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Uncertainty of Coordinates and Looking for Dispersion of GPS Receiver

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Abstract

The aim of the paper is to show some possible statistical solution of the estimation of the dispersion of the GPS receiver. The presented method (based on theory of linear model with additional constraints of type I) can serve for an improvement of the accuracy of estimators of coordinates acquired from the GPS receiver.

Key words: Two stage regression models; BLUE; uncertainty of the type A and B; confidence ellipsoids; variance components.

2000 Mathematics Subject Classification: 62J05, 62J10, 62F10

1 Introduction

The aim of this paper is to make one keep in view that the geographical coordinates, obtained with the help of a GPS receiver cannot be regarded as accurate data. Based on the results of one exemplary measurement, we will show that it is always necessary to take into account an uncertainty of data acquired from the GPS receiver. The user of the GPS receiver should always consider carefully if the measured values are sufficiently accurate with respect to the particular purposes. This conclusion can be drawn only in cases when an estimation of a dispersion of the GSP receiver is known in a given place and time.

In order to lower the uncertainty of the measurement, various measuring approaches are used. A repeated (multistage) measurement is one of such procedures. In addition, it is also well-known how to determine the estimation of the dispersion of the GPS receiver.

However, a possible situation can arise when the user of the device is not in a position to repeat the measurement several times during longer time interval. This can be caused either by a physical principle of a given design of the measurement or by practical aspects (e.g. expensiveness of the repeated measurement carried out for several days).

To avoid this difficulty, we will show another possible approach which leads to the estimation of the dispersion of the GPS receiver. Moreover, the presented method can serve for an improvement of the accuracy of data acquired from the GPS receiver.

In the following text, an algorithm based on the theory of estimation is introduced which would eventually decrease the uncertainty of the coordinates obtained from the GPS receiver with an utilization of an additional measurement (in our case, by a measuring tape). Even for an amateur measurement, the dispersion of the measured lengths is approximately about 0.1^2 m^2. From here and on, the uncertainty of the first-stage measurement is considered as the B-type uncertainty (in our case, the B-type uncertainty represents the uncertainty of the measurement by the measuring tape) and the lengths obtained in the first-stage measurement are denoted by a symbol Θ. On the contrary, the uncertainty of the second-stage mesurement is considered as the A-type uncertainty (in our case, the A-type uncertainty represents the uncertainty of the measurement by the GPS receiver) and the coordinates acquired in the second-stage measurement are denoted by a symbol β.

Motivation

Let us suppose the following situation. The goal was to determine a stochastic distribution of a chemical element in the soil. The coordinates of the positions, where the value of the chemical element was intended to be measured, have been acquired by the GPS receiver. The obtained values are depicted in Figure 1 where every point corresponds to the place where the sample was taken. According to the design of the measurement and principle of the utilized device, it was then expected that the acquired data would create an accumulation in the form of a ring.

As it is evident from Figure 1, the ring was generated from data for one "locality". However, the expected ring for the second locality was extended in comparison with the previous one. One may therefore ask the following questions. What were the reasons for such an anomalous behaviour of the measured data? Was it a consequence of the uncertainty of the acquired coordinates?

In the next example from another area of interest, it will be shown that the estimation of the dispersion of the GPS device is 0.354^2 m^2. This value may greatly differ depending on a number of available satellites, surrounding landscape and sedulity of the person performing the measurement. Therefore, the values acquired by the GPS receiver can exhibit different accuracy.

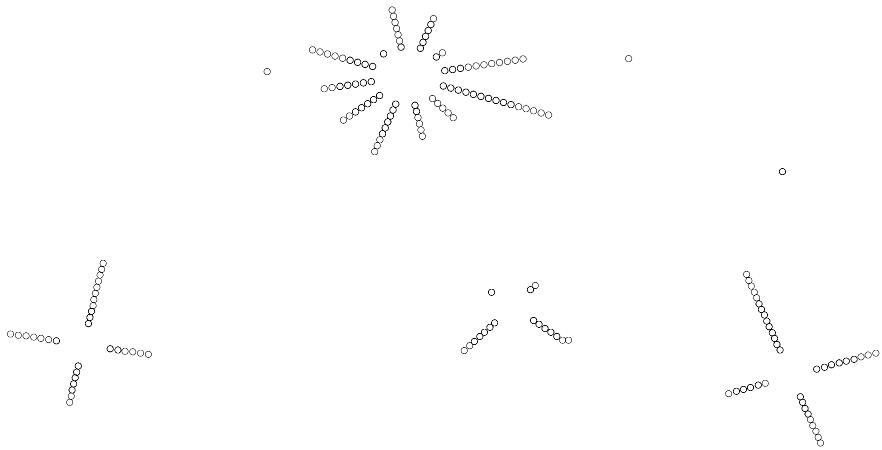


Figure 1: Coordinates of the measured points.

In the above-discussed example describing the measurement of the location points in the soil, it was found out that the student carrying out the measurement did not respect the instructions for a given measurement. The measurement was not performed all at once but there was a time delay between particular steps of the measurement.

Notation

The following notations will be used throughout the paper:

$\mathbb{R}^n$	space of all n-dimensional real vectors;
Θ	real column vector—from the first stage;
β	real column vector—from the second stage;
$\mathbf{I}_{m,m}$, $\mathbf{A}_{m,n}$	$m \times m$ identity matrix; real $m \times n$ matrix;
$\mathbf{A}_{r_1:s_1,r_2:s_2}$	$(s_1 - r_1) \times (s_2 - r_2)$ block matrix with elements of $\mathbf{A}$;
$\mathbf{A}', r(\mathbf{A}), \mathrm{Tr}(\mathbf{A})$	transpose, rank and trace of the matrix $\mathbf{A}$;
$\mathbf{A} = \mathrm{diag}(\mathbf{u})$	diagonal matrix with diagonal equal elements of vector $\mathbf{u}$;
$\mathcal{M}(\mathbf{A})$	column space of the matrix $\mathbf{A}$; $\mathcal{M}(\mathbf{A}) = \{\mathbf{Au} : \mathbf{u} \in \mathbb{R}^n\} \subset \mathbb{R}^m$;
$\mathrm{Ker}(\mathbf{A})$	null space of the matrix $\mathbf{A}$; $\mathrm{Ker}(\mathbf{A}) = \{\mathbf{u} : \mathbf{u} \in \mathbb{R}^n, \mathbf{Au} = \mathbf{0}\} \subset \mathbb{R}^n$;
$\mathbf{A}^-$	generalized inverse of the matrix $\mathbf{A}$ (satisfying $\mathbf{AA}^-\mathbf{A} = \mathbf{A}$), (see [4]);
$\mathrm{P}_{\mathbf{A}}$	orthogonal projector onto $\mathcal{M}(\mathbf{A})$ in Euclidean norm; $\mathrm{P}_{\mathbf{A}} = \mathbf{A}(\mathbf{A}'\mathbf{A})^-\mathbf{A}'$;
$\mathrm{M}_{\mathbf{A}}$	orthogonal projector onto $\mathcal{M}^\perp(\mathbf{A}) = \mathrm{Ker}(\mathbf{A}')$ in Euclidean norm; $\mathrm{M}_{\mathbf{A}} = \mathbf{I} - \mathrm{P}_{\mathbf{A}}$;
$\mathbf{Y} \sim (\mathbf{A}\Theta, \mathbf{T})$	observation vector $\mathbf{Y}$ with mean value $\mathbf{A}\Theta$ and covariance matrix $\mathbf{T}$.

