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Inversion of 3×3 Partitioned Matrices in Investigation of the Twoepoch Linear Model with the Nuisance Parameters

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Abstract

The estimation procedures in the multiepoch (and specially twoepoch) linear regression models with the nuisance parameters that were described in [2], Chapter 9, frequently need finding the inverse of a 3×3 partitioned matrix. We use different kinds of such inversion in dependence on simplicity of the result, similarly as in well known Rohde formula for 2×2 partitioned matrix. We will show some of these formulas, also methods how to get the other formulas, and then we applicate the formulas in estimation of the mean value parameters in the twoepoch linear regression model with the nuisance parameters.

Key words: Inversion of partitioned matrices; Rohde formula; twoepoch regression model; useful and nuisance parameters; best linear estimators of the mean value parameter.

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1 Notations

The following notation will be used throughout the paper:

$\mathbb{R}^n$	the space of all n-dimensional real vectors;
$\mathbf{u}$, $\mathbf{A}$	the real column vector, the real matrix;
$\mathbf{A}', r(\mathbf{A})$	the transpose, the rank of the matrix $\mathbf{A}$;

$\mathcal{M}(\mathbf{A})$, $\mathrm{Ker}(\mathbf{A})$	the range, the null space of the matrix $\mathbf{A}$;
$\mathbf{A}^-$	a generalized inverse of a matrix $\mathbf{A}$ (satisfying $\mathbf{A}\mathbf{A}^-\mathbf{A} = \mathbf{A}$);
$\mathbf{A}^+$	the Moore-Penrose generalized inverse of a matrix $\mathbf{A}$ (satisfying $\mathbf{A}\mathbf{A}^+\mathbf{A} = \mathbf{A}, \mathbf{A}^+\mathbf{A}\mathbf{A}^+ = \mathbf{A}^+$, $(\mathbf{A}\mathbf{A}^+)' = \mathbf{A}\mathbf{A}^+, (\mathbf{A}^+\mathbf{A})' = \mathbf{A}^+\mathbf{A}$);
$\mathbf{P}_A$	the orthogonal projector onto $\mathcal{M}(\mathbf{A})$ (in Euclidean sense);
$\mathbf{M}_A = \mathbf{I} - \mathbf{P}_A$	the orthogonal projector onto $\mathcal{M}^\perp(\mathbf{A}) = \mathrm{Ker}(\mathbf{A}')$;
$\mathbf{I}_k$	the $k \times k$ identity matrix;
$\mathbf{0}_{m,n}$	the $m \times n$ null matrix;
$\mathbf{1}_k$	$= (1, \ldots, 1)' \in \mathbb{R}^k$;
χ^2_r	random variable with chi squared distribution with r degrees of freedom;
$\chi^2_r(1-\alpha)$	$(1-\alpha)$-quantile of this distribution.

If $\mathcal{M}(\mathbf{A}) \subset \mathcal{M}(\mathbf{S})$, $\mathbf{S}$ positive semidefinite (p.s.d.), then the symbol $\mathbf{P}_A^{S^-}$ denotes the projector projecting vectors in $\mathcal{M}(\mathbf{S})$ onto $\mathcal{M}(\mathbf{A})$ along $\mathcal{M}(\mathbf{S}\mathbf{A}^\perp)$. A general representation of all such projectors $\mathbf{P}_A^{S^-}$ is given by

$$\mathbf{A}(\mathbf{A}'\mathbf{S}^-\mathbf{A})^-\mathbf{A}'\mathbf{S}^- + \mathbf{B}(\mathbf{I} - \mathbf{S}\mathbf{S}^-),$$

where $\mathbf{B}$ is arbitrary, (see [4], (2.14)). $\mathbf{M}_A^{S^-} = \mathbf{I} - \mathbf{P}_A^{S^-}$.

2 Inversion of partitioned matrices

Lemma 1 (Rohde) *Let*

$$\mathbf{D} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}' & \mathbf{C} \end{pmatrix}$$

be (symmetric) positive definite (p.d.). Then

$$\mathbf{D}^{-1} =$$

$$= \begin{pmatrix} (\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1} & -(\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1} \\ -\mathbf{C}^{-1}\mathbf{B}'(\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1} & \mathbf{C}^{-1} + \mathbf{C}^{-1}\mathbf{B}'(\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1} \end{pmatrix} \quad (1)$$

$$= \begin{pmatrix} \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}(\mathbf{C} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{B}'\mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{B}(\mathbf{C} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{B})^{-1} \\ -(\mathbf{C} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{B}'\mathbf{A}^{-1} & (\mathbf{C} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{B})^{-1} \end{pmatrix}. \quad (2)$$

Proof see [1, Theorem 8.5.11, p. 99].

Theorem 1 (Version I) *Let*

$$\mathbf{Q} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{D} \\ \mathbf{B}' & \mathbf{C} & \mathbf{F} \\ \mathbf{D}' & \mathbf{F}' & \mathbf{E} \end{pmatrix}$$

be p.d. Then

$$\mathbf{Q}^{-1} = \begin{pmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} & \mathbf{Q}_{13} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} & \mathbf{Q}_{23} \\ \mathbf{Q}_{31} & \mathbf{Q}_{32} & \mathbf{Q}_{33} \end{pmatrix},$$

where

$$\begin{aligned}
\mathbf{Q}_{11} &= [\mathbf{A} - \mathbf{D}\mathbf{E}^{-1}\mathbf{D}' - (\mathbf{B} - \mathbf{D}\mathbf{E}^{-1}\mathbf{F}')(\mathbf{C} - \mathbf{F}\mathbf{E}^{-1}\mathbf{F}')^{-1}(\mathbf{B}' - \mathbf{F}\mathbf{E}^{-1}\mathbf{D}')]^{-1}, \\
\mathbf{Q}_{12} &= -\mathbf{Q}_{11}(\mathbf{B} - \mathbf{D}\mathbf{E}^{-1}\mathbf{F}')(\mathbf{C} - \mathbf{F}\mathbf{E}^{-1}\mathbf{F}')^{-1}, \\
\mathbf{Q}_{13} &= -(\mathbf{Q}_{11}\mathbf{D} + \mathbf{Q}_{12}\mathbf{F})\mathbf{E}^{-1}, \\
\mathbf{Q}_{21} &= -(\mathbf{C} - \mathbf{F}\mathbf{E}^{-1}\mathbf{F}')^{-1}(\mathbf{B}' - \mathbf{F}\mathbf{E}^{-1}\mathbf{D}')\mathbf{Q}_{11} = (\mathbf{Q}_{12})', \\
\mathbf{Q}_{22} &= (\mathbf{C} - \mathbf{F}\mathbf{E}^{-1}\mathbf{F}')^{-1} + (\mathbf{C} - \mathbf{F}\mathbf{E}^{-1}\mathbf{F}')^{-1}(\mathbf{B}' - \mathbf{F}\mathbf{E}^{-1}\mathbf{D}')\mathbf{Q}_{11} \\
&\quad \times (\mathbf{B} - \mathbf{D}\mathbf{E}^{-1}\mathbf{F}')(\mathbf{C} - \mathbf{F}\mathbf{E}^{-1}\mathbf{F}')^{-1}, \\
\mathbf{Q}_{23} &= -(\mathbf{Q}_{21}\mathbf{D} + \mathbf{Q}_{22}\mathbf{F})\mathbf{E}^{-1}, \\
\mathbf{Q}_{31} &= -\mathbf{E}^{-1}(\mathbf{D}'\mathbf{Q}_{11} + \mathbf{F}'\mathbf{Q}_{21}) = (\mathbf{Q}_{13})', \\
\mathbf{Q}_{32} &= -\mathbf{E}^{-1}(\mathbf{D}'\mathbf{Q}_{12} + \mathbf{F}'\mathbf{Q}_{22}) = (\mathbf{Q}_{23})', \\
\mathbf{Q}_{33} &= \mathbf{E}^{-1} + \mathbf{E}^{-1}(\mathbf{D}'\mathbf{Q}_{11}\mathbf{D} + \mathbf{D}'\mathbf{Q}_{12}\mathbf{F} + \mathbf{F}'\mathbf{Q}_{21}\mathbf{D} + \mathbf{F}'\mathbf{Q}_{22}\mathbf{F})\mathbf{E}^{-1}.
\end{aligned}$$

