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2 – \( (n^2, 2n, 2n - 1) \) Designs Obtained from Affine Planes*

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Abstract

The simple incidence structure \( D(A, 2) \) formed by points and unordered pairs of distinct parallel lines of a finite affine plane \( A = (P, L) \) of order \( n > 2 \) is a \( 2 - (n^2, 2n, 2n - 1) \) design. If \( n = 3 \), \( D(A, 2) \) is the complementary design of \( A \). If \( n = 4 \), \( D(A, 2) \) is isomorphic to the geometric design \( AG_3(4, 2) \) (see [2; Theorem 1.2]). In this paper we give necessary and sufficient conditions for a \( 2 - (n^2, 2n, 2n - 1) \) design to be of the form \( D(A, 2) \) for some finite affine plane \( A \) of order \( n > 4 \). As a consequence we obtain a characterization of small designs \( D(A, 2) \).

Key words: \( 2 - (n^2, 2n, 2n - 1) \) designs; incidence structure; affine planes.

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By a \( 2 - (v, k, \lambda) \) design we mean a pair \( D = (P, B) \) where \( P \) is a set of \( v \) points and \( B \) is a collection of distinguished subsets of \( P \) called blocks such that each block contains \( k \) points and any two distinct points are contained in exactly \( \lambda \) common blocks\(^1\). Our main result is the following

Theorem 1 Let \( n \) be an integer with \( n > 4 \) and let \( D = (P, B) \) be a \( 2 - (n^2, 2n, 2n - 1) \) design. Then \( D \) is of the form \( D(A, 2) \) if and only if the following two conditions are satisfied: \((c_1)\) any three distinct points of \( D \)

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\(^1\)For further definitions (and basic results) about 2-designs see [1].
are contained in exactly 3 or \( n - 1 \) common blocks; (c2) if \( X_1, X_2, \ldots, X_{n-1} \) are \( n - 1 \) distinct blocks of \( D \) such that \( |X_1 \cap X_2 \cap \cdots \cap X_{n-1}| > 2 \), then \( X_1 \cap X_2 \cap \cdots \cap X_{n-1} = X_i \cap X_j \) whenever \( i \neq j \).

Before proving the theorem we need some preliminary results about 2 - \((n^2, 2n, 2n-1)\) designs.

**Lemma 1** Suppose \( A = (P, L) \) is a finite affine plane of order \( n > 4 \) and let \( D(A, 2) \) be the system of points and unordered pairs of distinct parallel lines of \( A \). Then \( D(A, 2) \) is a 2 - \((n^2, 2n, 2n-1)\) design satisfying the following properties:

1. any three distinct collinear points of \( A \) are contained in exactly \( n - 1 \) blocks of \( D(A, 2) \);
2. any three distinct non-collinear points of \( A \) are joined by precisely 3 blocks of \( D(A, 2) \);
3. if \( X_1, X_2, \ldots, X_{n-1} \) are \( n - 1 \) distinct blocks of \( D(A, 2) \) such that \( |X_1 \cap X_2 \cap \cdots \cap X_{n-1}| > 2 \), then \( X_1 \cap X_2 \cap \cdots \cap X_{n-1} = X_i \cap X_j \) whenever \( i \neq j \).

**Proof** This follows directly from the definition of \( D(A, 2) \). \( \square \)

**Lemma 2** Let \( n \) be an integer greater than 4 and let \( D = (P, B) \) be a 2 - \((n^2, 2n, 2n-1)\) design any three distinct points of which are contained in exactly 3 or \( n - 1 \) blocks. Then for any choice of two distinct points \( x, y \) in \( D \) there are precisely \( n - 2 \) points \( z \in P \setminus \{x, y\} \) with the property that \( x, y, z \) are joined by \( n - 1 \) distinct blocks of \( D \).