2 Model of measurements

Definition 1 Let us consider the following linear model $\mathbf{Y} - \mathbf{D}\hat{\Theta} \sim_n (\mathbf{X}\beta, \Sigma_0)$, where $\mathbf{\Sigma}_0 = \sigma^2\mathbf{V}_1 + \mathbf{D}\mathbf{V}_0\mathbf{D}'$ and where $\mathbf{Y} \sim_n (\mathbf{D}\Theta + \mathbf{X}\beta, \sigma^2\mathbf{V}_1)$ is a random observation vector, $\beta \in \mathbb{R}^k$ stands for a vector of the useful parameters and $\mathbf{X}_{n,k}$ denotes a design matrix belonging to the vector β. We suppose that an estimator $\widehat{\Theta} \sim_{k_1} (\Theta, \mathbf{V}_0)$ of Θ is at our disposal only.

Theorem 1 *The standard estimator $\hat{\sigma}^2$ of the parameter σ^2 for the model defined in Definition 1 is given by the expression in the form of*

$$\begin{aligned}\hat{\sigma}^2 = {} & \lambda[(\mathbf{Y} - \mathbf{D}\widehat{\Theta})'(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^+\mathbf{V}_1(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^+(\mathbf{Y} - \mathbf{D}\widehat{\Theta})] \\ & - \mathrm{Tr}[(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^+\mathbf{V}_1(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^+\mathbf{V}_0]\,,\end{aligned}$$

where the value of the parameter λ is expressed by the following equation

$$\mathbf{S}_{(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^+}\lambda = 1\,,$$

where the 1×1 matrix $\mathbf{S}_{(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^+}$ takes the form of

$$\mathbf{S}_{(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^+} = \mathrm{Tr}[(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^+\mathbf{V}_1(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^+\mathbf{V}_1]\,.$$

Proof Firstly, we have the model in the form of

$$\mathbf{Y} \sim (\mathbf{X}\beta, \mathbf{V}_0 + \sigma^2\mathbf{V}_1),$$

where the estimator $\hat{\sigma}^2$ of the parameter σ^2 takes the form of

$$\hat{\sigma}^2 = \mathbf{Y}'\mathbf{A}\mathbf{Y} + \mathbf{a},$$

where $\mathbf{A}$ is a suitable matrix. Let $\mathbf{E}[\hat{\sigma}^2] = \sigma^2$, which is equivalent to $\mathbf{E}[\hat{\sigma}^2] = \mathrm{Tr}(\mathbf{A}\mathbf{V}_0) + \sigma^2\,\mathrm{Tr}(\mathbf{A}\mathbf{V}_1) + \beta'\mathbf{X}'\mathbf{A}\mathbf{X}\beta + \mathbf{a} = \sigma^2$. This implies that $\mathbf{a} = -\,\mathrm{Tr}(\mathbf{A}\mathbf{V}_0)$, $\mathrm{Tr}(\mathbf{A}\mathbf{V}_1) = 1$ and $\mathbf{X}'\mathbf{A}\mathbf{X} = 0$.

It is known (see [1]) that the matrix $\mathbf{A}$ in the form of $\mathbf{A} = \mathrm{M}_{\mathbf{X}}\mathbf{S}\mathrm{M}_{\mathbf{X}}$, where $\mathbf{S} = \mathbf{S}'$, satisfies the conditions $\mathbf{A}\mathbf{X} = \mathbf{0}$ and $\mathbf{A} = \mathbf{A}'$. This leads to the minimalization of the functional Φ defined as $\Phi = \mathrm{Tr}(\mathbf{A}\mathbf{\Sigma}_0\mathbf{A}\mathbf{\Sigma}_0) - 2\lambda\,\mathrm{Tr}(\mathbf{A}\mathbf{V}_1)$. This can be rewritten as

$$\Phi(\mathbf{S}) = \mathrm{Tr}(\mathbf{S}\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}}\mathbf{S}\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}}) - 2\lambda\,\mathrm{Tr}(\mathbf{S}\mathrm{M}_{\mathbf{X}}\mathbf{V}_1\mathrm{M}_{\mathbf{X}}).$$

As $\frac{\partial\Phi(\mathbf{S})}{\partial\mathbf{S}} = 0$, we arrive at $4(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})\mathbf{S}(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}}) = 4\lambda\mathrm{M}_{\mathbf{X}}\mathbf{V}_1\mathrm{M}_{\mathbf{X}}$. Now we have the matrix system in the form of $\mathbf{A}\mathbf{X}\mathbf{B} = \mathbf{C}$. The general solution of this matrix system is $\mathbf{X} = \mathbf{A}^-\mathbf{C}\mathbf{B}^- + \mathbf{Z} - \mathbf{A}^-\mathbf{A}\mathbf{Z}\mathbf{B}\mathbf{B}^-$ It is possible to show, that $\mathbf{X} = \mathbf{A}^+\mathbf{C}\mathbf{B}^+$ is also the solution (see [4]).
As $\mathrm{M}_{\mathbf{X}}\mathbf{S}\mathrm{M}_{\mathbf{X}} = \mathbf{A}$, it follows that

$$\mathbf{A} = \lambda(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^+\mathbf{V}_1(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^+\,.$$