Proof Let us denote

$$\mathbf{U} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}' & \mathbf{C} \end{pmatrix}, \qquad \mathbf{V} = \begin{pmatrix} \mathbf{D} \\ \mathbf{F} \end{pmatrix}.$$

The matrix $\mathbf{U}$ is p.d. so that we get with use of Lemma 1, formula (1)

$$\mathbf{Q}^{-1} = \begin{pmatrix} \mathbf{U} & \mathbf{V} \\ \mathbf{V}' & \mathbf{E} \end{pmatrix}^{-1}$$

$$\overset{(1)}{=} \begin{pmatrix} (\mathbf{U} - \mathbf{V}\mathbf{E}^{-1}\mathbf{V}')^{-1} & -(\mathbf{U} - \mathbf{V}\mathbf{E}^{-1}\mathbf{V}')^{-1}\mathbf{V}\mathbf{E}^{-1} \\ -\mathbf{E}^{-1}\mathbf{V}'(\mathbf{U} - \mathbf{V}\mathbf{E}^{-1}\mathbf{V}')^{-1} & \mathbf{E}^{-1} + \mathbf{E}^{-1}\mathbf{V}'(\mathbf{U} - \mathbf{V}\mathbf{E}^{-1}\mathbf{V}')^{-1}\mathbf{V}\mathbf{E}^{-1} \end{pmatrix}$$

with p.d. matrix

$$(\mathbf{U} - \mathbf{V}\mathbf{E}^{-1}\mathbf{V}')^{-1} = \begin{pmatrix} \mathbf{A} - \mathbf{D}\mathbf{E}^{-1}\mathbf{D}' & \mathbf{B} - \mathbf{D}\mathbf{E}^{-1}\mathbf{F}' \\ \mathbf{B}' - \mathbf{F}\mathbf{E}^{-1}\mathbf{D}' & \mathbf{C} - \mathbf{F}\mathbf{E}^{-1}\mathbf{F}' \end{pmatrix}^{-1}.$$

An application of Rohde formula (1) again and arrangement give us the desired result. □

Corollary 1 *Inverse of partitioned p.d. matrix*

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{D} \\ \mathbf{B}' & \mathbf{C} & \mathbf{0} \\ \mathbf{D}' & \mathbf{0} & \mathbf{E} \end{pmatrix}$$

is equal to

$$\begin{pmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} & \mathbf{Q}_{13} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} & \mathbf{Q}_{23} \\ \mathbf{Q}_{31} & \mathbf{Q}_{32} & \mathbf{Q}_{33} \end{pmatrix} =$$

$$= \begin{pmatrix} \mathbf{Q}_{11} & -\mathbf{Q}_{11}\mathbf{B}\mathbf{C}^{-1} & -\mathbf{Q}_{11}\mathbf{D}\mathbf{E}^{-1} \\ -\mathbf{C}^{-1}\mathbf{B}'\mathbf{Q}_{11} & \mathbf{C}^{-1} + \mathbf{C}^{-1}\mathbf{B}'\mathbf{Q}_{11}\mathbf{B}\mathbf{C}^{-1} & -\mathbf{Q}_{21}\mathbf{D}\mathbf{E}^{-1} \\ -\mathbf{E}^{-1}\mathbf{D}'\mathbf{Q}_{11} & -\mathbf{E}^{-1}\mathbf{D}'\mathbf{Q}_{12} & \mathbf{E}^{-1} + \mathbf{E}^{-1}\mathbf{D}'\mathbf{Q}_{11}\mathbf{D}\mathbf{E}^{-1} \end{pmatrix},$$

where

$$\mathbf{Q}_{11} = (\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}' - \mathbf{D}\mathbf{E}^{-1}\mathbf{D}')^{-1}.$$

Theorem 2 (Version II) *Let*

$$\mathbf{Q} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{D} \\ \mathbf{B}' & \mathbf{C} & \mathbf{F} \\ \mathbf{D}' & \mathbf{F}' & \mathbf{E} \end{pmatrix}$$

be p.d. Then

$$\mathbf{Q}^{-1} = \begin{pmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} & \mathbf{Q}_{13} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} & \mathbf{Q}_{23} \\ \mathbf{Q}_{31} & \mathbf{Q}_{32} & \mathbf{Q}_{33} \end{pmatrix},$$

where

$$\begin{aligned}
\mathbf{Q}_{11} &= (\mathbf{A} - \mathbf{D}\mathbf{E}^{-1}\mathbf{D}')^{-1} + (\mathbf{A} - \mathbf{D}\mathbf{E}^{-1}\mathbf{D}')^{-1}(\mathbf{B} - \mathbf{D}\mathbf{E}^{-1}\mathbf{F}')\mathbf{Q}_{22} \\
&\quad \times (\mathbf{B}' - \mathbf{F}\mathbf{E}^{-1}\mathbf{D}')(\mathbf{A} - \mathbf{D}\mathbf{E}^{-1}\mathbf{D}')^{-1}, \\
\mathbf{Q}_{12} &= -(\mathbf{A} - \mathbf{D}\mathbf{E}^{-1}\mathbf{D}')^{-1}(\mathbf{B} - \mathbf{D}\mathbf{E}^{-1}\mathbf{F}')\mathbf{Q}_{22}, \\
\mathbf{Q}_{13} &= -(\mathbf{Q}_{11}\mathbf{D} + \mathbf{Q}_{12}\mathbf{F})\mathbf{E}^{-1}, \\
\mathbf{Q}_{21} &= -\mathbf{Q}_{22}(\mathbf{B}' - \mathbf{F}\mathbf{E}^{-1}\mathbf{D}')(\mathbf{A} - \mathbf{D}\mathbf{E}^{-1}\mathbf{D}')^{-1}, \\
\mathbf{Q}_{22} &= [\mathbf{C} - \mathbf{F}\mathbf{E}^{-1}\mathbf{F}' - (\mathbf{B}' - \mathbf{F}\mathbf{E}^{-1}\mathbf{D}')(\mathbf{A} - \mathbf{D}\mathbf{E}^{-1}\mathbf{D}')^{-1}(\mathbf{B} - \mathbf{D}\mathbf{E}^{-1}\mathbf{F}')]^{-1}, \\
\mathbf{Q}_{23} &= -(\mathbf{Q}_{21}\mathbf{D} + \mathbf{Q}_{22}\mathbf{F})\mathbf{E}^{-1}, \\
\mathbf{Q}_{31} &= -\mathbf{E}^{-1}(\mathbf{D}'\mathbf{Q}_{11} + \mathbf{F}'\mathbf{Q}_{21}), \\
\mathbf{Q}_{32} &= -\mathbf{E}^{-1}(\mathbf{D}'\mathbf{Q}_{12} + \mathbf{F}'\mathbf{Q}_{22}), \\
\mathbf{Q}_{33} &= \mathbf{E}^{-1} + \mathbf{E}^{-1}(\mathbf{D}'\mathbf{Q}_{11}\mathbf{D} + \mathbf{D}'\mathbf{Q}_{12}\mathbf{F} + \mathbf{F}'\mathbf{Q}_{21}\mathbf{D} + \mathbf{F}'\mathbf{Q}_{22}\mathbf{F})\mathbf{E}^{-1}.
\end{aligned}$$

