e. there exists a bijection  $f : \omega \rightarrow \omega$  such that  $f^\beta(p) = q$ .*

PROOF: This is routine using [6, Theorem 1.1]. □

**Theorem 1.** *For every countable group  $G$ , the following statements hold:*

- (i) *for every right cancellable ultrafilter  $p \in G^*$ , there exist  $X \in p$  and a bijection  $f : X \rightarrow \omega$  such that  $G(p)$  is homeomorphic to  $\text{Seq}(f^\beta(p))$ ;*
- (ii) *for every ultrafilter  $q \in \omega^*$ , there exists an injection  $h : \omega \rightarrow G$  such that  $h^\beta(q)$  is right cancellable and  $\text{Seq}(q)$  is homeomorphic to  $G(h^\beta(q))$ .*

**Theorem 2.** *Let  $G$  be a countable group,  $p_1$  and  $p_2$  be right cancellable ultrafilters from  $G^*$ . Then  $G(p_1)$  and  $G(p_2)$  are homeomorphic if and only if  $p_1$  and  $p_2$  are of the same type.*

PROOF OF THEOREM 1: (i) We use the following criterion [1, Theorem 8.11]: an ultrafilter  $p \in G^*$  is right cancellable if and only if there exists a family  $\{P_g : g \in G\}$  of members of  $p$  such that  $gP_g \cap hP_h = \emptyset$  for all distinct  $g, h \in G$ .

We need also the following description of topology of  $G(p)$  from [4, p. 12] in the form suggested by the referee. Given an indexed family  $\langle P_g \rangle_{g \in G}$  of members of  $p$  and  $h \in G$ , let  $U(\langle P_g \rangle_{g \in G}, h, 0) = \{h\} \cup hP_h$ , for  $n \in \omega$  let

$$U(\langle P_g \rangle_{g \in G}, h, n + 1) = \bigcup_{y \in U(\langle P_g \rangle_{g \in G}, h, n)} yP_y,$$

and let  $U(\langle P_g \rangle_{g \in G}, h) = \bigcup_{n=0}^\infty U(\langle P_g \rangle_{g \in G}, h, n)$ . Then  $U(\langle P_g \rangle_{g \in G}, h)$  is an open neighbourhood of  $h$  and, given any neighbourhood  $V$  of  $h$ , there is a choice of  $\langle P_g \rangle_{g \in G}$  such that  $U(\langle P_g \rangle_{g \in G}, h) \subseteq V$ .

We choose  $\langle P_g \rangle_{g \in G}$  such that each  $P_g \in p$ ,  $e \notin gP_g$  where  $e$  is the identity of  $G$ , and  $gP_g \cap hP_h = \emptyset$  whenever  $g \neq h$ . Fix a bijection  $f : P_e \rightarrow \omega$ , put  $X = P_e$ , and let  $q = f^\beta(p)$ . We show that if  $U$  is an open neighbourhood of  $e$  in  $G(p)$  and  $V$  is an open neighbourhood of the empty sequence in  $\text{Seq}(q)$ , then there exist a clopen subset  $S$  of  $U$  and  $\varphi : S \rightarrow V$  such that  $\varphi[S]$  is clopen in  $\text{Seq}(q)$  and  $\varphi$  is a homeomorphism.

Since  $U$  is an open neighbourhood of  $e$ , choose  $\langle Q_g \rangle_{g \in G}$  in  $P$  such that

$$U(\langle Q_g \rangle_{g \in G}, e) \subseteq U.$$

Since  $V$  is open in  $\text{Seq}(q)$ , if  $g \in P_e$  and  $f(g) \in V$ , then

$$f(g)^{-1}V = \{n \in \omega : f(g)n\} \in q,$$

so pick  $R_g \in p$  such that  $f[R_g] \subseteq f(g)^{-1}V$  (if  $g \in G \setminus P_e$  or  $f(g) \notin V$ , let  $R_g = G$ ). For  $g \in G$ , let  $P'_g = P_g \cap Q_g \cap R_g$ . We put  $S = U(\langle P'_g \rangle_{g \in G}, e)$ . Then

every element  $g \in S, g \neq e$  can be written as  $g = x_0x_1 \dots x_n$ , where  $x_0 \in P'_e$  and  $x_{k+1} \in P'_{x_0x_1 \dots x_k}$  for each  $k \in \{0, \dots, n-1\}$ . Since  $gP'_g \cap hP'_h = \emptyset$  whenever  $g \neq h$  and  $e \notin gP'_g$ , this representation of  $g$  is unique.