With regard to the condition that $\mathrm{Tr}(\mathbf{A}\mathbf{V}_1) = 1$, we arrive at

$$\lambda\,\mathrm{Tr}[(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^+\mathbf{V}_1(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^+\mathbf{V}_1] = 1.$$

From this relation, we can obtain the result for the matrix $\mathbf{A}$ and the equation for the Lagrange parameter λ. We then get

$$\lambda = \frac{1}{\mathrm{Tr}[(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^{+}\mathbf{V}_1(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^{+}\mathbf{V}_1]},$$

$$\mathbf{A} = \frac{(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^{+}\mathbf{V}_1(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^{+}}{\mathrm{Tr}[(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^{+}\mathbf{V}_1(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^{+}\mathbf{V}_1]}.$$

Finally, the estimator $\hat{\sigma}^2$ of the parameter σ^2 for the matrix $\mathbf{\Sigma}_0$ can be now written as

$$\hat{\sigma}^2 = \mathbf{Y}'\mathbf{A}\mathbf{Y} - \mathrm{Tr}(\mathbf{A}\mathbf{V}_0) = \frac{\mathbf{Y}'(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^{+}\mathbf{V}_1(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^{+}\mathbf{Y}}{\mathrm{Tr}[(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^{+}\mathbf{V}_1(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^{+}\mathbf{V}_1]}$$
$$- \frac{\mathrm{Tr}[(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^{+}\mathbf{V}_1(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^{+}\mathbf{V}_0]}{\mathrm{Tr}[(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^{+}\mathbf{V}_1(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^{+}\mathbf{V}_1]}. \qquad \square$$

Hereafter we will focus on the same model but from a different point of view. We will consider the model of the measurement and then we will present how to determine the estimators of the fundamental parameters.

Definition 2 The model of connecting measurement will be represented by

$$(i) \qquad \begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} \sim \left[\begin{pmatrix} \mathbf{X}_1, & \mathbf{0} \\ \mathbf{D}, & \mathbf{X}_2 \end{pmatrix} \begin{pmatrix} \Theta \\ \beta \end{pmatrix}, \begin{pmatrix} \mathbf{\Sigma}_{11}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{\Sigma}_{22} \end{pmatrix} \right],$$

where $\mathbf{X}_1, \mathbf{D}, \mathbf{X}_2$ are known $n_1 \times k_1, n_2 \times k_1, n_2 \times k_2$ matrices, respectively, such that $\mathrm{M}(\mathbf{D}') \subset \mathrm{M}(\mathbf{X}_1')$; Θ and β are unknown k_1- and k_2-dimensional vectors; $\mathbf{\Sigma}_{22} = \sigma^2\mathbf{V}_1$, where $\mathbf{\Sigma}_{11}$ and $\mathbf{V}_1$ are known matrices.

In this model, the parameter Θ is estimated on the basis of the vector $\mathbf{Y}_1$ of the first stage and parameter β on the basis of the vectors $\mathbf{Y}_2 - \mathbf{D}\hat{\Theta}$ and $\hat{\Theta}$.

At this point, it should be mentioned that the results of the measurement from the second stage (i.e. $\mathbf{Y}_2$) cannot be used for a modification of the estimator $\hat{\Theta}$.

The parametric space $\underline{\Theta}$ of this model of connecting measurement $\mathbf{Y}$ is defined as

$$(ii) \qquad \underline{\Theta} = \{(\Theta', \beta')' : \mathbf{B}\beta + \mathbf{C}\Theta + \mathbf{a} = \mathbf{0}\},$$

where $\mathbf{B}$ and $\mathbf{C}$ are $q \times k_2$ and $q \times k_1$ matrices, $\mathbf{a}$ is q-dimensional vector, $r(\mathbf{B}) = q < k_2$.

Definition 3 The model in the parametric space $\underline{\Theta}$ (see Definition 2) is regular provided that $r(\mathbf{X}_1) = k_1$, $r(\mathbf{X}_2) = k_2$, $\mathbf{\Sigma}_{11}, \mathbf{\Sigma}_{22}$ are positively definite matrices, $r(\mathbf{B}) = q$.

Remark 1 The vector Θ represents the parameter of the first stage (connecting) whereas the vector β denotes the parameter of the second stage (connected). In the second stage, we then start with the unbiased estimator $\widehat{\Theta} = (\mathbf{X}_1'\mathbf{\Sigma}_{11}^{-1}\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{\Sigma}_{11}^{-1}\mathbf{Y}_1$ originating from the first stage whose covariance matrix is expressed in the form of $\mathrm{Var}(\widehat{\Theta}) = \mathbf{V}_0 = (\mathbf{X}_1'\mathbf{\Sigma}_{11}^{-1}\mathbf{X}_1)^{-1}$.

Definition 4 The least-square estimator of the parameter β, obtained under the condition that $\mathbf{\Sigma}_{11} = \mathbf{0}$ ($\Rightarrow \mathrm{Var}(\widehat{\Theta}) = \mathbf{0}$), is called the standard estimator if the vector Θ is substituted by $\widehat{\Theta}$ in this estimator.

Theorem 2 *The standard estimator $\widehat{\beta}$ of the parameter β in the model (i) and (ii) postulated in Definition 2 and given by*

$$\begin{aligned}\widehat{\beta} = {} & (\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}(\mathbf{Y}_2 - \mathbf{D}\widehat{\Theta}) \\ & - (\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}'[\mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}']^{-1} \\ & \times \{\mathbf{a} + \mathbf{C}\widehat{\Theta} + \mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}(\mathbf{Y}_2 - \mathbf{D}\widehat{\Theta})\},\end{aligned}$$

is unbiased.

