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BIAS OF LS ESTIMATORS IN NONLINEAR REGRESSION MODELS WITH CONSTRAINTS. PART II: BIADDITIVE MODELS

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Abstract. General results giving approximate bias for nonlinear models with constrained parameters are applied to bilinear models in anova framework, called biadditive models. Known results on the information matrix and the asymptotic variance matrix of the parameters are summarized, and the Jacobians and Hessians of the response and of the constraints are derived. These intermediate results are the basis for any subsequent second order study of the model. Despite the large number of parameters involved, bias formulæ turn out to be quite simple due to the orthogonal structure of the model. In particular, the response estimators are shown to be approximately unbiased. Some simulations assess the validity of the approximations.

Keywords: asymptotic variance, bilinear model, nonlinear least squares, response function, second order approximation

MSC 2000: 62J02

1. Introduction

Bilinear models in anova framework date back to the work of Fisher and Mackenzie [10]. Analyzing a two-factor crossed experiment, these authors compare additive modelling $[\alpha_i + \beta_j]$ and multiplicative modelling $[\gamma_i\delta_j]$. Subscripts i and j denote the levels of the two factors of interest while Greek letters designate unknown parameters. The second step was made in 1936 by Eckart and Young [8]. They proposed the least-squares approximation of any matrix by a matrix of lower rank leading to the powerful tool of the singular value decomposition. Statistical models relying implicitly on this decomposition were independently proposed by Gilbert [12], Gollob [13], Mandel [16] and Johnson and Graybill [15] under the concept of multiplicative

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modelling of the interaction between two factors. These multiplicative models have proved to be very efficient tools for interpreting interactions between two factors, even when no replication is available. In the book by Gauch [11] an extensive list of references can be found about the subject. At the same time independent but consistent results about the asymptotic variances of the maximum likelihood estimators of the parameters appeared in Goodman and Haberman [14], Chadœuf and Denis [1], Denis and Gower [2, 4, 5] and Dorkenoo and Mathieu [7].

Biadditive terminology introduced in Denis and Gower [2, 3] highlights the bilinear nature of these models. Following their notation, we will deal here with models $B(m, a, b, \pi_r)$, i.e. models with an additive part; but our results remain almost unchanged for members of the biadditive family which are orthogonal. It is only the $B(m, *, *, \pi_r)$ models, promoted under the name of shifted multiplicative models by Seyedesadr and Cornelius [18], which are excluded here.

Here we apply to the $B(m, a, b, \pi_r)$ models the asymptotic bias formulæ proposed by Pázman and Denis [17] for general nonlinear models when the parameters are constrained by nonlinear equalities, continuing the path opened by Silvey [19].

Before deriving the bias, the model is presented and maximum likelihood estimators of the expectation parameters are given, as are the information and asymptotic variance matrices. Jacobians and Hessians for the response and constraint set are stated, and these in turn are of use for any further second order analyses of the model. Besides the basic parameterization proposed in Section 2.1, we present another commonly used parameterization (Section 4.1). Bias functions are obtained for both types of parameterization and for the response function. Interestingly, although most of the developments presented here are very technical, the final results are surprisingly simple. For example, the bias of the nonlinear parameters of the model is given by simple formulæ in Theorem 9 and Proposition 13: parts of bias are colinear to the corresponding vectors of parameters. Another nice result is that the approximate bias of the response function is zero (Proposition 14).

Full numerical checking of the formulæ have been carried out, most of them are presented elsewhere (see Denis and Pázman [6]). Finally, with some simulations the validity range of the approximations proposed is studied. SPLUS functions of all results presented in the paper are available under request from the first author.

2. Model

2.1. Definition

Biadditive models considered here read

$$
(1)\quad \begin{cases}
\textit{formula} & y_{(i,j)} = \mu + \alpha_i + \beta_j + \sum_{u=1}^{r} \gamma_{iu}\delta_{ju} + \varepsilon_{(i,j)} \\
\textit{moments} & E[\varepsilon_{(i,j)}] = 0; \quad \mathrm{Cov}[\varepsilon_{(i,j)}, \varepsilon_{(i,'j')}] = \begin{cases} \sigma^2 \text{ when } (i,j) = (i',j') \\ 0 \text{ otherwise} \end{cases} \\
 & (\varphi_\alpha^C):\quad \sum_i \alpha_i = 0 \qquad 1 \\
 & (\varphi_\beta^C):\quad \sum_j \beta_j = 0 \qquad 1 \\
 & (\varphi_{\gamma,u}^C):\quad \sum_i \gamma_{iu} = 0 \quad \forall u \qquad r \\
\textit{constraints} & (\varphi_{\delta,u}^C):\quad \sum_j \delta_{ju} = 0 \quad \forall u \qquad r \\
 & (\varphi_u^N):\quad \frac{1}{2}\left(\sum_i \gamma_{iu}^2 - \sum_j \delta_{ju}^2\right) = 0 \quad \forall u \qquad r \\
 & (\varphi_{\gamma,u,v}^O):\quad \sum_i \gamma_{iu}\gamma_{iv} = 0 \quad \forall u < v \qquad \frac{r(r-1)}{2} \\
 & (\varphi_{\delta,u,v}^O):\quad \sum_j \delta_{ju}\delta_{jv} = 0 \quad \forall u < v \qquad \frac{r(r-1)}{2}
\end{cases}
$$

where $i \in \{1 \dots I\}$ and $j \in \{1 \dots J\}$ are the levels of two factors, say the row-factor column-factor, respectively, having effect on the variate of interest y; and r, the number of multiplicative terms, is less than or equal to $\min(I-1, J-1)$. The first three terms $(\mu + \alpha_i + \beta_j)$ correspond to additive modelling (linear part of the model), the other terms, $\sum_{u=1}^{r} \gamma_{iu}\delta_{ju}$, correspond to the modelled interaction (the nonlinear part of the model, in fact bilinear). By φ_α^C, φ_β^C, $\varphi_{\gamma,u}^C$, $\varphi_{\delta,u}^C$, φ_u^N, $\varphi_{\gamma,u,v}^O$ and $\varphi_{\delta,u,v}^O$ we denote the constraints on the parameters. The numbers following each constraint definition indicate the number of constraints generated (by varying the subscripts u and v).