Proof follows directly from the proof of Theorem 1, if we use Rohde formula (2) instead of (1) in inverting p.d. matrix

$$\begin{pmatrix} \mathbf{A} - \mathbf{D}\mathbf{E}^{-1}\mathbf{D}' & \mathbf{B} - \mathbf{D}\mathbf{E}^{-1}\mathbf{F}' \\ \mathbf{B}' - \mathbf{F}\mathbf{E}^{-1}\mathbf{D}' & \mathbf{C} - \mathbf{F}\mathbf{E}^{-1}\mathbf{F}' \end{pmatrix}^{-1}. \qquad \square$$

Remark 1 (Version III & Version IV) We use (1) and (2) in inverting p.d. matrix

$$\mathbf{U} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}' & \mathbf{C} \end{pmatrix}$$

in

$$\mathbf{Q}^{-1} = \begin{pmatrix} \mathbf{U} & \mathbf{V} \\ \mathbf{V}' & \mathbf{E} \end{pmatrix}^{-1}$$

$$\stackrel{(2)}{=} \begin{pmatrix} \mathbf{U}^{-1} + \mathbf{U}^{-1}\mathbf{V}(\mathbf{E} - \mathbf{V}'\mathbf{U}^{-1}\mathbf{V})^{-1}\mathbf{V}'\mathbf{U}^{-1} & -\mathbf{U}^{-1}\mathbf{V}(\mathbf{E} - \mathbf{V}'\mathbf{U}^{-1}\mathbf{V})^{-1} \\ -(\mathbf{E} - \mathbf{V}'\mathbf{U}^{-1}\mathbf{V})^{-1}\mathbf{V}'\mathbf{U}^{-1} & (\mathbf{E} - \mathbf{V}'\mathbf{U}^{-1}\mathbf{V})^{-1} \end{pmatrix},$$

where $\mathbf{V} = (\mathbf{D}', \mathbf{F}')'$.

Remark 2 (Version V & Version VI) Let us denote

$$\mathbf{W} = (\mathbf{B}, \mathbf{D}), \qquad \mathbf{Z} = \begin{pmatrix} \mathbf{C} & \mathbf{F} \\ \mathbf{F}' & \mathbf{E} \end{pmatrix}$$

in

$$\mathbf{Q} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{D} \\ \mathbf{B}' & \mathbf{C} & \mathbf{F} \\ \mathbf{D}' & \mathbf{F}' & \mathbf{E} \end{pmatrix}.$$

The matrix $\mathbf{Z}$ is p.d. and using (1) we get

$$\mathbf{Q}^{-1} = \begin{pmatrix} \mathbf{A} & \mathbf{W} \\ \mathbf{W}' & \mathbf{Z} \end{pmatrix}^{-1}$$

$$\stackrel{(1)}{=} \begin{pmatrix} (\mathbf{A} - \mathbf{W}\mathbf{Z}^{-1}\mathbf{W}')^{-1} & -(\mathbf{A} - \mathbf{W}\mathbf{Z}^{-1}\mathbf{W}')^{-1}\mathbf{W}\mathbf{Z}^{-1} \\ -\mathbf{Z}^{-1}\mathbf{W}'(\mathbf{A} - \mathbf{W}\mathbf{Z}^{-1}\mathbf{W}')^{-1} & \mathbf{Z}^{-1} + \mathbf{Z}^{-1}\mathbf{W}'(\mathbf{A} - \mathbf{W}\mathbf{Z}^{-1}\mathbf{W}')^{-1}\mathbf{W}\mathbf{Z}^{-1} \end{pmatrix}.$$

The only thing that remains is to invert $\mathbf{Z}$ by (1) and (2).

Remark 3 (Version VII & Version VIII) Using Rohde formula (2) in p.d. matrix inversion

$$\mathbf{Q}^{-1} = \begin{pmatrix} \mathbf{A} & \mathbf{W} \\ \mathbf{W}' & \mathbf{Z} \end{pmatrix}^{-1}$$

we obtain

$$\begin{pmatrix} \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{W}(\mathbf{Z} - \mathbf{W}'\mathbf{A}^{-1}\mathbf{W})^{-1}\mathbf{W}'\mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{W}(\mathbf{Z} - \mathbf{W}'\mathbf{A}^{-1}\mathbf{W})^{-1} \\ -(\mathbf{Z} - \mathbf{W}'\mathbf{A}^{-1}\mathbf{W})^{-1}\mathbf{W}'\mathbf{A}^{-1} & (\mathbf{Z} - \mathbf{W}'\mathbf{A}^{-1}\mathbf{W})^{-1} \end{pmatrix}$$

with p.d. matrix

$$(\mathbf{Z} - \mathbf{W}'\mathbf{A}^{-1}\mathbf{W})^{-1} = \begin{pmatrix} \mathbf{C} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{B} & \mathbf{F} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{D} \\ \mathbf{F}' - \mathbf{D}'\mathbf{A}^{-1}\mathbf{B} & \mathbf{E} - \mathbf{D}'\mathbf{A}^{-1}\mathbf{D} \end{pmatrix}^{-1}.$$

An application of (1) and (2) again give us the result. For

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{D} \\ \mathbf{B}' & \mathbf{C} & \mathbf{F} \\ \mathbf{D}' & \mathbf{F}' & \mathbf{E} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} & \mathbf{Q}_{13} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} & \mathbf{Q}_{23} \\ \mathbf{Q}_{31} & \mathbf{Q}_{32} & \mathbf{Q}_{33} \end{pmatrix},$$