Proof See [3], p. 72–73. □

Theorem 3 *If* $\mathrm{Var}(\widehat{\Theta}) \neq \mathbf{0}$ *then the covariance matrix of the standard estimator $\widehat{\beta}$ is composed of two uncertainties, i.e. the "uncertainty of type A" and "uncertainty of type B", as*

$$\mathrm{Var}(\widehat{\beta}) = \underbrace{\mathrm{Var}_0(\widehat{\beta})}_{\substack{\textit{uncertainty of} \\ \textit{type A}}} \underbrace{\begin{aligned}& + \langle\{\mathbf{I} - (\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}'[\mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}']^{-1}\mathbf{B}\} \\ & \times (\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{D} - (\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1} \\ & \times \mathbf{B}'[\mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}']^{-1}\mathbf{C}\rangle \\ & \times \mathrm{Var}(\widehat{\Theta}) \\ & \times \langle\{\mathbf{I} - (\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}'[\mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}']^{-1}\mathbf{B}\} \\ & \times (\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{D} - (\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1} \\ & \times \mathbf{B}'[\mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}']^{-1}\mathbf{C}\rangle'\end{aligned}}_{\substack{\textit{uncertainty of} \\ \textit{type B}}}$$

where

$$\begin{aligned}\mathrm{Var}_0(\widehat{\beta}) = {} & (\mathrm{M}_{\mathbf{B}'}\mathbf{X}_2'\mathbf{\Sigma}_{22}\mathbf{X}_2\mathrm{M}_{\mathbf{B}'})^{+} = (\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1} - (\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}' \\ & \times[\mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}']^{-1}\mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}.\end{aligned}$$

Proof See [3], p. 74. □

Corollary 1 *For the case of the model with* $\mathbf{X}_2 = \mathbf{I}$ *and* $\mathbf{D} = \mathbf{0}$, *the covariance matrix of the standard estimator is given by*

$$\begin{aligned}\mathrm{Var}(\widehat{\beta}) = {} & [\mathbf{I} - \mathbf{\Sigma}_{22}\mathbf{B}'(\mathbf{B}\mathbf{\Sigma}_{22}\mathbf{B}')^{-1}\mathbf{B}]\mathbf{\Sigma}_{22}[\mathbf{I} - \mathbf{B}'(\mathbf{B}\mathbf{\Sigma}_{22}\mathbf{B}')^{-1}\mathbf{B}\mathbf{\Sigma}_{22}] \\ & + \mathbf{\Sigma}_{22}\mathbf{B}'(\mathbf{B}\mathbf{\Sigma}_{22}\mathbf{B}')^{-1}\mathbf{C}\,\mathrm{Var}(\hat{\Theta})\mathbf{C}'(\mathbf{B}\mathbf{\Sigma}_{22}\mathbf{B}')^{-1}\mathbf{B}\mathbf{\Sigma}_{22}\,.\end{aligned}$$

Proof See [3], p. 73–74. □

Theorem 4 *The* $(1-\alpha)$*-confidence domain for the parameter* β, $\beta \in \underline{\Theta}$ *(see Definition 2), based on the standard* BLUE $\widehat{\beta}$, *is a set expressed by*

$$\mathcal{E}_{1-\alpha}(\beta) = \left\{ \mathbf{u} : \mathbf{u} \in \underline{\Theta}_\beta \subset \mathbb{R}^{k_2}, (\mathbf{u} - \widehat{\beta})'[\mathrm{Var}(\widehat{\beta})]^-(\mathbf{u} - \widehat{\beta}) \leq \chi^2_{r[\mathrm{Var}(\widehat{\beta})]}(1-\alpha) \right\}.$$

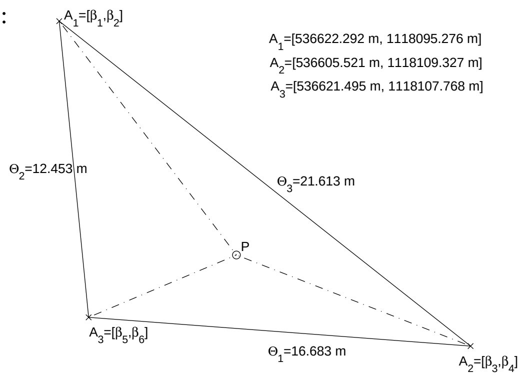
Here the symbol $\chi^2_{r[\mathrm{Var}(\widehat{\beta})]}(1-\alpha)$ *denotes* $(1-\alpha)$*-quantile of* χ^2*-distribution with* $r[\mathrm{Var}(\widehat{\beta})]$ *degrees of freedom.*

Proof See [2], p. 158–159. □

3 Illustrative example

The aim of this example is to find a dispersion for a CARMIN GPS 12XL navigator and estimate the plane coordinates β of the points A_1, A_2, A_3 in the Situation I and plane coordinates of the points A_1, A_2, A_3 and P in the Situation II using the theory of basic linear models of the measurement.

Situation I:



Situation II:

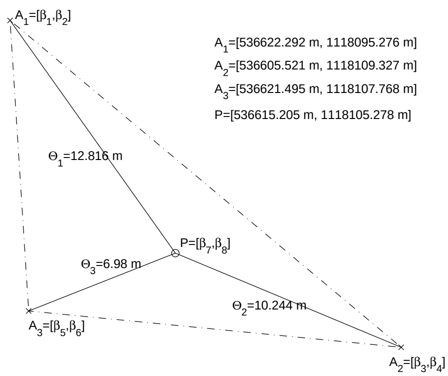


Figure 2: The polygonometric measurement.

We have given four points A_1, A_2, A_3 and P and their geographical specifications, i.e. their latitudes and longitudes, which have been obtained from a CARMIN GPS 12XL navigator. All points have been visualized on Fig. 2.

For our purposes, the geographical coordinates were transformed to the plane system known as S-JTSK (where $+x$-axes ... south, $+y$-axes ... west). For details on S-JTSK coordinates, see [5].

So we have estimated values of $A_i = (Y_{2i-1}, Y_{2i})$, $i = 1,2,3$ and measured values of $\widehat{\Theta}^I = (\widehat{\Theta}_1^I, \widehat{\Theta}_2^I, \widehat{\Theta}_3^I)'$ in the Situation I or we have estimated values of $A_i = (Y_{2i-1}, Y_{2i})$, $i = 1,2,3$, and $P = (Y_7, Y_8)$ and measured values of $\widehat{\Theta}^{II} = (\widehat{\Theta}_1^{II}, \widehat{\Theta}_2^{II}, \widehat{\Theta}_3^{II})'$ in the Situation II.