This model is a special case of Model (1-2) in [17]: the number of parameters is $p = 1+(r+1)(I+J)$, the number of observations is $n = IJ$ and the number of constraints is $q = 2 + 2r + r^2$. For the sake of simplicity of notation, let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_I)^T$, $\boldsymbol{\beta} = (\beta_1, \beta_2, \dots, \beta_J)^T$, $\boldsymbol{\gamma}_u = (\gamma_{1u}, \gamma_{2u}, \dots, \gamma_{Iu})^T$ and $\boldsymbol{\delta}_u = (\delta_{1u}, \delta_{2u}, \dots, \delta_{Ju})^T$ $\forall u = 1, \dots, r$. The parameters will be ordered in the following way:

$$\theta = (\mu, \boldsymbol{\alpha}^T, \boldsymbol{\beta}^T, \boldsymbol{\gamma}_1^T, \boldsymbol{\delta}_1^T, \boldsymbol{\gamma}_2^T, \boldsymbol{\delta}_2^T, \dots, \boldsymbol{\gamma}_r^T, \boldsymbol{\delta}_r^T)^T$$

and we will distinguish between the additive parameters

$$\theta^A = (\mu, \boldsymbol{\alpha}^T, \boldsymbol{\beta}^T)^T \text{ of size } p^A$$

and the biadditive parameters

$$\theta^B = (\boldsymbol{\gamma}_1^T, \boldsymbol{\delta}_1^T, \boldsymbol{\gamma}_2^T, \boldsymbol{\delta}_2^T, \ldots, \boldsymbol{\gamma}_r^T, \boldsymbol{\delta}_r^T)^T \text{ of size } p^B.$$

It will be shown in Section 2.2.1 by the ranks of the Jacobians that the constraints on the parameters are only identifiable and are independent. There are two centering constraints, (φ_α^C) and (φ_β^C), for the additive part and $r(2+r)$ constraints associated with the multiplicative terms:

- centering: $(\varphi_{\gamma,u}^C)$ and $(\varphi_{\delta,u}^C)$
- equality of norms: (φ_u^N)
- orthogonality between the $\boldsymbol{\gamma}$: $(\varphi_{\gamma,u,v}^O)$
- orthogonality between the $\boldsymbol{\delta}$: $(\varphi_{\delta,u,v}^O)$

Although these constraints ensure the local identifiability of the model, global identifiability requires more properties. The $\boldsymbol{\gamma}$'s (and consequently also the $\boldsymbol{\delta}$'s) are ordered according to their norm:

$$\boldsymbol{\gamma}_1^T \boldsymbol{\gamma}_1 \geqslant \boldsymbol{\gamma}_2^T \boldsymbol{\gamma}_2 \geqslant \ldots \geqslant \boldsymbol{\gamma}_r^T \boldsymbol{\gamma}_r.$$

Even so, global identifiability is not guaranteed. For instance, the signs of any $\boldsymbol{\gamma}_u$ and $\boldsymbol{\delta}_u$ can be simultaneously changed without modifying the response. Moreover, if some norms are equal ($\boldsymbol{\gamma}_u^T \boldsymbol{\gamma}_u = \boldsymbol{\gamma}_{u+1}^T \boldsymbol{\gamma}_{u+1}$) any rotations on these vectors and the associated rotations on $\boldsymbol{\delta}_u$ and $\boldsymbol{\delta}_{u+1}$ can be performed without changing the response. Hence, we will suppose that

$$\boldsymbol{\gamma}_1^T \boldsymbol{\gamma}_1 > \boldsymbol{\gamma}_2^T \boldsymbol{\gamma}_2 > \ldots > \boldsymbol{\gamma}_r^T \boldsymbol{\gamma}_r > 0.$$

Even more, to maintain the compactness of the parameter space which is required in Pázman and Denis [17], we will suppose that

$$\frac{1}{\kappa} \geqslant \boldsymbol{\gamma}_1^T \boldsymbol{\gamma}_1 + \kappa \geqslant \boldsymbol{\gamma}_2^T \boldsymbol{\gamma}_2 + \kappa \geqslant \ldots \geqslant \boldsymbol{\gamma}_r^T \boldsymbol{\gamma}_r + \kappa \geqslant 2\kappa$$

for some small $\kappa > 0$. If κ is very small, this has no noticeable influence on statistical considerations.

2.2. Structure

Ordering the subscripts (i, j) by varying first i, we can write Model (1) in a vector form:

$$E[\mathbf{y}] = \eta(\theta) = (\mathbf{1}_J \otimes \mathbf{1}_I)\, \mu + \mathbf{1}_J \otimes \boldsymbol{\alpha} + \boldsymbol{\beta} \otimes \mathbf{1}_I + \sum_{u=1}^{r} \boldsymbol{\delta}_u \otimes \boldsymbol{\gamma}_u. \tag{2}$$

Here $\mathbf{1}_s$ denotes the column vector of $\mathbb{R}^s$ with all entries equal to 1.

2.2.1. Jacobians and Hessians.

Proposition 1. *The Jacobian* $J(\theta) = \frac{\partial \eta(\theta)}{\partial \theta^T}$ *of Model* (1) *is given by*

$$(\mathbf{1}_J \otimes \mathbf{1}_I, \mathbf{1}_J \otimes \mathbf{I}_I, \mathbf{I}_J \otimes \mathbf{1}_I, \boldsymbol{\delta}_1 \otimes \mathbf{I}_I, \mathbf{I}_J \otimes \boldsymbol{\gamma}_1, \ldots, \boldsymbol{\delta}_r \otimes \mathbf{I}_I, \mathbf{I}_J \otimes \boldsymbol{\gamma}_r). \tag{3}$$

Its rank is $(1+r)(I+J-(1+r))$

Proof. $J(\theta) = \frac{\partial \eta}{\partial \theta^T}$ is an IJ times $(1+I+J)+r(I+J)$ matrix. Following the distinction previously made between the additive and biadditive parameters, it is convenient to consider separately the part corresponding to the additive parameters denoted by $J(\theta^A)$ and the part corresponding to the biadditive parameters denoted by $J(\theta^B)$. Of course the complete Jacobian is given by

$$J(\theta) = \left(J(\theta^A), J(\theta^B)\right).$$

Straightforward derivations give

$$J(\theta^A) = (\mathbf{1}_J \otimes \mathbf{1}_I, \mathbf{1}_J \otimes \mathbf{I}_I, \mathbf{I}_J \otimes \mathbf{1}_I).$$