it is interesting to compare Version VIII,

$$\begin{aligned}
\mathbf{Q}_{11} &= \mathbf{A}^{-1} + \mathbf{A}^{-1}(\mathbf{B}\mathbf{Q}_{22}\mathbf{B}' + \mathbf{B}\mathbf{Q}_{23}\mathbf{D}' + \mathbf{D}\mathbf{Q}_{32}\mathbf{B}' + \mathbf{D}\mathbf{Q}_{33}\mathbf{D}')\mathbf{A}^{-1}, \\
\mathbf{Q}_{12} &= -\mathbf{A}^{-1}(\mathbf{B}\mathbf{Q}_{22} + \mathbf{D}\mathbf{Q}_{32}), \\
\mathbf{Q}_{13} &= -\mathbf{A}^{-1}(\mathbf{B}\mathbf{Q}_{23} + \mathbf{D}\mathbf{Q}_{33}), \\
\mathbf{Q}_{21} &= -(\mathbf{Q}_{22}\mathbf{B}' + \mathbf{Q}_{23}\mathbf{D}')\mathbf{A}^{-1}, \\
\mathbf{Q}_{22} &= (\mathbf{C} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{B})^{-1} + (\mathbf{C} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{B})^{-1}(\mathbf{F} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{D})\mathbf{Q}_{33} \\
&\quad \times (\mathbf{F}' - \mathbf{D}'\mathbf{A}^{-1}\mathbf{B})(\mathbf{C} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{B})^{-1}, \\
\mathbf{Q}_{23} &= -(\mathbf{C} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{B})^{-1}(\mathbf{F} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{D})\mathbf{Q}_{33}, \\
\mathbf{Q}_{31} &= -(\mathbf{Q}_{32}\mathbf{B}' + \mathbf{Q}_{33}\mathbf{D}')\mathbf{A}^{-1}, \\
\mathbf{Q}_{32} &= -\mathbf{Q}_{33}(\mathbf{F}' - \mathbf{D}'\mathbf{A}^{-1}\mathbf{B})(\mathbf{C} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{B})^{-1}, \\
\mathbf{Q}_{33} &= [\mathbf{E} - \mathbf{D}'\mathbf{A}^{-1}\mathbf{D} - (\mathbf{F}' - \mathbf{D}'\mathbf{A}^{-1}\mathbf{B})(\mathbf{C} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{B})^{-1}(\mathbf{F} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{D})]^{-1},
\end{aligned}$$

with Version I—it's in the certain sense "dual" form of Version VIII. Similar comparisons can be done with other couples of formulas.

3 Twoepoch linear model

The theory of the linear regression models is one of the established statistical disciplines and it may seem that nearly all has been investigated there. But this is valid only for the simplest structures of the linear models. In the practice we need to solve more and more complicated problems and investigation of corresponding structures of models is at the beginning. The formulas are quite complicated there but easy programmable and it enables us to get the estimations of unknown parameters in linear models.

The estimation procedures in multiepoch linear regression models with nuisance parameters and its application in geodesy were described in [2, Chapter 9]. But in the twoepoch case we can derive the estimations using convenient inverse of 3×3 partitioned matrices much easily so it legitimates to deal with them specially.

We derive optimum estimators of the useful mean value within a linear twoepoch model with the stable and variable (nonstable) parameters, when the data are affected by a systematic (deterministic) influence, i.e. by a noise which can be described by a linear model and whose parameters called nuisance, are estimable from results of the measurement. The subject of an interpretation are changes of the useful parameters in the single epochs and their characteristics of accuracy.

Sometimes the dimension of the useful mean value parameters is essentially smaller than that one of the nuisance parameter. In connection with this fact the problem occurs how to determine the optimum estimators of the useful parameters and their accuracy without evaluating in each epoch the large vector of the nuisance parameters.

One of the fundamental types of multiepoch and specially twoepoch model (which may exist also in the form with the nuisance parameters) was described in [2, p. 366].

Replicated measurements studying existence of deformation of some object and its course (if it exists) are realized in separate networks especially constructed for this purpose. It consists of a group of supporting points, whose position is assumed to be stable (this assumption—hypothesis—is verified during the measurement), and a group of points, whose movements related to the position of the stable points, are investigated (the coordinates of the group of the stable points are a priori unknown). As far as the processing of the measured results is concerned this means, that in the framework of each epoch and after finishing each epoch both the coordinates of the supporting points and the coordinates of the investigated points, are to be determined. The former serve to verify the above-mentioned hypothesis on the stableness of the group of supporting points.

Let us describe another example from the microeconomics practice. The progress of daily receipts in retail trade in the same months of two following years is observed. This progress usually consists of weekly period part and trend part. The weekly period doesn't change a lot because of conservative behaviour of the shoppers (i.e. useful stable parameters in expression of the entire linear model modelling the situation) in contrast to the trend. There is an influence of the commercial offers, inflation etc. (i.e. variable parameters; we suppose that the annual changes are not dramatical). The trend can be quite complicated and we need often only a small fraction of information that it contains. Here, the nonstable parameters in case of quadratic trend can be divided into the useful linear term parameter, that gives some pieces of information about increase or decrease of receipts, and two nuisance parameters (absolute term and quadratic term). The data in the above mentioned problem are usually characterized by a large dispersion and dependence among them.

The result of the measurement at the i-th time point in the first epoch could be described as

$$Y_{1i} = \beta_1 \cos \lambda t_{1i} + \beta_2 \sin \lambda t_{1i} + \gamma_1 t_{1i} + \kappa_{11} + \kappa_{12} t_{1i}^2 + \varepsilon_{1i}, \quad i = 1, \ldots, n_1$$

(λ is known from periodogram, see [5, p. 92]) and

$$Y_{2i} = \beta_1 \cos \lambda t_{2i} + \beta_2 \sin \lambda t_{2i} + \gamma_2 t_{2i} + \kappa_{21} + \kappa_{22} t_{2i}^2 + \varepsilon_{2i}, \quad i = 1, \ldots, n_2$$

in the second epoch. Here $\beta_1 \cos \lambda t_{ji} + \beta_2 \sin \lambda t_{ji}$ describes the weekly period (the measurements must begin with respect to this period in both epochs) and $\gamma_j t_{ji} + \kappa_{j1} + \kappa_{j2} t_{ji}^2$, $j = 1, 2$ the quadratical trend in the first and second epoch, respectively.

Let us consider the observation vector $\mathbf{Y} = (\mathbf{Y}_1', \mathbf{Y}_2')'$. The model described above could be rewritten in the form

$$\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{X}_1 & \mathbf{W}_1 & \mathbf{0} & \mathbf{Z}_1 & \mathbf{0} \\ \mathbf{X}_2 & \mathbf{0} & \mathbf{W}_2 & \mathbf{0} & \mathbf{Z}_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta} \\ \boldsymbol{\gamma} \\ \boldsymbol{\kappa} \end{pmatrix} + \begin{pmatrix} \boldsymbol{\varepsilon}_1 \\ \boldsymbol{\varepsilon}_2 \end{pmatrix}, \tag{3}$$

where

$$\mathbf{X}_1 = \begin{pmatrix} \cos \lambda t_{11} & \sin \lambda t_{11} \\ \vdots & \vdots \\ \cos \lambda t_{1n_1} & \sin \lambda t_{1n_1} \end{pmatrix}, \qquad \mathbf{X}_2 = \begin{pmatrix} \cos \lambda t_{21} & \sin \lambda t_{21} \\ \vdots & \vdots \\ \cos \lambda t_{2n_2} & \sin \lambda t_{2n_2} \end{pmatrix},$$

$$\mathbf{W}_1 = (t_{11}, \ldots, t_{1n_1})', \qquad \mathbf{W}_2 = (t_{21}, \ldots, t_{2n_2})',$$

$$\mathbf{Z}_1 = \begin{pmatrix} 1 & t_{11}^2 \\ \vdots & \vdots \\ 1 & t_{1n_1}^2 \end{pmatrix}, \qquad \mathbf{Z}_2 = \begin{pmatrix} 1 & t_{21}^2 \\ \vdots & \vdots \\ 1 & t_{2n_2}^2 \end{pmatrix},$$

$$\boldsymbol{\beta} = (\beta_1, \beta_2)', \qquad \boldsymbol{\gamma} = (\gamma_1, \gamma_2)', \qquad \boldsymbol{\kappa} = (\kappa_{11}, \kappa_{12}, \kappa_{21}, \kappa_{22})'.$$

The matrices $\mathbf{X}_1, \mathbf{X}_2, \mathbf{W}_1, \mathbf{W}_2, \mathbf{Z}_1, \mathbf{Z}_2$ are known, the vector $\boldsymbol{\beta}$ is a vector of the useful stable parameters, $\boldsymbol{\gamma}$ is a vector of the useful variable parameters and $\boldsymbol{\kappa}$ is a vector of the nuisance variable parameters.