**Proof** Let \( x, y \) be any two distinct points of \( D \) and denote by \( c \) the number of points \( z \in P \setminus \{x, y\} \) with the property that \( x, y, z \) are joined by \( n - 1 \) blocks of \( D \). Then \( 0 \leq c \leq n^2 - 2 \) and \( n^2 - 2 - c \) is the number of points \( w \in P \setminus \{x, y\} \) with the property that \( x, y, w \) are joined by exactly 3 blocks of \( D \). Thus, counting the point block pairs \((p, C)\) with \( x \neq p \neq y \) and \( \{x, y, p\} \subset C \), we find \( 3(n^2 - 2 - c) + (n - 1)c = (2n - 2)(2n - 1) \) which can be written as \((n - 4)c = (n - 4)(n - 2)\). Hence, since \( n - 4 \neq 0 \), \( c = n - 2 \) and the lemma is proved. \( \square \)

**Lemma 3** Let \( n \) be an integer with \( n > 4 \) and let \( D = (P, B) \) be a 2 - \((n^2, 2n, 2n-1)\) design. If \( X_1, X_2, \ldots, X_{n-1} \) are \( n - 1 \) distinct blocks of \( D \) such that \( X_1 \cap X_2 \cap \cdots \cap X_{n-1} = X_i \cap X_j \) whenever \( i \neq j \), then \( |X_1 \cap X_2 \cap \cdots \cap X_{n-1}| \geq n \) with equality if and only if \( X_1 \cup X_2 \cup \ldots \cup X_{n-1} = P \).

**Proof** Write \( X_1 \cup X_2 \cup \cdots \cup X_{n-1} = l \cup (X_1 \setminus l) \cup (X_2 \setminus l) \cup \cdots \cup (X_{n-1} \setminus l) \), where \( l = X_1 \cap X_2 \cap \cdots \cap X_{n-1} \). Then \( |X_1 \cup X_2 \cup \cdots \cup X_{n-1}| = a + (n-1)(2n-a) = n^2 + (n-2)(n-a) \) with \( a = |l| \). Thus, since \( D \) has \( n^2 \) points, we obtain \( n^2 \geq n^2 + (n-2)(n-a) \) which, since \( n > 4 \), gives \( n \leq a \). Moreover \( n = a \) is
equivalent to ask \(|X_1 \cup X_2 \cup \cdots \cup X_{n-1}| = n^2\), i.e. \(X_1 \cup X_2 \cup \cdots \cup X_{n-1} = \mathcal{P}\), and the lemma is proved. \(\square\)