For the biadditive part the basic derivation is $\frac{\partial(\boldsymbol{\delta}_u \otimes \boldsymbol{\gamma}_u)}{\partial(\boldsymbol{\gamma}_u^T, \boldsymbol{\delta}_u^T)}$. It is immediate, once one has established the order of subscripts. We obtain

$$\begin{aligned}\frac{\partial(\boldsymbol{\delta}_u \otimes \boldsymbol{\gamma}_u)}{\partial \boldsymbol{\gamma}_u^T} &= \frac{\partial\left(\delta_{1u}\boldsymbol{\gamma}_u^T, \delta_{2u}\boldsymbol{\gamma}_u^T, \ldots, \delta_{Ju}\boldsymbol{\gamma}_u^T\right)^T}{\partial \boldsymbol{\gamma}_u^T} \\ &= (\delta_{1u}\mathbf{I}_I, \delta_{2u}\mathbf{I}_I, \ldots, \delta_{Ju}\mathbf{I}_I)^T \\ &= \boldsymbol{\delta}_u \otimes \mathbf{I}_I.\end{aligned}$$

Hence

$$\frac{\partial(\boldsymbol{\delta}_u \otimes \boldsymbol{\gamma}_u)}{\partial(\boldsymbol{\gamma}_u^T, \boldsymbol{\delta}_u^T)} = (\boldsymbol{\delta}_u \otimes \mathbf{I}_I, \mathbf{I}_J \otimes \boldsymbol{\gamma}_u),$$

which produces the proposed expression for the Jacobian.

Let us now establish the rank of Matrix (3). Reordering the columns of a matrix does not modify its rank, so we can look for the rank of

$$(\mathbf{1}_J \otimes \mathbf{1}_I, (\mathbf{1}_J, \boldsymbol{\delta}_1, \ldots, \boldsymbol{\delta}_r) \otimes \mathbf{I}_I, \mathbf{I}_J \otimes (\mathbf{1}_I, \boldsymbol{\gamma}_1, \ldots, \boldsymbol{\gamma}_r)).$$

Denote by $\mathcal{M}[\mathbf{A}]$ the vector space generated by the columns of any matrix $\mathbf{A}$. We have

$$\operatorname{rk}(\mathbf{1}_I, \boldsymbol{\gamma}_1, \ldots, \boldsymbol{\gamma}_r) = \operatorname{rk}(\mathbf{1}_J, \boldsymbol{\delta}_1, \ldots, \boldsymbol{\delta}_r) = r+1 \tag{4}$$

since the centering and the orthogonality constraints hold. One can find $I - r - 1$ independent vectors, say

$$(\boldsymbol{\gamma}_{r+1}, \dots, \boldsymbol{\gamma}_{I-1}), \tag{5}$$

such that they generate the orthocomplement vector subspace to $\mathcal{M}[(\mathbf{1}_I, \boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_r)]$, and similarly $J - r - 1$ independent vectors, say

$$(\boldsymbol{\delta}_{r+1}, \dots, \boldsymbol{\delta}_{J-1}), \tag{6}$$

generating the orthocomplement vector subspace to $\mathcal{M}[(\mathbf{1}_J, \boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_r)]$. Since

$$\begin{aligned}
\mathcal{M}[\mathbf{I}_I] &= \mathcal{M}[(\mathbf{1}_I, \boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_r)] \oplus \mathcal{M}[(\boldsymbol{\gamma}_{r+1}, \dots, \boldsymbol{\gamma}_{I-1})], \\
\mathcal{M}[\mathbf{I}_J] &= \mathcal{M}[(\mathbf{1}_J, \boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_r)] \oplus \mathcal{M}[(\boldsymbol{\delta}_{r+1}, \dots, \boldsymbol{\delta}_{J-1})]
\end{aligned}$$

we have

$$\begin{aligned}
&\mathcal{M}[(\mathbf{1}_J \otimes \mathbf{1}_I, (\mathbf{1}_J, \boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_r) \otimes \mathbf{I}_I, \mathbf{I}_J \otimes (\mathbf{1}_I, \boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_r))] \\
&= \mathcal{M}[((\mathbf{1}_J, \boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_r) \otimes \mathbf{I}_I, \mathbf{I}_J \otimes (\mathbf{1}_I, \boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_r))] \\
&= \mathcal{M}[((\mathbf{1}_J, \boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_r) \otimes (\mathbf{1}_I, \boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_r))] \\
&\quad \oplus \mathcal{M}[((\mathbf{1}_J, \boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_r) \otimes (\boldsymbol{\gamma}_{r+1}, \dots, \boldsymbol{\gamma}_{I-1}))] \\
&\quad \oplus \mathcal{M}[((\boldsymbol{\delta}_{r+1}, \dots, \boldsymbol{\delta}_{J-1}) \otimes (\mathbf{1}_I, \boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_r))],
\end{aligned}$$

hence

$$\begin{aligned}
&\operatorname{rk}\,(\mathbf{1}_J \otimes \mathbf{1}_I, (\mathbf{1}_J, \boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_r) \otimes \mathbf{I}_I, \mathbf{I}_J \otimes (\mathbf{1}_I, \boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_r)) \\
&= \operatorname{rk}((\mathbf{1}_J, \boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_r) \otimes (\mathbf{1}_I, \boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_r)) \\
&\quad + \operatorname{rk}((\mathbf{1}_J, \boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_r) \otimes (\boldsymbol{\gamma}_{r+1}, \dots, \boldsymbol{\gamma}_{I-1})) \\
&\quad + \operatorname{rk}((\boldsymbol{\delta}_{r+1}, \dots, \boldsymbol{\delta}_{J-1}) \otimes (\mathbf{1}_I, \boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_r)) \\
&= (1 + r)^2 + (1 + r)(I - r - 1) + (J - r - 1)(1 + r) \\
&= (1 + r)(I + J - (1 + r)).
\end{aligned}$$

□

Note that, with the constraints in (1), $(\mathbf{1}_I, \boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_r)$ and $(\mathbf{1}_J, \boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_r)$ are orthogonal bases of $\mathcal{M}[(\mathbf{1}_I, \boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_r)]$ and $\mathcal{M}[(\mathbf{1}_J, \boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_r)]$.