With respect to above mentioned, let us consider the linear model (3), called the twoepoch model with the stable and nonstable parameters and with the nuisance parameters. We suppose that

- $(\mathbf{Y}_1', \mathbf{Y}_2')'$ is a (n_1+n_2)-dimensional random observation vector after the second epoch of measurement,
- $\boldsymbol{\beta} \in \mathbb{R}^k$ is a vector of the useful stable parameters, the same in both epochs,
- $\gamma = (\gamma_1', \gamma_2')' \in \mathbb{R}^{l_1+l_2}$ is a vector of the useful nonstable parameters in the first and the second epoch of measurement,
- $\boldsymbol{\kappa} = (\boldsymbol{\kappa}_1', \boldsymbol{\kappa}_2')' \in \mathbb{R}^{s_1+s_2}$ is a vector of the nuisance nonstable parameters in first and second epoch,
- $\mathbf{X}_1$, $\mathbf{X}_2$ are $n_1 \times k$, $n_2 \times k$ design matrices belonging to the vector $\boldsymbol{\beta}$,
- $\mathbf{W}_1$ is a $n_1 \times l_1$ design matrix belonging to the vector $\boldsymbol{\gamma}_1$,
- $\mathbf{W}_2$ is a $n_2 \times l_2$ design matrix belonging to the vector $\boldsymbol{\gamma}_2$,
- $\mathbf{Z}_1$ is a $n_1 \times s_1$ design matrix belonging to the vector $\boldsymbol{\kappa}_1$,
- $\mathbf{Z}_2$ is a $n_2 \times s_2$ design matrix belonging to the vector $\boldsymbol{\kappa}_2$.

We suppose that

1. $\mathrm{E}(\mathbf{Y}_1) = \mathbf{X}_1\boldsymbol{\beta} + \mathbf{W}_1\gamma_1 + \mathbf{Z}_1\boldsymbol{\kappa}_1$, $\mathrm{E}(\mathbf{Y}_2) = \mathbf{X}_2\boldsymbol{\beta} + \mathbf{W}_2\gamma_2 + \mathbf{Z}_2\boldsymbol{\kappa}_2$,
   $\forall\boldsymbol{\beta} \in \mathbb{R}^k, \forall\boldsymbol{\gamma}_1 \in \mathbb{R}^{l_1}, \forall\boldsymbol{\gamma}_2 \in \mathbb{R}^{l_2}, \forall\boldsymbol{\kappa}_1 \in \mathbb{R}^{s_1}, \forall\boldsymbol{\kappa}_2 \in \mathbb{R}^{s_2}$;
2. $\mathrm{var}\left[\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix}\right] = \begin{pmatrix} \boldsymbol{\Sigma}_1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_2 \end{pmatrix}$,
3. the matrix $\boldsymbol{\Sigma}_i$ is not a function of the vector $(\boldsymbol{\beta}', \gamma_i', \boldsymbol{\kappa}_i')'$ for $i = 1, 2$.

If the matrix $\begin{pmatrix} \boldsymbol{\Sigma}_1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_2 \end{pmatrix}$ is p.d. and

$$r\left[\begin{pmatrix} \mathbf{X}_1 & \mathbf{W}_1 & \mathbf{0} & \mathbf{Z}_1 & \mathbf{0} \\ \mathbf{X}_2 & \mathbf{0} & \mathbf{W}_2 & \mathbf{0} & \mathbf{Z}_2 \end{pmatrix}\right] = k + l_1 + l_2 + s_1 + s_2 < n_1 + n_2,$$

the model is said to be *regular* (see [2, p. 13]).

The described model arises by sequential realizations of the linear partial regression models,

$$\mathbf{Y}_1 = (\mathbf{X}_1, \mathbf{W}_1, \mathbf{Z}_1)\begin{pmatrix} \boldsymbol{\beta} \\ \gamma_1 \\ \boldsymbol{\kappa}_1 \end{pmatrix} + \boldsymbol{\varepsilon}_1, \quad \mathrm{var}(\mathbf{Y}_1) = \boldsymbol{\Sigma}_1 \tag{4}$$

and

$$\mathbf{Y}_2 = (\mathbf{X}_2, \mathbf{W}_2, \mathbf{Z}_2) \begin{pmatrix} \boldsymbol{\beta} \\ \gamma_2 \\ \boldsymbol{\kappa}_2 \end{pmatrix} + \varepsilon_2, \quad \mathrm{var}(\mathbf{Y}_2) = \boldsymbol{\Sigma}_2, \tag{5}$$

representing the model of the measurement within the first and second epoch, respectively.

Theorem 3 *The BLUE, i.e. the best linear unbiased estimator, of the parameters* $\boldsymbol{\beta}, \gamma_i, \boldsymbol{\kappa}_i$, $i = 1, 2$ *in the single first and second epoch modelled by (4) and (5), respectively, are*

$$\begin{aligned}
\widehat{\boldsymbol{\beta}}^{(i)} &= (\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{M}_{W_i}^{\Sigma_i^{-1}M_{Z_i}^{\Sigma_i^{-1}}}\mathbf{X}_i)^{-1}\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{M}_{W_i}^{\Sigma_i^{-1}M_{Z_i}^{\Sigma_i^{-1}}}\mathbf{Y}_i, \\
\widehat{\gamma_i}^{(i)} &= (\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i)^{-1}\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}(\mathbf{Y}_i - \mathbf{X}_i\widehat{\boldsymbol{\beta}}^{(i)}), \\
\widehat{\boldsymbol{\kappa}_i}^{(i)} &= (\mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i)^{-1}\mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}(\mathbf{Y}_i - \mathbf{X}_i\widehat{\boldsymbol{\beta}}^{(i)} - \mathbf{W}_i\widehat{\gamma}_i^{(i)}),
\end{aligned}$$

(Version I) and equivalently

$$\begin{aligned}
\widehat{\boldsymbol{\beta}}^{(i)} &= (\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i)^{-1}\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}(\mathbf{Y}_i - \mathbf{W}_i\widehat{\gamma}_i^{(i)}), \\
\widehat{\gamma_i}^{(i)} &= (\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{M}_{X_i}^{\Sigma_i^{-1}M_{Z_i}^{\Sigma_i^{-1}}}\mathbf{W}_i)^{-1}\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{M}_{X_i}^{\Sigma_i^{-1}M_{Z_i}^{\Sigma_i^{-1}}}\mathbf{Y}_i, \\
\widehat{\boldsymbol{\kappa}_i}^{(i)} &= (\mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i)^{-1}\mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}(\mathbf{Y}_i - \mathbf{X}_i\widehat{\boldsymbol{\beta}}^{(i)} - \mathbf{W}_i\widehat{\gamma}_i^{(i)}),
\end{aligned}$$

(Version II) for $i = 1, 2$.