Then we extend  $f$  to an injection  $\varphi : S \rightarrow \text{Seq}(q)$  defined by the rule:  $\varphi(e) = \emptyset$  where  $\emptyset$  is an empty sequence and, for every  $g \in S, g \neq e, g = x_1x_2 \dots x_k$ ,

$$\varphi(g) = f(x_1)f(x_2) \dots f(x_k).$$

Given any  $h \in S$ , we have  $U(\langle P'_g \rangle_{g \in G}, h) \subseteq S$  so  $S$  is open. Assume that  $h \in \text{cl} S$  and pick  $m \in \omega$  such that  $U(\langle P'_g \rangle_{g \in G}, h, m) \cap S \neq \emptyset$ . Then there exist  $y_0, y_1, \dots, y_m$  and  $x_0, x_1, \dots, x_n$  such that

$$\begin{aligned} hy_0y_1 \dots y_m &= x_0x_1 \dots x_n, \quad y_0 \in P'_h, \quad x_0 \in P'_e, \\ y_{i+1} &\in P'_{hy_0 \dots y_i}, \quad x_{j+1} = P_{x_0 \dots x_j} \end{aligned}$$

for all  $i \in \{0, \dots, m-1\}, j \in \{0, \dots, n-1\}$ . By the choice of  $\langle P_g \rangle_{g \in G}$ , we have  $hy_0 \dots y_{m-1} = x_0x_1 \dots x_{n-1}$ . Repeating this argument, we conclude that  $h \in S$ , so  $S$  is closed. To see that  $\varphi[S]$  is clopen and  $\varphi$  is a homeomorphism, it suffices to notice that  $\varphi(gh) = \varphi(g)\varphi(h)$  whenever  $g \in S, h \in P'_g$ , and repeat above arguments.

Let  $g \in G(p), t \in \text{Seq}(q)$  and  $U, V$  be open neighbourhoods of  $g$  and  $t$ . The space  $G(p)$  is homogeneous by definition,  $\text{Seq}(q)$  is homogeneous by [6, Theorem 1.2]. Hence, we can choose the clopen homeomorphic subset  $S$  and  $T$  such that  $g \in S \subseteq U, t \in T \subseteq V$ . To conclude the proof, we partition  $G(p)$  and  $\text{Seq}(q)$  in  $\omega$  clopen subsets  $\{S_i : i \in \omega\}$  and  $\{T_i : i \in \omega\}$  such that  $S_i$  and  $T_i$  are homeomorphic for each  $i \in \omega$ . We enumerate  $G(p) = \{g_n : n \in \omega\}, \text{Seq}(q) = \{t_n : n \in \omega\}$  and choose the clopen homeomorphic neighbourhoods  $S_0$  and  $T_0$  of  $g_0$  and  $t_0$  such that  $G(p) \setminus S_0$  and  $\text{Seq}(q) \setminus T_0$  are infinite. Assume that we have chosen the clopen subsets  $S_0, \dots, S_n$  and  $T_0, \dots, T_n$  of  $G(p)$  and  $\text{Seq}(q)$  such that  $G(p) \setminus (S_0 \cup \dots \cup S_n)$  and  $\text{Seq}(q) \setminus (T_0 \cup \dots \cup T_n)$  are infinite,  $S_i, T_i$  are homeomorphic for each  $i \in \{0, \dots, n\}$ , and  $S_i \cap S_j = \emptyset, T_i \cap T_j = \emptyset$  for all distinct  $i, j \in \{0, \dots, n\}$ . We choose the minimal  $k \in \omega$  and  $m \in \omega$  such that  $g_k \notin S_0 \cup \dots \cup S_n, t_m \notin T_0 \cup \dots \cup T_n$ . Then we choose the clopen homeomorphic neighbourhoods  $S_{n+1}$  and  $T_{n+1}$  of  $g_k$  and  $t_m$  such that  $S_{n+1} \cap S_i = \emptyset, T_{n+1} \cap T_i = \emptyset$  for each  $i \in \{0, \dots, n\}$ , and  $G(p) \setminus (S_0 \cup \dots \cup S_{n+1}), \text{Seq}(q) \setminus (T_0 \cup \dots \cup T_{n+1})$  are infinite. After  $\omega$  steps we get the partition  $G(p) = \bigcup_{i \in \omega} S_i, \text{Seq}(q) = \bigcup_{i \in \omega} T_i$ .