Proposition 2. *The Hessian of Model* (1) *is given by*

$$H_{**}^{(i,j)}(\theta) = \frac{\partial^2 \eta_{(i,j)}(\theta)}{\partial\theta\partial\theta^T} = \begin{pmatrix} \mathbf{0}_{p^A \times p^A} & \mathbf{0}_{p^A \times p^B} \\ \mathbf{0}_{p^B \times p^A} & \mathbf{I}_r \otimes \begin{pmatrix} \mathbf{0} & \mathbf{e}_i \mathbf{f}_j^T \\ \mathbf{f}_j \mathbf{e}_i^T & \mathbf{0} \end{pmatrix} \end{pmatrix}$$

where $\mathbf{e}_i$ *and* $\mathbf{f}_j$ *are the* i*th and* j*th canonical vectors of* $\mathbb{R}^I$ *and* $\mathbb{R}^J$*. Every* $H_{**}^{(i,j)}(\theta)$ *is a symmetrical matrix of rank* $2r$*.*

Proof. The Hessian of the response function is a four dimensional array *row factor* $\times$ *column factor* $\times$ *parameters* $\times$ *parameters*; it is presented here as a series of symmetrical matrices for each couple of factor levels (i,j).

From Proposition 1, $\frac{\partial \eta_{(i,j)}}{\partial \theta^T}$ is given by

$$\frac{\partial \eta_{(i,j)}}{\partial \theta^T} = \left(1, \mathbf{e}_i^T, \mathbf{f}_j^T, \delta_{j1}\mathbf{e}_i^T, \gamma_{i1}\mathbf{f}_j^T, \ldots, \delta_{jr}\mathbf{e}_i^T, \gamma_{ir}\mathbf{f}_j^T\right).$$

All terms involving linear parameters vanish and consequently

$$\frac{\partial^2 \eta_{(i,j)}}{\partial \theta^A \partial \left(\theta^A\right)^T} = \mathbf{0}_{p^A \times p^A} \quad \text{and} \quad \frac{\partial^2 \eta_{(i,j)}}{\partial \theta^B \partial \left(\theta^A\right)^T} = \mathbf{0}_{p^B \times p^A}.$$

For the remaining $\frac{\partial \eta_{(i,j)}}{\partial \theta^B \partial \theta^{BT}}$ block, it suffices to check that

$$\frac{\partial\left(\delta_{ju}\mathbf{e}_i^T\right)}{\partial \boldsymbol{\delta}_u} = \mathbf{f}_j\mathbf{e}_i^T \text{ and } \frac{\partial\left(\gamma_{iu}\mathbf{f}_j^T\right)}{\partial \boldsymbol{\gamma}_u} = \mathbf{e}_i\mathbf{f}_j^T$$

to obtain that

$$\frac{\partial^2 \eta_{(i,j)}}{\partial \theta^B \partial \left(\theta^B\right)^T} = \mathbf{I}_r \otimes \begin{pmatrix} \mathbf{0} & \mathbf{e}_i\mathbf{f}_j^T \\ \mathbf{f}_j\mathbf{e}_i^T & \mathbf{0} \end{pmatrix}.$$

The rank of $H_{**}^{(i,j)}(\theta)$ is equal to the rank of its second diagonal block because the other blocks are null. On the other hand,

$$\begin{aligned} \operatorname{rk}\left(\mathbf{I}_r \otimes \begin{pmatrix} \mathbf{0} & \mathbf{e}_i\mathbf{f}_j^T \\ \mathbf{f}_j\mathbf{e}_i^T & \mathbf{0} \end{pmatrix}\right) &= \operatorname{rk}(\mathbf{I}_r)\operatorname{rk}\left(\begin{pmatrix} \mathbf{0} & \mathbf{e}_i\mathbf{f}_j^T \\ \mathbf{f}_j\mathbf{e}_i^T & \mathbf{0} \end{pmatrix}\right) \\ &= r\operatorname{rk}\left(\begin{pmatrix} \mathbf{0} & \mathbf{e}_i\mathbf{f}_j^T \\ \mathbf{f}_j\mathbf{e}_i^T & \mathbf{0} \end{pmatrix}\right). \end{aligned}$$

The rank of the remaining matrix is obviously two. □

Proposition 3. *The rows of the Jacobian for the constraints,* $L(\theta) = \frac{\partial \varphi(\theta)}{\partial \theta^T}$*, are given by the expressions*

$$\begin{array}{llll}
1 & L_\alpha^C & = \left(0, \mathbf{1}_I^T, \mathbf{0}_{1\times J}, \mathbf{0}_{1\times p^B}\right) & \\
1 & L_\beta^C & = \left(0, \mathbf{0}_{1\times I}, \mathbf{1}_J^T, \mathbf{0}_{1\times p^B}\right) & \\
r & L_{\gamma,u}^C & = \left(\mathbf{0}_{1\times p^A}, \mathbf{g}_u^T \otimes \left(\mathbf{1}_I^T, \mathbf{0}_{1\times J}\right)\right) & \forall u \\
r & L_{\delta,u}^C & = \left(\mathbf{0}_{1\times p^A}, \mathbf{g}_u^T \otimes \left(\mathbf{0}_{1\times I}, \mathbf{1}_J^T\right)\right) & \forall u \\
r & L_u^N & = \left(\mathbf{0}_{r\times p^A}, \mathbf{g}_u^T \otimes \left(\boldsymbol{\gamma}_u^T, -\boldsymbol{\delta}_u^T\right)\right) & \forall u \\
r(r-1)/2 & L_{\gamma,u,v}^O & = \left(\mathbf{0}_{r\times p^A}, \mathbf{g}_u^T \otimes \left(\boldsymbol{\gamma}_v^T, \mathbf{0}_{1\times J}\right) + \mathbf{g}_v^T \otimes \left(\boldsymbol{\gamma}_u^T, \mathbf{0}_{1\times J}\right)\right) & \forall u < v \\
r(r-1)/2 & L_{\delta,u,v}^O & = \left(\mathbf{0}_{r\times p^A}, \mathbf{g}_u^T \otimes \left(\mathbf{0}_{1\times I}, \boldsymbol{\delta}_v^T\right) + \mathbf{g}_v^T \otimes \left(\mathbf{0}_{1\times I}, \boldsymbol{\delta}_u^T\right)\right) & \forall u < v
\end{array}$$

where the first column indicates the number of rows of $L(\theta)$ *involved by the formula in the line and* $\mathbf{g}_u$ *is the* u*th canonical vector of* $\mathbb{R}^r$*. The rank of* $L(\theta)$ *is* $2 + 2r + r^2$*.*