Let the result from the first and the second stage of measurement in the Situation I be $(\widehat{\Theta}_1^I, \widehat{\Theta}_2^I, \widehat{\Theta}_3^I)' = (16.683 \text{ m}, 12.453 \text{ m}, 21.613 \text{ m})'$ and

$$\mathbf{Y}^{Ig} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \end{pmatrix} = \begin{pmatrix} 49°38'02.2'' \\ 17°23'35.1'' \\ 49°38'01.8'' \\ 17°23'36.0'' \\ 49°38'01.8'' \\ 17°23'35.2'' \end{pmatrix} \rightarrow \mathbf{Y}^I = \begin{pmatrix} 536622.292 \text{ m} \\ 1118095.276 \text{ m} \\ 536605.521 \text{ m} \\ 1118109.327 \text{ m} \\ 536621.495 \text{ m} \\ 1118107.768 \text{ m} \end{pmatrix}.$$

In the Situation II, let the result from the first and the second stage of measurement be $(\widehat{\Theta}_1^{II}, \widehat{\Theta}_2^{II}, \widehat{\Theta}_3^{II})' = (12.816 \text{ m}, 10.244 \text{ m}, 6.980 \text{ m})'$ and

$$\mathbf{Y}^{IIg} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \\ Y_7 \\ Y_8 \end{pmatrix} = \begin{pmatrix} 49°38'02.2'' \\ 17°23'35.1'' \\ 49°38'01.8'' \\ 17°23'36.0'' \\ 49°38'01.8'' \\ 17°23'35.2'' \\ 49°38'01.9'' \\ 17°23'35.5'' \end{pmatrix} \rightarrow \mathbf{Y}^{II} = \begin{pmatrix} 536622.292 \text{ m} \\ 1118095.276 \text{ m} \\ 536605.521 \text{ m} \\ 1118109.327 \text{ m} \\ 536621.495 \text{ m} \\ 1118107.768 \text{ m} \\ 536614.788 \text{ m} \\ 1118105.885 \text{ m} \end{pmatrix}.$$

The accuracy of the measurement is given by the covariance matrix $\begin{pmatrix} \mathbf{V}_0 & \mathbf{0} \\ \mathbf{0} & \sigma^2\mathbf{V}_1 \end{pmatrix}$. Let $\widehat{\Theta}_1$, $\widehat{\Theta}_2$, $\widehat{\Theta}_3$ be the random variables with the mean values Θ_1, Θ_2, Θ_3, then

$$\mathbf{Y}_1 = \begin{pmatrix} \widehat{\Theta}_1 \\ \widehat{\Theta}_2 \\ \widehat{\Theta}_3 \end{pmatrix} \sim N_3 \left[\mathbf{X}_1 \begin{pmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \end{pmatrix} ; \mathbf{V}_0 \right].$$

In our case, we will consider the covariance matrices in the form of

$$\mathbf{V}_0 = \sigma_d^2 \times \begin{pmatrix} 1, & 0, & 0 \\ 0, & 1, & 0 \\ 0, & 0, & 1 \end{pmatrix},$$

where $\sigma_d^2 = 0.01^2 \text{m}^2$ and $\mathbf{X}_1 = \mathbf{I}_{3,3}$. Note that $\sigma_d^2 = 0.01^2 \text{m}^2$, especially the value of 0.01 m, is usually used for the value of the standard deviation of the measuring tape.

Let $Y_1, Y_2, Y_3, Y_4, Y_5, Y_6$ be the random variables with the mean values β_1, β_2, β_3, β_4, β_5, β_6, respectively, and dispersions $\sigma^2\mathbf{V}_1$.

$$\mathbf{Y}_2 = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \end{pmatrix} \sim N_6 \left[\mathbf{X}_2 \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \end{pmatrix} ; \sigma^2\mathbf{V}_1 \right].$$

We can use the covariance matrix in the form of

$$\mathbf{\Sigma}_{22} = \sigma^2\mathbf{V}_1 = \begin{pmatrix} \cos^2\varphi, & 0, & 0, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0, & 0, & 0 \\ 0, & 0, & \cos^2\varphi, & 0, & 0, & 0 \\ 0, & 0, & 0, & 1, & 0, & 0 \\ 0, & 0, & 0, & 0, & \cos^2\varphi, & 0 \\ 0, & 0, & 0, & 0, & 0, & 1 \end{pmatrix},$$

where $\sigma^2 = 3.1^2\mathrm{m}^2$, $\cos(\varphi) = \cos(49°) = 0.6564$ and $\mathbf{X}_2 = \mathbf{I}_{6,6}$. For the parameter σ^2 we will use the following value, calculated from

$$\sigma^2_{GPS} = \frac{2 \cdot \pi \cdot 6378 \cdot 1000}{360 \cdot 60 \cdot 60 \cdot 10} = 3.1^2\mathrm{m}^2,$$

where the expression above, especially the value of 3.1 m, denotes the standard deviation, derived from the smallest decimal digit which the GPS reciever displays.

The angle $\varphi = 49°$ stands for the value of the latitude where the measurement has been carried out.

Finally, we have the model given by

$$\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} = \begin{pmatrix} \widehat{\Theta}_1 \\ \widehat{\Theta}_2 \\ \widehat{\Theta}_3 \\ Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \end{pmatrix} \sim N_9 \left[\begin{pmatrix} \mathbf{X}_1, & \mathbf{0} \\ \mathbf{0}, & \mathbf{X}_2 \end{pmatrix} \begin{pmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \end{pmatrix} ; \begin{pmatrix} \mathbf{V}_0, & \mathbf{0} \\ \mathbf{0}, & \sigma^2\mathbf{V}_1 \end{pmatrix} \right].$$

Now, we can briefly describe the core of the example. We are in the position when we have the model expressed by

$$\mathbf{Y} = \mathbf{f}(\theta) + \epsilon, \tag{1}$$

$$\mathrm{Var}(\varepsilon) = \mathbf{\Sigma}_0, \tag{2}$$

where $\mathbf{\Sigma}_0 = \sigma^2 \mathbf{V}_1 + \mathbf{V}_0$. Here we can more closely rewrite the relation (1), i.e.

$$\mathbf{Y} = \mathbf{f}(\beta) + \varepsilon = \begin{pmatrix} \sqrt{(\beta_1-\beta_3)^2+(\beta_2-\beta_4)^2} \\ \sqrt{(\beta_3-\beta_5)^2+(\beta_4-\beta_6)^2} \\ \sqrt{(\beta_1-\beta_5)^2+(\beta_2-\beta_6)^2} \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \end{pmatrix} + \varepsilon. \tag{3}$$

In our example, we will consider the covariance matrices $\mathbf{W}_0 = (0.1)^2 \times \begin{pmatrix} \mathbf{I}_{3,3}, & \mathbf{0}_{3,6} \\ \mathbf{0}_{6,3}, & \mathbf{0}_{6,6} \end{pmatrix}$ and $\sigma^2 \mathbf{W}_1 = \sigma^2 \times \mathrm{diag}((0,0,0,1,\cos^2\varphi,1,\cos^2\varphi,1,\cos^2\varphi)')$ with $\sigma^2 = 3.1^2 \mathrm{m}^2$ and $\cos(\varphi) = \cos(49°) = 0.6564$.