Proof According to [2, Theorem 1.1.1, p. 13], the BLUE of the vector parameter $(\boldsymbol{\beta}', \gamma_i', \boldsymbol{\kappa}_i')'$, $i = 1, 2$, in each epoch separately, is given by

$$\begin{pmatrix} \widehat{\boldsymbol{\beta}}^{(i)} \\ \widehat{\gamma_i}^{(i)} \\ \widehat{\boldsymbol{\kappa}_i}^{(i)} \end{pmatrix} = \left[\begin{pmatrix} \mathbf{X}_i' \\ \mathbf{W}_i' \\ \mathbf{Z}_i' \end{pmatrix} \boldsymbol{\Sigma}_i^{-1} (\mathbf{X}_i, \mathbf{W}_i, \mathbf{Z}_i) \right]^{-1} \begin{pmatrix} \mathbf{X}_i' \\ \mathbf{W}_i' \\ \mathbf{Z}_i' \end{pmatrix} \boldsymbol{\Sigma}_i^{-1}\mathbf{Y}_i.$$

Using Theorem 1 and Theorem 2, the crucial point of the proof consists in the fact that

$$\left[\begin{pmatrix} \mathbf{X}_i' \\ \mathbf{W}_i' \\ \mathbf{Z}_i' \end{pmatrix} \boldsymbol{\Sigma}_i^{-1} (\mathbf{X}_i, \mathbf{W}_i, \mathbf{Z}_i) \right]^{-1} =$$

$$= \begin{pmatrix} \mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{X}_i & \mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{W}_i & \mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i \\ \mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{X}_i & \mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{W}_i & \mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i \\ \mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{X}_i & \mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{W}_i & \mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} & \mathbf{Q}_{13} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} & \mathbf{Q}_{23} \\ \mathbf{Q}_{31} & \mathbf{Q}_{32} & \mathbf{Q}_{33} \end{pmatrix},$$

where $(\mathbf{M}_{Z_i}^{\Sigma_i^{-1}} = \mathbf{I} - \mathbf{P}_{Z_i}^{\Sigma_i^{-1}} = \mathbf{I} - \mathbf{Z}_i(\mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i)^{-1}\mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1})$

$$
\begin{aligned}
\mathbf{Q}_{11} &\overset{\text{T.1}}{=} [\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i - \mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i(\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i)^{-1} \\
&\quad \times \mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i]^{-1} = (\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{M}_{W_i}^{\Sigma_i^{-1}M_{Z_i}^{\Sigma_i^{-1}}}\mathbf{X}_i)^{-1}, \\
\mathbf{Q}_{12} &= -\mathbf{Q}_{11}\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i(\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i)^{-1}, \\
\mathbf{Q}_{13} &= -\mathbf{Q}_{11}[\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i - \mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i(\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i)^{-1}\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i] \\
&\quad \times (\mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i)^{-1}, \\
\mathbf{Q}_{21} &= -(\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i)^{-1}\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i\mathbf{Q}_{11}, \\
\mathbf{Q}_{22} &= (\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i)^{-1} + (\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i)^{-1}\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i \\
&\quad \times \mathbf{Q}_{11}\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i(\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i)^{-1}, \\
\mathbf{Q}_{23} &= -(\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i)^{-1}[\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i - \mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i\mathbf{Q}_{11}\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i \\
&\quad + \mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i\mathbf{Q}_{11}\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i(\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i)^{-1} \\
&\quad \times \mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i](\mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i)^{-1}, \\
\mathbf{Q}_{31} &= -(\mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i)^{-1}[\mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{X}_i - \mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{W}_i(\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i)^{-1} \\
&\quad \times \mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i]\mathbf{Q}_{11}, \\
\mathbf{Q}_{32} &= -(\mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i)^{-1}[\mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{W}_i - \mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{X}_i\mathbf{Q}_{11}\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i \\
&\quad + \mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{W}_i(\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i)^{-1}\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}} \\
&\quad \times \mathbf{X}_i\mathbf{Q}_{11}\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i](\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i)^{-1}, \\
\mathbf{Q}_{33} &= (\mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i)^{-1} + (\mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i)^{-1}[\mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{X}_i\mathbf{Q}_{11}\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i \\
&\quad - \mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{X}_i\mathbf{Q}_{11}\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i(\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i)^{-1}\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i \\
&\quad - \mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{W}_i(\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i)^{-1}\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i\mathbf{Q}_{11}\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i \\
&\quad + \mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{W}_i(\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i)^{-1}\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i \\
&\quad + \mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{W}_i(\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i)^{-1}\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i\mathbf{Q}_{11}\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i \\
&\quad \times (\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i)^{-1}\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i](\mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i)^{-1}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{Q}_{11} &\overset{\text{T.2}}{=} (\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i)^{-1} + (\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i)^{-1}\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i\mathbf{Q}_{22} \\
&\quad \times \mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i(\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i)^{-1}, \\
\mathbf{Q}_{12} &= -(\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i)^{-1}\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i\mathbf{Q}_{22},
\end{aligned}
$$

$$\begin{aligned}
\mathbf{Q}_{13} &= -(\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i)^{-1}[\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Z}_i - \mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i\mathbf{Q}_{22}\mathbf{W}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Z}_i \\
&\quad + \mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i\mathbf{Q}_{22}\mathbf{W}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i(\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i)^{-1} \\
&\quad \times \mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Z}_i](\mathbf{Z}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Z}_i)^{-1}, \\
\mathbf{Q}_{21} &= -\mathbf{Q}_{22}\mathbf{W}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i(\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i)^{-1}, \\
\mathbf{Q}_{22} &= (\mathbf{W}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{M}_{X_i}^{\Sigma_i^{-1}M_{Z_i}^{\Sigma_i^{-1}}}\mathbf{W}_i)^{-1}, \\
\mathbf{Q}_{23} &= -\mathbf{Q}_{22}[\mathbf{W}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Z}_i - \mathbf{W}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i(\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i)^{-1}\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Z}_i] \\
&\quad \times (\mathbf{Z}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Z}_i)^{-1}, \\
\mathbf{Q}_{31} &= -(\mathbf{Z}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Z}_i)^{-1}[\mathbf{Z}_i'\mathbf{\Sigma}_i^{-1}\mathbf{X}_i - \mathbf{Z}_i'\mathbf{\Sigma}_i^{-1}\mathbf{W}_i\mathbf{Q}_{22}\mathbf{W}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i + \\
&\quad \mathbf{Z}_i'\mathbf{\Sigma}_i^{-1}\mathbf{X}_i(\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i)^{-1}\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i\mathbf{Q}_{22}\mathbf{W}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i] \\
&\quad \times (\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i)^{-1}, \\
\mathbf{Q}_{32} &= -(\mathbf{Z}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Z}_i)^{-1}[\mathbf{Z}_i'\mathbf{\Sigma}_i^{-1}\mathbf{W}_i - \mathbf{Z}_i'\mathbf{\Sigma}_i^{-1}\mathbf{X}_i(\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i)^{-1} \\
&\quad \times \mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i]\mathbf{Q}_{22}, \\
\mathbf{Q}_{33} &= (\mathbf{Z}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Z}_i)^{-1} + (\mathbf{Z}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Z}_i)^{-1}[\mathbf{Z}_i'\mathbf{\Sigma}_i^{-1}\mathbf{W}_i\mathbf{Q}_{22}\mathbf{W}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Z}_i \\
&\quad - \mathbf{Z}_i'\mathbf{\Sigma}_i^{-1}\mathbf{W}_i\mathbf{Q}_{22}\mathbf{W}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i(\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i)^{-1}\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Z}_i \\
&\quad - \mathbf{Z}_i'\mathbf{\Sigma}_i^{-1}\mathbf{X}_i(\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i)^{-1}\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i\mathbf{Q}_{22}\mathbf{W}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Z}_i \\
&\quad + \mathbf{Z}_i'\mathbf{\Sigma}_i^{-1}\mathbf{X}_i(\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i)^{-1}\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Z}_i \\
&\quad + \mathbf{Z}_i'\mathbf{\Sigma}_i^{-1}\mathbf{X}_i(\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i)^{-1}\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i\mathbf{Q}_{22}\mathbf{W}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i \\
&\quad \times (\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i)^{-1}\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Z}_i](\mathbf{Z}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Z}_i)^{-1},
\end{aligned}$$