P r o o f. It is straightforward to derive the different blocks of $\frac{\partial\varphi(\theta)}{\partial\theta^T}$ once we recall that $\varphi(\theta)$ and θ are defined in Section 2.1.

The derivation of the rank can be done in several steps:

1. Constraints on additive parameters give null components for the θ^B part of the Jacobian. Similarly, constraints on biadditive parameters give null components for the θ^A part of the Jacobian. It follows that
$$\operatorname{rk}\left(\frac{\partial\varphi(\theta)}{\partial\theta^T}\right) = \operatorname{rk}\left(\frac{\partial\varphi(\theta)}{\partial(\theta^A)^T}\right) + \operatorname{rk}\left(\frac{\partial\varphi(\theta)}{\partial(\theta^B)^T}\right).$$
2. Obviously
$$\operatorname{rk}\left(\frac{\partial\varphi(\theta)}{\partial(\theta^A)^T}\right) = 2.$$
3. From the centering constraints it follows that $\mathbf{1}_I$ and $\boldsymbol{\gamma}_u$ are linearly independent, and so are $\mathbf{1}_J$ and $\boldsymbol{\delta}_u$. Hence
$$\emptyset = (\mathcal{R}_{C\gamma} \cup \mathcal{R}_{C\delta}) \cap (\mathcal{R}_N \cup \mathcal{R}_{O\gamma} \cup \mathcal{R}_{O\delta})$$
where
$$\begin{aligned}
\mathcal{R}_{C\gamma} &= \mathcal{M}\left[\mathbf{g}_1 \otimes \left(\mathbf{1}_I^T, \mathbf{0}_{1\times J}\right)^T, \ldots, \mathbf{g}_r \otimes \left(\mathbf{1}_I^T, \mathbf{0}_{1\times J}\right)^T\right] \\
\mathcal{R}_{C\delta} &= \mathcal{M}\left[\mathbf{g}_1 \otimes \left(\mathbf{0}_{1\times I}, \mathbf{1}_J^T\right)^T, \ldots, \mathbf{g}_r \otimes \left(\mathbf{0}_{1\times I}, \mathbf{1}_J^T\right)^T\right] \\
\mathcal{R}_N &= \mathcal{M}\left[\mathbf{g}_1 \otimes \left(\boldsymbol{\gamma}_1^T, -\boldsymbol{\delta}_1^T\right)^T, \ldots, \mathbf{g}_r \otimes \left(\boldsymbol{\gamma}_r^T, -\boldsymbol{\delta}_r^T\right)^T\right] \\
\mathcal{R}_{O\gamma} &= \mathcal{M}\left[\mathbf{g}_u \otimes \left(\boldsymbol{\gamma}_v^T, \mathbf{0}_{1\times J}\right)^T + \mathbf{g}_v \otimes \left(\boldsymbol{\gamma}_u^T, \mathbf{0}_{1\times J}\right)^T, \quad \forall u < v\right] \\
\mathcal{R}_{O\delta} &= \mathcal{M}\left[\mathbf{g}_u \otimes \left(\mathbf{0}_{1\times I}, \boldsymbol{\delta}_v^T\right)^T + \mathbf{g}_v \otimes \left(\mathbf{0}_{1\times I}, \boldsymbol{\delta}_u^T\right)^T, \quad \forall u < v\right],
\end{aligned}$$
so
$$\operatorname{rk}\left(\frac{\partial\varphi(\theta)}{\partial(\theta^B)^T}\right) = \dim(\mathcal{R}_{C\gamma} \cup \mathcal{R}_{C\delta}) + \dim(\mathcal{R}_N \cup \mathcal{R}_{O\gamma} \cup \mathcal{R}_{O\delta}).$$
4. It is easy to see that
$$\begin{aligned}
\dim(\mathcal{R}_{C\gamma} \cup \mathcal{R}_{C\delta}) &= 2r, \\
\dim(\mathcal{R}_N) &= r, \\
\dim(\mathcal{R}_{O\gamma}) &= \frac{r(r-1)}{2}, \\
\dim(\mathcal{R}_{O\delta}) &= \frac{r(r-1)}{2}.
\end{aligned}$$

5. It is easy to check that $\mathcal{R}_{O\gamma} \perp \mathcal{R}_{O\delta}$, hence

$$\dim\left(\mathcal{R}_{O\gamma} \cup \mathcal{R}_{O\delta}\right) = \dim\left(\mathcal{R}_{O\gamma}\right) + \dim\left(\mathcal{R}_{O\delta}\right).$$

6. Due to (4) no vectors of the form $\mathbf{g}_u \otimes \left(\boldsymbol{\gamma}_u^T, -\boldsymbol{\delta}_u^T\right)^T$ can belong to $\mathcal{R}_{O\gamma} \cup \mathcal{R}_{O\delta}$ since only generators of the form $\mathbf{g}_u \otimes \left(\boldsymbol{\gamma}_v^T, \mathbf{0}_{1\times J}\right)^T$ with $u \neq v$ are available. Now vectors $\mathbf{g}_1 \otimes \left(\boldsymbol{\gamma}_1^T, -\boldsymbol{\delta}_1^T\right)^T, \ldots, \mathbf{g}_r \otimes \left(\boldsymbol{\gamma}_r^T, -\boldsymbol{\delta}_r^T\right)^T$ have nonnull components in the same position, consequently

$$\dim\left(\mathcal{R}_N \cup \left(\mathcal{R}_{O\gamma} \cup \mathcal{R}_{O\delta}\right)\right) = \dim\left(\mathcal{R}_N\right) + \dim\left(\mathcal{R}_{O\gamma} \cup \mathcal{R}_{O\delta}\right).$$