For the function $\mathbf{f}$, we will generate the Taylor expansion at the suitable point which is given by

$$\mathbf{f}(\beta^1) = \mathbf{f}(\beta^0) + \mathbf{A}(\beta^1 - \beta^0).$$

According to the theory of the measurement, we have to define the matrix $\mathbf{A}$ that is given by $\mathbf{A} = \left(\frac{\partial \mathbf{f}}{\partial \Theta'}\right)$. As an illustration, the expression for $A_{3,6}$ takes the form of $A_{3,6} = \frac{\beta_6 - \beta_2}{\sqrt{\beta_5^2 - 2\beta_5\beta_1 + \beta_1^2 + \beta_6^2 - 2\beta_6\beta_2 + \beta_2^2}}$. The derivation of the other elements, i.e., $A_{1,1}$, $A_{1,2}$, $A_{1,3}$, $A_{1,4}$, $A_{2,3}$, $A_{2,4}$, $A_{2,5}$, $A_{2,6}$, $A_{3,1}$, $A_{3,2}$ and $A_{3,5}$, is analogous.

Now we will determine the estimator $\hat{\sigma}^2$ of the parameter σ^2 according to the Theorem 2.1. The whole process of determining the estimator $\hat{\sigma}^2$ can be now, according to the Theorem 2.1, written as

$$\begin{aligned} \hat{\sigma}^2 &= \lambda\{[(\mathbf{Y} - \mathbf{D}\widehat{\Theta})'(\mathrm{M}_\mathbf{A}\mathbf{\Sigma}_0\mathrm{M}_\mathbf{A})^+\mathbf{W}_1(\mathrm{M}_\mathbf{A}\mathbf{\Sigma}_0\mathrm{M}_\mathbf{A})^+(\mathbf{Y} - \mathbf{D}\widehat{\Theta})] && (4) \\ &\quad - \mathrm{Tr}[(\mathrm{M}_\mathbf{A}\mathbf{\Sigma}_0\mathrm{M}_\mathbf{A})^+\mathbf{W}_1(\mathrm{M}_\mathbf{A}\mathbf{\Sigma}_0\mathrm{M}_\mathbf{A})^+\mathbf{W}_0]\}, && (5) \end{aligned}$$

where the value of the parameter λ is expressed by the following equation

$$\mathbf{S}_{(\mathrm{M}_\mathbf{A}\mathbf{\Sigma}_0\mathrm{M}_\mathbf{A})^+}\lambda = 1, \tag{6}$$

where the 1×1 matrix $\mathbf{S}_{(\mathrm{M}_\mathbf{A}\mathbf{\Sigma}_0\mathrm{M}_\mathbf{A})^+}$ takes the form of

$$\mathbf{S}_{(\mathrm{M}_\mathbf{A}\mathbf{\Sigma}_0\mathrm{M}_\mathbf{A})^+} = \mathrm{Tr}[(\mathrm{M}_\mathbf{A}\mathbf{\Sigma}_0\mathrm{M}_\mathbf{A})^+\mathbf{W}_1(\mathrm{M}_\mathbf{A}\mathbf{\Sigma}_0\mathrm{M}_\mathbf{A})^+\mathbf{W}_1]. \tag{7}$$

By solving equations (5),(6) and (7) we have obtained $\lambda = 4.1751e^{-27}$ and $\hat{\sigma}^2 = (0.3540\ \mathrm{m})^2$.

We can say that the estimator of the uncertainty in GPS–coordinates is $\hat{\sigma}^2 = 0.3540^2\ \mathrm{m}^2$. Hereafter, we will focus on the same model but from a different point of view. We will consider the model of the measurement (i) and condition (ii) from Definition 2.2. Finally, we have in the Situation I the model given by

$$\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} = \begin{pmatrix} \widehat{\Theta}_1 \\ \widehat{\Theta}_2 \\ \widehat{\Theta}_3 \\ Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \end{pmatrix} \sim N_9 \left[\begin{pmatrix} \mathbf{X}_1, & \mathbf{0} \\ \mathbf{0}, & \mathbf{X}_2 \end{pmatrix} \begin{pmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \end{pmatrix}; \begin{pmatrix} \mathbf{\Sigma}_{11}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{\Sigma}_{22} \end{pmatrix} \right].$$

In our case, $\mathbf{X}_1 = \mathbf{I}$, $\mathbf{X}_2 = \mathbf{I}$, $\mathbf{\Sigma}_{11} = (\mathbf{W}_0)_{1:3,1:3}$ and $\mathbf{\Sigma}_{22} = (\sigma^2 \mathbf{W}_1)_{4:9,4:9}$ (see $\mathbf{W}_0$ and $\sigma^2 \mathbf{W}_1$ on p. 168).

One can observe from Figure 2 in the Situation I that the condition $\mathbf{g}(\Theta, \beta) = \mathbf{0}$ is implied for the parameters Θ and β, where

$$\begin{aligned} g_1(\Theta, \beta) &= (\beta_5 - \beta_3)^2 + (\beta_6 - \beta_4)^2 - \Theta_1^2, \\ g_2(\Theta, \beta) &= (\beta_5 - \beta_1)^2 + (\beta_6 - \beta_2)^2 - \Theta_2^2, \\ g_3(\Theta, \beta) &= (\beta_3 - \beta_1)^2 + (\beta_4 - \beta_2)^2 - \Theta_3^2. \end{aligned}$$

The linear version of the condition $\mathbf{g}(\Theta, \beta) = \mathbf{0}$, obtained using the Taylor expansion at the approximate point $(\Theta^0, \beta^0) = (\widehat{\Theta}_1, \widehat{\Theta}_2, \widehat{\Theta}_3, Y_1, Y_2, Y_3, Y_4, Y_5, Y_6)$, is in the form of $\mathbf{B}\delta\beta + \mathbf{C}\delta\Theta + \mathbf{a} = \mathbf{0}$, where $\delta\beta = \beta - \beta^0$, $\delta\Theta = \Theta - \Theta^0$, $\mathbf{B} = \frac{\partial \mathbf{g}(\Theta^0, \beta^0)}{\partial \beta'}$, $\mathbf{C} = \frac{\partial \mathbf{g}(\Theta^0, \beta^0)}{\partial \Theta'}$ and $\mathbf{a} = \mathbf{g}(\Theta^0, \beta^0)$.