(ii) We enumerate  $G = \{g_n : n \in \omega\}$  with  $g_0 = e$ , put  $K_n = \{g_i : i \leq n\}$  and choose inductively a sequence  $(x_n)_{n \in \omega}$  in  $G$  such that the subsets  $\{K_n x_n : n \in \omega\}$  are pairwise disjoint. We put  $X = \{x_n : n \in \omega\}$  and note that  $gX \cap X$  is finite for each  $g \in G, g \neq e$ . Given any ultrafilter  $r \in G^*$  with  $X \in r$ , we can choose inductively a sequence  $\langle R_n \rangle_{n \in \omega}$  of members of  $r$  such that the subsets  $\{g_n R_n : n \in \omega\}$  are pairwise disjoint. By [1, Theorem 8.11],  $r$  is right cancellable.

We fix an arbitrary bijection  $h : \omega \rightarrow X$  and put  $p = h^\beta(q)$ . Since  $p$  is right cancellable, we can choose  $\langle P_n \rangle_{n \in \omega}$  such that each  $P_n \in p$ ,  $P_0 \subseteq X$ ,  $e \notin g_n P_n$  and  $g_n P_n \cap g_m P_m = \emptyset$  whenever  $n \neq m$ . Put  $f = h^{-1}|_{P_0}$ . Then  $f^\beta(p) = q$  and (see proof of (i))  $G(p)$  is homeomorphic to  $\text{Seq}(q)$ .  $\square$

**PROOF OF THEOREM 2:** By Theorem 1(i), there exist  $q_1$  and  $q_2$  from  $\omega^*$  such that, for  $i \in \{1, 2\}$ ,  $p_i$  and  $q_i$  are of the same type, and  $G(p_i)$  is homeomorphic to  $\text{Seq}(q_i)$ . By Lemma 1,  $\text{Seq}(q_1)$  and  $\text{Seq}(q_2)$  are homeomorphic if and only if  $q_1$  and  $q_2$  are of the same type.  $\square$

**Acknowledgment.** The author would like to thank the referee for his/her suggestions which helped to improve the paper substantially.

#### REFERENCES

- [1] Hindman N., Strauss D., *Algebra in the Stone-Čech Compactification: Theory and Applications*, Walter de Gruyter, Berlin, 1998.
- [2] Hindman N., Protasov I., Strauss D., *Topologies on  $S$  determined by idempotents in  $\beta S$* , Topology Proc. **23** (1998), 155–190.
- [3] Protasov I.V., *Maximal topologies on groups*, Siberian Math. J. **39** (1998), 1184–1194.
- [4] Protasov I.V., *Extremal topologies on groups*, Mat. Stud. **15** (2001), 9–22.
- [5] Protasov I.V., *Remarks on extremally disconnected semitopological groups*, Comment. Math. Univ. Carolin. **43** (2002), 343–347.
- [6] Vaughan J.E., *Two spaces homeomorphic to  $\text{Seq}(p)$* , Comment. Math. Univ. Carolin. **42** (2001), 209–218.

DEPARTMENT OF CYBERNETICS, KIEV UNIVERSITY, VOLODIMIRSKA 64, KIEV 01033, UKRAINE  
*E-mail:* protasov@unicyb.kiev.ua

(Received May 21, 2008, revised September 11, 2008)