□

Proposition 4. *The matrix* $\begin{pmatrix} J(\theta) \\ L(\theta) \end{pmatrix}$ *is of full column rank.*

Proof. According to Propositions 1 and 3, the matrix $\begin{pmatrix} J(\theta) \\ L(\theta) \end{pmatrix}$ reads

$$\left(\begin{array}{cccccccc}
\mathbf{1}_J \otimes \mathbf{1}_I & \mathbf{1}_J \otimes \mathbf{I}_I & \mathbf{I}_J \otimes \mathbf{1}_I & \boldsymbol{\delta}_1 \otimes \mathbf{I}_I & \mathbf{I}_J \otimes \boldsymbol{\gamma}_1 & \ldots & \boldsymbol{\delta}_r \otimes \mathbf{I}_I & \mathbf{I}_J \otimes \boldsymbol{\gamma}_r \\
\hline
- & \mathbf{1}_I^T & - & - & - & - & - & - \\
- & - & \mathbf{1}_J^T & - & - & - & - & - \\
- & - & - & \mathbf{1}_I^T & - & - & - & - \\
- & - & - & - & \mathbf{1}_J^T & - & - & - \\
- & - & - & - & - & \ddots & - & - \\
- & - & - & - & - & - & \mathbf{1}_I^T & - \\
- & - & - & - & - & - & - & \mathbf{1}_J^T \\
\hline
- & - & - & \boldsymbol{\gamma}_1^T & -\boldsymbol{\delta}_1^T & - & - & - \\
- & - & - & - & - & \ddots & - & - \\
- & - & - & - & - & - & \boldsymbol{\gamma}_r^T & -\boldsymbol{\delta}_r^T \\
\hline
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
- & - & - & \boldsymbol{\gamma}_r^T & - & - & \boldsymbol{\gamma}_1^T & - \\
- & - & - & - & \boldsymbol{\delta}_r^T & - & - & \boldsymbol{\delta}_1^T \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{array}\right)$$

where for the sake of clarity, null block matrices have been indicated by "–" and null lines drawn out to better display the structure. Postmultiplying this matrix by the block diagonal matrix

$$\begin{pmatrix} 1 & \mathbf{0}_{1\times(p-1)} \\ \mathbf{0}_{(p-1)\times 1} & \mathbf{I}_{(r+1)} \otimes \begin{pmatrix} \mathbf{Q}_\gamma & \mathbf{0}_{I\times J} \\ \mathbf{0}_{J\times I} & \mathbf{Q}_\delta \end{pmatrix} \end{pmatrix}$$

where $\mathbf{Q}_\gamma = \left((\mathbf{1}_I, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3, \ldots, \boldsymbol{\gamma}_{I-1})^T\right)^{-1}$ and $\mathbf{Q}_\delta = \left((\mathbf{1}_J, \boldsymbol{\delta}_1, \boldsymbol{\delta}_2, \boldsymbol{\delta}_3, \ldots, \boldsymbol{\delta}_{J-1})^T\right)^{-1}$ with complementing vectors defined in Expressions (5) and (6), one obtains

$$
\left(\begin{array}{cccccccc}
\mathbf{1}_J \otimes \mathbf{1}_I & \mathbf{1}_J \otimes \mathbf{Q}_\gamma & \mathbf{Q}_\delta \otimes \mathbf{1}_I & \boldsymbol{\delta}_1 \otimes \mathbf{Q}_\gamma & \mathbf{Q}_\delta \otimes \boldsymbol{\gamma}_1 & \ldots & \boldsymbol{\delta}_r \otimes \mathbf{Q}_\gamma & \mathbf{Q}_\delta \otimes \boldsymbol{\gamma}_r \\
\hline
- & \mathbf{e}_1^T & - & - & - & - & - & - \\
- & - & \mathbf{f}_1^T & - & - & - & - & - \\
- & - & - & \mathbf{e}_1^T & - & - & - & - \\
- & - & - & - & \mathbf{f}_1^T & - & - & - \\
- & - & - & - & - & \ddots & - & - \\
- & - & - & - & - & - & \mathbf{e}_1^T & - \\
- & - & - & - & - & - & - & \mathbf{f}_1^T \\
\hline
- & - & - & \mathbf{e}_2^T & -\mathbf{f}_2^T & - & - & - \\
- & - & - & - & - & \ddots & - & - \\
- & - & - & - & - & - & \mathbf{e}_{r+1}^T & -\mathbf{f}_{r+1}^T \\
\hline
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
- & - & - & \mathbf{e}_{r+1}^T & - & - & \mathbf{e}_2^T & - \\
- & - & - & - & \mathbf{f}_{r+1}^T & - & - & \mathbf{f}_2^T \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{array}\right).
$$

It can be checked that all multiple columns of the transformed $J(\theta)$, namely $\mathbf{1}_J \otimes \mathbf{1}_I$, $\boldsymbol{\delta}_1 \otimes \mathbf{1}_I$, $\boldsymbol{\delta}_2 \otimes \mathbf{1}_I$, $\mathbf{1}_J \otimes \boldsymbol{\gamma}_1$, $\mathbf{1}_J \otimes \boldsymbol{\gamma}_2$, $\boldsymbol{\delta}_1 \otimes \boldsymbol{\gamma}_1$, $\boldsymbol{\delta}_2 \otimes \boldsymbol{\gamma}_2$, $\boldsymbol{\delta}_1 \otimes \boldsymbol{\gamma}_2$ and $\boldsymbol{\delta}_1 \otimes \boldsymbol{\gamma}_2$, are distinguished by the canonical vectors in the transformed $L(\theta)$. Consequently all columns of this matrix are linearly independent. $\square$