**Proof of Theorem 1** In view of Lemma 1, we have only to prove that \(\mathcal{D} = \mathcal{D}(\mathcal{A}, 2)\) for some affine plane \(\mathcal{A}\) (of order \(n\)), provided conditions (\(c_1\)) and (\(c_2\)) hold. Define \(\mathcal{A} = (\mathcal{P}, \mathcal{L})\) by taking \(\mathcal{P}\) as the set of points and the set \(\mathcal{L} = \{l \in \mathcal{P} : |l| > 2, l = L_1 \cap L_2 \cap \cdots \cap L_{n-1}\) with \(L_1, L_2, \ldots, L_{n-1}\) distinct blocks of \(\mathcal{D}\) as the set of lines. By Lemma 2, \(\mathcal{L}\) is non-empty. Let \(l \in \mathcal{L}\) and let \(L_1, L_2, \ldots, L_{n-1}\) be the \(n-1\) distinct blocks of \(\mathcal{D}\) such that \(l = L_1 \cap L_2 \cap \cdots \cap L_{n-1}\). Then condition (\(c_2\)) gives \(l = L_i \cap L_j\) whenever \(i \neq j\) so that, by Lemma 3, \(l\) contains at least \(n\) points. On the other hand, as any three distinct points of \(l\) are joined by the \(n-1\) blocks \(L_i (i = 1, 2, \ldots, n-1)\), it follows from Lemma 2 that \(l\) contains at most \(2 + (n-2) = n\) points. Thus we must have \(n \leq |l| \leq n\) and consequently \(|l| = n\). Let \(x, y\) be any two distinct points of \(\mathcal{D}\). By Lemma 2 we may choose a point \(z \in \mathcal{P} \setminus \{x, y\}\) and \(n-1\) distinct blocks \(Z_1, Z_2, \ldots, Z_{n-1} \in \mathcal{B}\) such that \(\{x, y, z\} \subseteq Z_1 \cap Z_2 \cap \cdots \cap Z_{n-1}\). Therefore \(h = Z_1 \cap Z_2 \cap \cdots \cap Z_{n-1}\) belongs to \(\mathcal{L}\) and passes through both \(x\) and \(y\). Assume that \(\{x, y\} \subseteq k\) for some \(k \in \mathcal{L}\) with \(k \neq h\). Writing \(k\) as the intersection \(k = W_1 \cap W_2 \cap \cdots \cap W_{n-1}\) of \(n-1\) distinct blocks \(W_1, W_2, \ldots, W_{n-1} \in \mathcal{B}\) we obtain \(\{x, y, p\} \subseteq Z_1 \cap Z_2 \cap \cdots \cap Z_{n-1}\) or \(\{x, y, p\} \subseteq W_1 \cap W_2 \cap \cdots \cap W_{n-1}\) whenever \(p \in h \cup k\) is a point such that \(x \neq p \neq y\). Then from Lemma 2 we deduce \(|h \cup k| \leq 2 + (n-2) = n\) which contradicts our assumption \(k \neq h\) and shows that \(h\) is the unique element in \(\mathcal{L}\) containing \(\{x, y\}\). Thus each \(l \in \mathcal{L}\) has \(n\) points and each pair of points is on exactly one common point set \(m \in \mathcal{L}\): this is sufficient to conclude that \(\mathcal{A} = (\mathcal{P}, \mathcal{L})\) is a finite affine plane of order \(n\). Note that such a plane \(\mathcal{A} = (\mathcal{P}, \mathcal{L})\) has the properties: (i) for any line \(l \in \mathcal{L}\) and any point \(x \in \mathcal{P}, x \notin l\), there is just one block of \(\mathcal{D}\) containing both \(l\) and \(x\); (ii) if a block \(C \in \mathcal{B}\) contains a line \(h \in \mathcal{L}\) and if \(y \in C\) is a point not on \(h\), then \(C = h \cup k\) where \(k \in \mathcal{L}\) is the only line of \(\mathcal{A}\) through \(y\) not intersecting \(h\). Property (i) follows from the fact that (by condition (\(c_2\)) and Lemma 3) the point set \(\mathcal{P}\) can be written as disjoint union \(\mathcal{P} = l \cup (L_1 \setminus l) \cup (L_2 \setminus l) \cup \cdots \cup (L_{n-1} \setminus l)\), if \(L_1, L_2, \ldots, L_{n-1}\) are the \(n-1\) distinct blocks of \(\mathcal{D}\) through the line \(l \in \mathcal{L}\). To show (ii) we proceed as follows. Denote by \(k\) the line of \(\mathcal{A}\) through \(y\) parallel to \(h\). Let \(z \in C \cap h\) be a point distinct from \(y\) and denote by \(l\) the line of \(\mathcal{A}\) joining \(y\) to \(z\). We claim that \(l = k\). In fact \(l \neq h\) and \(l = W_1 \cap W_2 \cap \cdots \cap W_{n-1}\) for suitable \(n-1\) distinct blocks \(W_1, W_2, \ldots, W_{n-1} \in \mathcal{B}\). Suppose there is a point \(w \in h \cap l\). Then \(y, z, w\) are three distinct points belonging to \(l\) and, by condition (\(c_1\)), there is no block in \(\mathcal{D}\) containing \(\{y, z, w\}\), apart from the blocks \(W_i\). But \(h \subseteq C\) forces \(w \in C\) and consequently \(\{y, z, w\} \subseteq C\). Thus we have \(C = W_i\) for some \(i \in \{1, 2, \ldots, n-1\}\) so that \(l \subseteq C\). Then \(l \cap h \subseteq C\) and there is just one point \(p \in C\) such that \(p \notin l \cup h\), since \(|C| = 2n = 1 + |l \cup h|\). As \(p\) belongs to \(n+1\) lines of \(\mathcal{A}\), we may choose a line \(s \in \mathcal{L}\) through \(p\) such that \(w \notin s\) and \(s\) meets both \(l\) and \(h\). Since \(C = \{p\} \cup l \cup h\), we have that \(s\) intersects \(C\) in exactly three points, namely \(p, l \cap s\) and \(h \cap s\). On the other hand, if \(S_1, S_2, \ldots, S_{n-1}\) are the \(n-1\) distinct blocks of \(\mathcal{D}\) such that \(s = S_1 \cap S_2 \cap \cdots \cap S_{n-1}\), we infer from condition (\(c_1\)) that \(S_1, S_2, \ldots, S_{n-1}\) are the only blocks of \(\mathcal{D}\) containing \(p, l \cap s, h \cap s\). Since
\{p, l \cap s, h \cap s\} \subset C$, we obtain $C = S_j$ for some $j \in \{1, 2, \ldots, n - 1\}$ and hence $s \subset C$. Therefore $s = s \cap C$ consists of three points, a contradiction. Thus $l$ and $h$ do not intersect and $l$ is the unique line of $A$ through $y$ not intersecting $h$, i.e. $l = k$. Therefore $z \in k$. As this is true for every point $z \in C \setminus h$ distinct from $y$ and $|C \setminus h| = n = |k|$, we may conclude that $C \setminus h = k$. So $C = h \cup k$ and (ii) holds.