respectively. Regarding that

$$\begin{aligned}
\widehat{\boldsymbol{\beta}}^{(i)} &= \mathbf{Q}_{11}\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Y}_i + \mathbf{Q}_{12}\mathbf{W}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Y}_i + \mathbf{Q}_{13}\mathbf{Z}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Y}_i, \\
\widehat{\boldsymbol{\gamma}_i}^{(i)} &= \mathbf{Q}_{21}\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Y}_i + \mathbf{Q}_{22}\mathbf{W}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Y}_i + \mathbf{Q}_{23}\mathbf{Z}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Y}_i, \\
\widehat{\boldsymbol{\kappa}_i}^{(i)} &= \mathbf{Q}_{31}\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Y}_i + \mathbf{Q}_{32}\mathbf{W}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Y}_i + \mathbf{Q}_{33}\mathbf{Z}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Y}_i,
\end{aligned}$$

$i = 1, 2$, the proof is complete. □

Notation 1 The model (3) can be rewritten as

$$\mathbf{Y} = (\mathbf{W}, \mathbf{Z})\begin{pmatrix} \boldsymbol{\delta} \\ \boldsymbol{\kappa} \end{pmatrix} + \boldsymbol{\varepsilon}, \tag{6}$$

where

$$\mathbf{Y} = \begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix}, \ \mathbf{W} = \begin{pmatrix} \mathbf{X}_1 & \mathbf{W}_1 & \mathbf{0} \\ \mathbf{X}_2 & \mathbf{0} & \mathbf{W}_2 \end{pmatrix}, \ \mathbf{Z} = \begin{pmatrix} \mathbf{Z}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{Z}_2 \end{pmatrix}, \ \boldsymbol{\delta} = \begin{pmatrix} \boldsymbol{\beta} \\ \boldsymbol{\gamma} \end{pmatrix}, \ \boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}$$

and

$$\boldsymbol{\Sigma} = \mathrm{var}(\mathbf{Y}) = \begin{pmatrix} \boldsymbol{\Sigma}_1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_2 \end{pmatrix},$$

so we get the (ordinary) linear model with nuisance parameters.

Proposition 1 *In the regular model (6) the BLUE of the parameter $(\boldsymbol{\delta}', \boldsymbol{\kappa}')'$ is given as*

$$\begin{pmatrix} \widehat{\boldsymbol{\delta}} \\ \widehat{\boldsymbol{\kappa}} \end{pmatrix} = \begin{pmatrix} (\mathbf{W}'\boldsymbol{\Sigma}^{-1}\mathbf{M}_Z^{\Sigma^{-1}}\mathbf{W})^{-1}\mathbf{W}'\boldsymbol{\Sigma}^{-1}\mathbf{M}_Z^{\Sigma^{-1}} \\ (\mathbf{Z}'\boldsymbol{\Sigma}^{-1}\mathbf{Z})^{-1}\mathbf{Z}'\boldsymbol{\Sigma}^{-1}\mathbf{M}_W^{\Sigma^{-1}M_Z^{\Sigma^{-1}}} \end{pmatrix} \mathbf{Y}. \tag{7}$$

Proof See [3, Theorem 1].

Theorem 4 *In the regular model (3) the BLUEs of the parameters $\boldsymbol{\beta}, \gamma_1, \gamma_2, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2$ are given as*

$$\begin{aligned}
\widehat{\boldsymbol{\beta}} &= (\mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}}\mathbf{M}_{W_1}^{\Sigma_1^{-1}M_{Z_1}^{\Sigma_1^{-1}}}\mathbf{X}_1 + \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}\mathbf{M}_{W_2}^{\Sigma_2^{-1}M_{Z_2}^{\Sigma_2^{-1}}}\mathbf{X}_2)^{-1} \\
&\quad \times (\mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}}\mathbf{M}_{W_1}^{\Sigma_1^{-1}M_{Z_1}^{\Sigma_1^{-1}}}\mathbf{Y}_1 + \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}\mathbf{M}_{W_2}^{\Sigma_2^{-1}M_{Z_2}^{\Sigma_2^{-1}}}\mathbf{Y}_2), \\
\widehat{\gamma_1} &= (\mathbf{W}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}}\mathbf{W}_1)^{-1}\mathbf{W}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}}(\mathbf{Y}_1 - \mathbf{X}_1\widehat{\boldsymbol{\beta}}), \\
\widehat{\gamma_2} &= (\mathbf{W}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}\mathbf{W}_2)^{-1}\mathbf{W}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}(\mathbf{Y}_2 - \mathbf{X}_2\widehat{\boldsymbol{\beta}}), \\
\widehat{\boldsymbol{\kappa}_1} &= (\mathbf{Z}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{Z}_1)^{-1}\mathbf{Z}_1'\boldsymbol{\Sigma}_1^{-1}(\mathbf{Y}_1 - \mathbf{X}_1\widehat{\boldsymbol{\beta}} - \mathbf{W}_1\widehat{\gamma_1}), \\
\widehat{\boldsymbol{\kappa}_2} &= (\mathbf{Z}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{Z}_2)^{-1}\mathbf{Z}_2'\boldsymbol{\Sigma}_2^{-1}(\mathbf{Y}_2 - \mathbf{X}_2\widehat{\boldsymbol{\beta}} - \mathbf{W}_2\widehat{\gamma_2}).
\end{aligned}$$