Here we present the values of the vector of the estimator $\widehat{\beta^I}$ (calculated according to Theorem 2.2) based on the model with the measurement of all triangular lengths by the measuring tape. They are as follows:

$$\widehat{\beta^I} = \begin{pmatrix} 536621.930 \text{ m} \\ 1118095.923 \text{ m} \\ 536604.643 \text{ m} \\ 1118108.123 \text{ m} \\ 536622.735 \text{ m} \\ 1118108.324 \text{ m} \end{pmatrix}.$$

Its covariance matrix was calculated (see Corollary 2.1) leading to

$$\mathrm{Var}(\widehat{\beta^I}) = \begin{pmatrix} 1.2455 & 0.5064 & 0.0300 & -0.9437 & 0.1645 & 0.4373 \\ 0.5064 & 3.3361 & -0.2980 & 2.3743 & -0.2084 & 3.2896 \\ 0.0300 & -0.2980 & 0.7453 & 0.5556 & 0.6647 & -0.2576 \\ -0.9437 & 2.3743 & 0.5556 & 4.1672 & 0.3881 & 2.4585 \\ 0.1645 & -0.2084 & 0.6647 & 0.3881 & 0.6108 & -0.1797 \\ 0.4373 & 3.2896 & -0.2576 & 2.4585 & -0.1797 & 3.2519 \end{pmatrix}.$$

As $\mathrm{Tr}[\mathrm{Var}(\widehat{\beta}^I)] < \mathrm{Tr}(\mathbf{\Sigma}_{22})$ (see p. 169) it is evident that we get a better estimator of the coordinates of points A_1, A_2 and A_3.

Furthermore, in the same way, we will find estimator $\widehat{\beta}^{II}$ for model for the Situation II. In this situation, one can observe that the condition $\mathbf{g}(\Theta, \beta) = \mathbf{0}$ is implied for the parameters Θ and β, where

$$\begin{aligned} g_1(\Theta,\beta) &= (\beta_1-\beta_7)^2 + (\beta_2-\beta_8)^2 - \Theta_1^2, \\ g_2(\Theta,\beta) &= (\beta_3-\beta_7)^2 + (\beta_4-\beta_8)^2 - \Theta_2^2, \\ g_3(\Theta,\beta) &= (\beta_5-\beta_7)^2 + (\beta_6-\beta_8)^2 - \Theta_3^2. \end{aligned}$$

The linear version of the condition $\mathbf{g}(\Theta, \beta) = \mathbf{0}$, obtained using the Taylor expansion at the approximate point $(\Theta^0, \beta^0) = (\widehat{\Theta}_1, \widehat{\Theta}_2, \widehat{\Theta}_3, Y_1, Y_2, Y_3, Y_4, Y_5, Y_6)$, is in the form of $\mathbf{B}\delta\beta + \mathbf{C}\delta\Theta + \mathbf{a} = \mathbf{0}$, where $\delta\beta = \beta - \beta^0$, $\delta\Theta = \Theta - \Theta^0$, $\mathbf{B} = \frac{\partial \mathbf{g}(\Theta^0,\beta^0)}{\partial \beta'}$, $\mathbf{C} = \frac{\partial \mathbf{g}(\Theta^0,\beta^0)}{\partial \Theta'}$ and $\mathbf{a} = \mathbf{g}(\Theta^0, \beta^0)$.

Here we present the values of the vector of the estimator $\widehat{\beta}^{II}$ (calculated according to Theorem 2.2) based on the model with the measurement of 3 distances from triangular points to the inner point P by the measuring tape. The result is

$$\widehat{\beta}^{II} = \begin{pmatrix} 536622.416 \text{ m} \\ 1118094.184 \text{ m} \\ 536605.578 \text{ m} \\ 1118109.178 \text{ m} \\ 536621.752 \text{ m} \\ 1118108.403 \text{ m} \\ 536614.768 \text{ m} \\ 1118105.885 \text{ m} \end{pmatrix}.$$

Its covariance matrix (see Corollary 2.1) is given by

$$\mathrm{Var}(\widehat{\beta}^{II}) =$$
$$= \begin{pmatrix} 1.3685 & 0.6308 & 0.0797 & -0.2083 & -0.0443 & -0.1096 & 0.0361 & -0.3129 \\ 0.6308 & 3.4356 & -0.7031 & 1.8373 & 0.3907 & 0.9667 & -0.3185 & 2.7604 \\ 0.0797 & -0.7031 & 1.0072 & 1.1309 & 0.0464 & 0.1147 & 0.3067 & -0.5426 \\ -0.2083 & 1.8373 & 1.1309 & 6.0447 & -0.1212 & -0.2998 & -0.8014 & 1.4179 \\ -0.0443 & 0.3907 & 0.0464 & -0.1212 & 1.0490 & -0.9674 & 0.3889 & 0.6979 \\ -0.1096 & 0.9667 & 0.1147 & -0.2998 & -0.9674 & 6.6065 & 0.9623 & 1.7267 \\ 0.0361 & -0.3185 & 0.3067 & -0.8014 & 0.3889 & 0.9623 & 0.7083 & 0.1576 \\ -0.3129 & 2.7604 & -0.5426 & 1.4179 & 0.6979 & 1.7267 & 0.1576 & 3.0951 \end{pmatrix}.$$

As it has already been said before, we can use the outputs from the second step of our example as the input for the third part of the computation. This innovation of the algorithm could result in better estimators of our parameters.

Our task is to find the estimator of the coordinates of the point P and their covariance matrix and to determine the confidence ellipses for the coordinates of all points.