As any parallel class of the affine plane $A = (P, L)$ consists of $n$ lines and $A$ has $n + 1$ parallel classes, we infer from (i) and (ii) that $D = (P, B)$ contains exactly $(n + 1)\frac{n(n - 1)}{2}$ blocks $X$ of the form $X = l \cup m$ with $l, m$ distinct parallel lines of $A$. But any $2 - (n^2, 2n, 2n - 1)$ design has precisely $b = (n + 1)\frac{n(n - 1)}{2}$ blocks. Then we must have

$$B = \{X \subset P : X = l \cup m \text{ with } l, m \text{ distinct parallel lines of } A\}$$

and hence $D = D(A, 2)$. The theorem is proved. \hfill \Box

Since up to isomorphism there is just one affine plane of order 5, 7 or 8 we have the following characterization of small designs $D(A, 2)$.

**Corollary 1** Suppose $n$ is one of the numbers 5, 7, 8 and let $A(n)$ be the desarguesian affine plane of order $n$. There exists up to isomorphisms exactly one $2 - (n^2, 2n, 2n - 1)$ design $D = (P, B)$ satisfying conditions (c$_1$), (c$_2$) of Theorem 1, namely the 2-design $D(A(n), 2)$.

We end our investigation with a few remarks

**Remark 1** If $A = (P, L)$ is a finite affine plane of order $n > 4$, then $0, 4, n$ are the intersection numbers of the $2 - (n^2, 2n, 2n - 1)$ design $D(A, 2)$: i.e. $\{0, 4, n\} = \{|X \cap Y| : X, Y \text{ are two distinct blocks of } D(A, 2)\}$.

**Remark 2** There is no plane of order $n = 6$, but there is an example of a $2 - (36, 12, 11)$ design produced by H. Hanany [3], Table 5.23, p. 343. The $2 - (25, 10, 9)$ design $D = (P, B)$ exhibited by H. Hanany, loc. cit. Table 5.23, p. 334 is not of the form $D(A, 2)$: since $D = (P, B)$ admits 8 as an intersection number (i.e. $|X \cap Y| = 8$ for suitable distinct blocks $X, Y \in B$).

**References**