Proposition 5. *The Hessian for the constraints, $K(\theta) = \frac{\partial^2 \varphi(\theta)}{\partial\theta\partial\theta^T}$, is given by the following symmetrical matrices, one for each constraint.*

$$
\begin{array}{lll}
1 & K_\alpha^C = \mathbf{0}_{p\times p} & \\
1 & K_\beta^C = \mathbf{0}_{p\times p} & \\
r & K_{\gamma,u}^C = \mathbf{0}_{p\times p} & \forall u \\
r & K_{\delta,u}^C = \mathbf{0}_{p\times p} & \forall u \\
r & K_u^N = \begin{pmatrix} \mathbf{0}_{p^A\times p^A} & \mathbf{0}_{p^A\times p^B} \\ \mathbf{0}_{p^B\times p^A} & \mathbf{g}_u\mathbf{g}_u^T \otimes \begin{pmatrix} \mathbf{I}_I & \mathbf{0}_{I,J} \\ \mathbf{0}_{J,I} & -\mathbf{I}_J \end{pmatrix} \end{pmatrix} & \forall u \\
r(r-1)/2 & K_{\gamma,u,v}^O = \begin{pmatrix} \mathbf{0}_{p^A\times p^A} & \mathbf{0}_{p^A\times p^B} \\ \mathbf{0}_{p^B\times p^A} & \left(\mathbf{g}_v\mathbf{g}_u^T + \mathbf{g}_u\mathbf{g}_v^T\right) \otimes \begin{pmatrix} \mathbf{I}_I & \mathbf{0}_{I,J} \\ \mathbf{0}_{J,I} & \mathbf{0}_{J,J} \end{pmatrix} \end{pmatrix} & \forall u < v \\
r(r-1)/2 & K_{\delta,u,v}^O = \begin{pmatrix} \mathbf{0}_{p^A\times p^A} & \mathbf{0}_{p^A\times p^B} \\ \mathbf{0}_{p^B\times p^A} & \left(\mathbf{g}_v\mathbf{g}_u^T + \mathbf{g}_u\mathbf{g}_v^T\right) \otimes \begin{pmatrix} \mathbf{0}_{I,I} & \mathbf{0}_{I,J} \\ \mathbf{0}_{J,I} & \mathbf{I}_J \end{pmatrix} \end{pmatrix} & \forall u < v
\end{array}
$$

where the first number is the number of constraints described in the line.

P r o o f. There are no technical difficulties in obtaining these matrices starting from the Jacobian given in Proposition 3. □

2.2.2. The information matrix and a related matrix.

The following derivations are given without proofs: either they are simple products of matrices or they are proved in Appendix A.1.

Proposition 6. *The information matrix $M(\theta)$ of Model (1) is given by*

$$J(\theta)^T J(\theta) = \begin{pmatrix} & \underbrace{\mu, \boldsymbol{\alpha}^T, \boldsymbol{\beta}^T} & \underbrace{\boldsymbol{\gamma}_1^T, \boldsymbol{\delta}_1^T} & \underbrace{\boldsymbol{\gamma}_2^T, \boldsymbol{\delta}_2^T} & \cdots & \underbrace{\boldsymbol{\gamma}_r^T, \boldsymbol{\delta}_r^T} \\ \left.\begin{matrix}\mu\\ \boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{matrix}\right\} & \mathbf{A}^{\nabla} & \mathbf{B}_1 & \mathbf{B}_2 & \cdots & \mathbf{B}_r \\ \left.\begin{matrix}\boldsymbol{\gamma}_1\\ \boldsymbol{\delta}_1\end{matrix}\right\} & \mathbf{B}_1^T & C_1\mathbf{I}_{I+J} + \boldsymbol{\Gamma}_{11} & \boldsymbol{\Gamma}_{21} & \cdots & \boldsymbol{\Gamma}_{r1} \\ \left.\begin{matrix}\boldsymbol{\gamma}_2\\ \boldsymbol{\delta}_2\end{matrix}\right\} & \mathbf{B}_2^T & \boldsymbol{\Gamma}_{12} & C_2\mathbf{I}_{I+J} + \boldsymbol{\Gamma}_{22} & \cdots & \boldsymbol{\Gamma}_{2r} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \left.\begin{matrix}\boldsymbol{\gamma}_r\\ \boldsymbol{\delta}_r\end{matrix}\right\} & \mathbf{B}_r^T & \boldsymbol{\Gamma}_{1r} & \boldsymbol{\Gamma}_{2r} & \cdots & C_r\mathbf{I}_{I+J} + \boldsymbol{\Gamma}_{rr} \end{pmatrix}$$

where $C_u = \boldsymbol{\gamma}_u^T \boldsymbol{\gamma}_u$ $\forall u$ and

$$\mathbf{A}^{\nabla} = \begin{pmatrix} IJ & J\mathbf{1}_I^T & I\mathbf{1}_J^T \\ J\mathbf{1}_I & J\mathbf{I}_I & \mathbf{1}_I\mathbf{1}_J^T \\ I\mathbf{1}_J & \mathbf{1}_J\mathbf{1}_I^T & I\mathbf{I}_J \end{pmatrix}, \quad \mathbf{B}_u = \begin{pmatrix} \mathbf{0}_{1\times(I+J)} \\ \boldsymbol{\Gamma}_{u0} \end{pmatrix}, \quad \boldsymbol{\Gamma}_{uv} = \begin{pmatrix} \mathbf{0}_{I\times I} & \boldsymbol{\gamma}_u\boldsymbol{\delta}_v^T \\ \boldsymbol{\delta}_u\boldsymbol{\gamma}_v^T & \mathbf{0}_{J\times J} \end{pmatrix}.$$