Proof According to Notation 1 we can use (7) to get the result. Here

$$\boldsymbol{\Sigma}^{-1}\mathbf{M}_Z^{\Sigma^{-1}} = \boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\mathbf{Z}(\mathbf{Z}'\boldsymbol{\Sigma}^{-1}\mathbf{Z})^{-1}\mathbf{Z}'\boldsymbol{\Sigma}^{-1} = \begin{pmatrix} \boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}} \end{pmatrix}$$

thus (we have used Corollary 1)

$$(\mathbf{W}'\boldsymbol{\Sigma}^{-1}\mathbf{M}_Z^{\Sigma^{-1}}\mathbf{W})^{-1} =$$

$$= \begin{pmatrix} \mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}}\mathbf{X}_1 + \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}\mathbf{X}_2 & \mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}}\mathbf{W}_1 & \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}\mathbf{W}_2 \\ \mathbf{W}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}}\mathbf{X}_1 & \mathbf{W}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}}\mathbf{W}_1 & \mathbf{0} \\ \mathbf{W}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}\mathbf{X}_2 & \mathbf{0} & \mathbf{W}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}\mathbf{W}_2 \end{pmatrix}^{-1}$$

$$\overset{\text{C.1}}{=} \begin{pmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} & \mathbf{Q}_{13} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} & \mathbf{Q}_{23} \\ \mathbf{Q}_{31} & \mathbf{Q}_{32} & \mathbf{Q}_{33} \end{pmatrix},$$

where

$$
\begin{aligned}
\mathbf{Q}_{11} &= (\mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}}\mathbf{M}_{W_1}^{\Sigma_1^{-1}M_{Z_1}^{\Sigma_1^{-1}}}\mathbf{X}_1 + \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}\mathbf{M}_{W_2}^{\Sigma_2^{-1}M_{Z_2}^{\Sigma_2^{-1}}}\mathbf{X}_2)^{-1},\\
\mathbf{Q}_{12} &= -\mathbf{Q}_{11}\mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}}\mathbf{W}_1(\mathbf{W}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}}\mathbf{W}_1)^{-1},\\
\mathbf{Q}_{13} &= -\mathbf{Q}_{11}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}\mathbf{W}_2(\mathbf{W}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}\mathbf{W}_2)^{-1},\\
\mathbf{Q}_{21} &= -(\mathbf{W}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}}\mathbf{W}_1)^{-1}\mathbf{W}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}}\mathbf{X}_1\mathbf{Q}_{11},\\
\mathbf{Q}_{22} &= (\mathbf{W}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}}\mathbf{W}_1)^{-1} + (\mathbf{W}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}}\mathbf{W}_1)^{-1}\mathbf{W}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}}\mathbf{X}_1\\
&\quad \times \mathbf{Q}_{11}\mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}}\mathbf{W}_1(\mathbf{W}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}}\mathbf{W}_1)^{-1},\\
\mathbf{Q}_{23} &= (\mathbf{W}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}}\mathbf{W}_1)^{-1}\mathbf{W}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}}\mathbf{X}_1\\
&\quad \times \mathbf{Q}_{11}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}\mathbf{W}_2(\mathbf{W}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}\mathbf{W}_2)^{-1},\\
\mathbf{Q}_{31} &= -(\mathbf{W}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}\mathbf{W}_2)^{-1}\mathbf{W}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}\mathbf{X}_2\mathbf{Q}_{11},\\
\mathbf{Q}_{32} &= (\mathbf{W}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}\mathbf{W}_2)^{-1}\mathbf{W}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}\mathbf{X}_2\\
&\quad \times \mathbf{Q}_{11}\mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}}\mathbf{W}_1(\mathbf{W}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}}\mathbf{W}_1)^{-1},\\
\mathbf{Q}_{33} &= (\mathbf{W}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}\mathbf{W}_2)^{-1} + (\mathbf{W}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}\mathbf{W}_2)^{-1}\mathbf{W}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}\mathbf{X}_2\\
&\quad \times \mathbf{Q}_{11}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}\mathbf{W}_2(\mathbf{W}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}\mathbf{W}_2)^{-1}.
\end{aligned}
$$

Utilizing that

$$
\mathbf{W}'\boldsymbol{\Sigma}^{-1}\mathbf{M}_Z^{\Sigma^{-1}} = \begin{pmatrix} \mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}} & \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}\\ \mathbf{W}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}} & \mathbf{0}\\ \mathbf{0} & \mathbf{W}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}} \end{pmatrix},
$$

we get (after some calculations) the BLUEs of the useful parameters $\boldsymbol{\beta}, \gamma_1, \gamma_2$. To get the same for the nuisance parameters $\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2$ it is sufficient to realize that

$$
(\mathbf{Z}'\boldsymbol{\Sigma}^{-1}\mathbf{Z})^{-1}\mathbf{Z}'\boldsymbol{\Sigma}^{-1} = \begin{pmatrix} (\mathbf{Z}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{Z}_1)^{-1}\mathbf{Z}_1'\boldsymbol{\Sigma}_1^{-1} & \mathbf{0}\\ \mathbf{0} & (\mathbf{Z}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{Z}_2)^{-1}\mathbf{Z}_2'\boldsymbol{\Sigma}_2^{-1} \end{pmatrix}
$$

and

$$
\begin{aligned}
\mathbf{M}_W^{\Sigma^{-1}M_Z^{\Sigma^{-1}}}\mathbf{Y} &= \mathbf{Y} - \mathbf{W}(\mathbf{W}'\boldsymbol{\Sigma}^{-1}\mathbf{M}_Z^{\Sigma^{-1}}\mathbf{W})^{-1}\mathbf{W}'\boldsymbol{\Sigma}^{-1}\mathbf{M}_Z^{\Sigma^{-1}}\mathbf{Y}\\
&= \begin{pmatrix} \mathbf{Y}_1\\ \mathbf{Y}_2 \end{pmatrix} - \begin{pmatrix} \mathbf{X}_1 & \mathbf{W}_1 & \mathbf{0}\\ \mathbf{X}_2 & \mathbf{0} & \mathbf{W}_2 \end{pmatrix}\begin{pmatrix} \widehat{\boldsymbol{\beta}}\\ \widehat{\gamma_1}\\ \widehat{\gamma_2} \end{pmatrix} = \begin{pmatrix} \mathbf{Y}_1 - \mathbf{X}_1\widehat{\boldsymbol{\beta}} - \mathbf{W}_1\widehat{\gamma_1}\\ \mathbf{Y}_2 - \mathbf{X}_2\widehat{\boldsymbol{\beta}} - \mathbf{W}_2\widehat{\gamma_2} \end{pmatrix}. \qquad \square
\end{aligned}
$$

Remark 4 Regarding that $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ are supposed to be positive definite, we can write (see [2, Lemma 10.1.35, p. 441])

$$\begin{aligned}
\mathbf{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}} &= \mathbf{\Sigma}_1^{-1} - \mathbf{\Sigma}_1^{-1}\mathbf{Z}_1(\mathbf{Z}_1'\mathbf{\Sigma}_1^{-1}\mathbf{Z}_1)^{-1}\mathbf{Z}_1'\mathbf{\Sigma}_1^{-1} = (\mathbf{M}_{Z_1}\mathbf{\Sigma}_1\mathbf{M}_{Z_1})^{+},\\
\mathbf{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}} &= \mathbf{\Sigma}_2^{-1} - \mathbf{\Sigma}_2^{-1}\mathbf{Z}_2(\mathbf{Z}_2'\mathbf{\Sigma}_2^{-1}\mathbf{Z}_2)^{-1}\mathbf{Z}_2'\mathbf{\Sigma}_2^{-1} = (\mathbf{M}_{Z_2}\mathbf{\Sigma}_2\mathbf{M}_{Z_2})^{+},
\end{aligned}$$

respectively.

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