We now apply the same model like in the Situation II. We can consider another covariance matrix $\mathrm{Var}(\widehat{\beta}^I)$—the result from the Situation I, where we have better estimator of parameters than in the first stage of the measurement because of $\mathrm{Tr}[\mathrm{Var}(\widehat{\beta}^I)] < \mathrm{Tr}(\mathbf{\Sigma}_{22})$. Here we present the values of the vector of the estimator $\widehat{\beta}^{II'}$ (calculated according to Theorem 2.2). They are as follows:

$$\widehat{\beta}^{II'} = \begin{pmatrix} 536622.473 \text{ m} \\ 1118094.363 \text{ m} \\ 536605.361 \text{ m} \\ 1118109.341 \text{ m} \\ 536622.071 \text{ m} \\ 1118107.489 \text{ m} \\ 536614.607 \text{ m} \\ 1118106.183 \text{ m} \end{pmatrix}.$$

In this case, Corollary 2.1 gives the covariance matrix in the form of

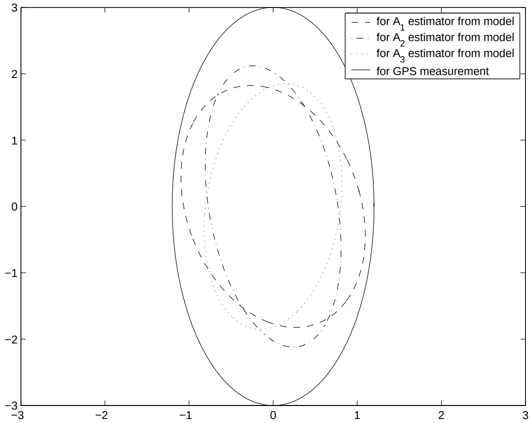
$$\mathrm{Var}(\widehat{\beta}^{II'}) =$$
$$= \begin{pmatrix} 1.1699 & 0.5461 & -0.0752 & -0.9390 & 0.0624 & 0.4752 & 0.2828 & -0.0823 \\ 0.5461 & 2.6197 & -0.2931 & 1.6152 & -0.2011 & 2.5707 & -0.0519 & 2.1943 \\ -0.0752 & -0.2931 & 0.5958 & 0.5062 & 0.5187 & -0.2560 & 0.4006 & 0.0429 \\ -0.9390 & 1.6152 & 0.5062 & 3.3453 & 0.3448 & 1.6967 & 0.0880 & 2.3429 \\ 0.0624 & -0.2011 & 0.5187 & 0.3448 & 0.4706 & -0.1745 & 0.3882 & 0.0308 \\ 0.4752 & 2.5707 & -0.2560 & 1.6967 & -0.1745 & 2.5309 & -0.0447 & 2.2017 \\ 0.2828 & -0.0519 & 0.4006 & 0.0880 & 0.3882 & -0.0447 & 0.3684 & 0.0086 \\ -0.0823 & 2.1943 & 0.0429 & 2.3429 & 0.0308 & 2.2017 & 0.0086 & 2.2611 \end{pmatrix}.$$

As we can see, it is possible to use estimator $\widehat{\beta}^I$ from the model for the Situation I for finding the estimator in the model for the Situation II.

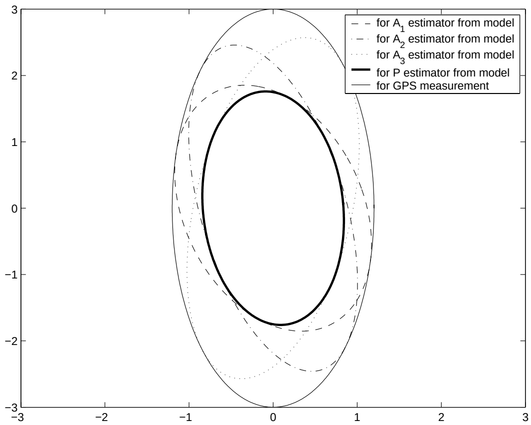
We have taken into account three different cases in which we have determined the possible way, how to obtain the coordinates from the GPS receiver, which shows a lesser uncertainty. These results, especially variances and residuals, for the first calculated situation are quite satisfactory. In the second situation we have not obtain better results because we have measured shorter distances. We have corrected this imperfection in the Situation II', where we have arrived at the best results. The essence of this method is based on the use of outputs of the Situation II as the input for the Situation II'.

The confidence ellipses obtained from calculated covariance matrix (based on Theorem 2.4) for $\alpha = 5$ % are depicted in Fig. 3.

Situation I:



Situation II:



Situation II':

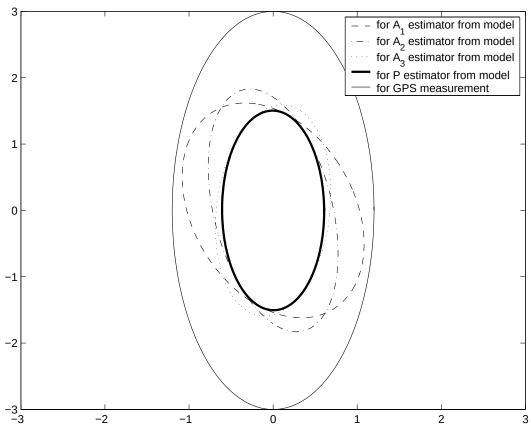


Figure 3: The (1-α) confidence ellipses for points A_1, A_2 and A_3 (solid line), for point $\hat{P}$ (bold solid line), for point $\hat{A}_1$ (dash line), for point $\hat{A}_2$ (dashdot line) and for point $\hat{A}_3$ (dot line).

4 Concluding remarks

We hope that our contribution has evidently pointed out a necessity to investigate the dispersion of the measuring device (the GPS receiver in our case) before the initiation of the measurement itself. In reality, a finding of the estimation of the dispersion can be complicated and infeasible in some cases. It may happen that the measurement cannot be repeated several times. Our proposed procedure, however, allows to estimate the dispersion without the measurement being repeated but with the help of the additional measurement (in our case, by a measure tape).

In the example worked out in this paper, we have calculated the values of the uncertainty of the GPS receiver which may have at the latitude of $\varphi = 49°$. Furthermore, our contribution have shown how the theory of estimation is a powerful tool for a modification of inaccurate data acquired by a measuring device (the GPS receiver in our case) with the utilization of the additional measurement. The example has also demonstrated a possibility of a successive improvement of the estimation by a further additional measurement.

References

[1] Kubáček, L., Kubáčková, L.: Statistics and Metrology. *Publishing House of Palacký University, Olomouc,* 2000, (in Czech)

[2] Kubáčková, L.: *Joint confidence and threshold ellipsoids in regression models.* Tatra Mountains Math. Publ. **7** (1996), 157–160.

[3] Marek, J.: *Estimation in connecting measurements.* Acta Univ. Palacki. Olomuc., Fac. rer. nat., Math. **42** (2003), 69–86.

[4] Rao, C. R., Mitra, S. K.: Generalized Inverse of Matrices and its Applications. *J. Wiley, New York,* 1971.

[5] Ryšavý, J.: Geodesy, *SNTL, Praha,* 1953 (in Czech).