Proposition 7. *The inverse of $M(\theta) + L^T(\theta) L(\theta)$ is given by the matrix*

$$\begin{pmatrix} & \underbrace{\mu, \boldsymbol{\alpha}^T, \boldsymbol{\beta}^T} & \underbrace{\boldsymbol{\gamma}_1^T, \boldsymbol{\delta}_1^T} & \underbrace{\boldsymbol{\gamma}_2^T, \boldsymbol{\delta}_2^T} & \cdots & \underbrace{\boldsymbol{\gamma}_r^T, \boldsymbol{\delta}_r^T} \\ \left.\begin{matrix}\mu\\ \boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{matrix}\right\} & \mathbf{A}^* & \mathbf{B}_1^* & \mathbf{B}_2^* & \cdots & \mathbf{B}_r^* \\ \left.\begin{matrix}\boldsymbol{\gamma}_1\\ \boldsymbol{\delta}_1\end{matrix}\right\} & (\mathbf{B}_1^*)^T & \mathbf{C}_{11}^* & \frac{-(\boldsymbol{\Delta}_{21}+\boldsymbol{\Gamma}_{21})}{(C_1-C_2)^2} & \cdots & \frac{-(\boldsymbol{\Delta}_{r1}+\boldsymbol{\Gamma}_{r1})}{(C_1-C_r)^2} \\ \left.\begin{matrix}\boldsymbol{\gamma}_2\\ \boldsymbol{\delta}_2\end{matrix}\right\} & (\mathbf{B}_2^*)^T & \frac{-(\boldsymbol{\Delta}_{12}+\boldsymbol{\Gamma}_{12})}{(C_2-C_1)^2} & \mathbf{C}_{22}^* & \cdots & \frac{-(\boldsymbol{\Delta}_{r2}+\boldsymbol{\Gamma}_{r2})}{(C_2-C_r)^2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \left.\begin{matrix}\boldsymbol{\gamma}_r\\ \boldsymbol{\delta}_r\end{matrix}\right\} & (\mathbf{B}_r^*)^T & \frac{-(\boldsymbol{\Delta}_{1r}+\boldsymbol{\Gamma}_{1r})}{(C_r-C_1)^2} & \frac{-(\boldsymbol{\Delta}_{2r}+\boldsymbol{\Gamma}_{2r})}{(C_r-C_2)^2} & \cdots & \mathbf{C}_{rr}^* \end{pmatrix}$$

where

$$\mathbf{A}^{*}=\begin{pmatrix} (IJ)^{-1}+I^{-2}+J^{-2} & -I^{-2}\mathbf{1}_I^T & -J^{-2}\mathbf{1}_J^T \\ -I^{-2}\mathbf{1}_I & \begin{array}{c} J^{-1}\mathbf{I}_I+\frac{J-I}{JI^2}\mathbf{1}_I\mathbf{1}_I^T \\ +J^{-2}\sum\limits_{u=1}^{r}\boldsymbol{\gamma}_u\boldsymbol{\gamma}_u^T \end{array} & \mathbf{0}_{I\times J} \\ -J^{-2}\mathbf{1}_J & \mathbf{0}_{J\times I} & \begin{array}{c} I^{-1}\mathbf{I}_J+\frac{I-J}{IJ^2}\mathbf{1}_J\mathbf{1}_J^T \\ +I^{-2}\sum\limits_{u=1}^{r}\boldsymbol{\delta}_u\boldsymbol{\delta}_u^T \end{array} \end{pmatrix},$$

$$\mathbf{B}_u^{*}=-\begin{pmatrix} \mathbf{0}_{1\times I} & \mathbf{0}_{1\times J} \\ \mathbf{0}_{I\times I} & J^{-2}\boldsymbol{\gamma}_u\mathbf{1}_J^T \\ I^{-2}\boldsymbol{\delta}_u\mathbf{1}_I^T & \mathbf{0}_{J\times J} \end{pmatrix},$$

$$\mathbf{C}_{uu}^{*}=C_u^{-1}\mathbf{I}_{I+J}+\frac{C_u^2-I^2}{C_uI^2\left(I+C_u\right)}\mathbf{E}_1+\frac{C_u^2-J^2}{C_uJ^2\left(J+C_u\right)}\mathbf{E}_2$$

$$-\sum_{v=1}^{r}\frac{1}{C_u\left(C_u+C_v\right)}\boldsymbol{\Delta}_{vv}+\sum_{\substack{v=1\\v\neq u}}^{r}\frac{2C_u}{\left(C_u+C_v\right)\left(C_u-C_v\right)^2}\left(\boldsymbol{\Delta}_{vv}+\boldsymbol{\Gamma}_{vv}\right)$$

with

$$\mathbf{E}_1=\begin{pmatrix} \mathbf{I}_{I\times I} & \mathbf{0}_{I\times J} \\ \mathbf{0}_{J\times I} & \mathbf{0}_{J\times J} \end{pmatrix},\quad \mathbf{E}_2=\begin{pmatrix} \mathbf{0}_{I\times I} & \mathbf{0}_{I\times J} \\ \mathbf{0}_{J\times I} & \mathbf{I}_{J\times J} \end{pmatrix},\quad \boldsymbol{\Delta}_{uv}=\begin{pmatrix} \boldsymbol{\gamma}_u\boldsymbol{\gamma}_v^T & \mathbf{0}_{I\times J} \\ \mathbf{0}_{J\times I} & \boldsymbol{\delta}_u\boldsymbol{\delta}_v^T \end{pmatrix},$$

$\boldsymbol{\Gamma}_{vv}$ *being defined in Proposition* 6.

2.3. Least squares estimators

It is convenient to distinguish the estimates of the linear part which can be obtained in closed form from the estimates of the bilinear part which are the solution of the eigenvalue equations.

2.3.1. Parameters of the linear terms.

L.S. estimators for the linear parameters (μ, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$) are given by identical formulæ whatever is r, the number of multiplicative terms, even if it is null. This good property is due to the centering constraints $\left(\varphi_{\gamma,u}^{C}\right)$ and $\left(\varphi_{\delta,u}^{C}\right)$ of Model (1). These estimators are especially simple, namely linear combinations of the observations. If $\mathbf{Y}$ is the I by J matrix of generic component $y_{(i,j)}$ then

$$\widehat{\mu}=\frac{1}{IJ}\mathbf{1}_I^T\mathbf{Y}\mathbf{1}_J,$$

$$\left(\widehat{\alpha}\right)_i=\frac{1}{J}\mathbf{Y}\mathbf{1}_J-\mathbf{1}_I\widehat{\mu},$$

$$\left(\widehat{\beta}\right)_j=\frac{1}{I}\mathbf{Y}^T\mathbf{1}_I-\mathbf{1}_J\widehat{\mu}.$$

They are unbiased.

2.3.2. Parameters of the bilinear terms.

Let $\mathbf{P}_I = \mathbf{1}_I \left(\mathbf{1}_I^T \mathbf{1}_I\right)^{-1} \mathbf{1}_I^T$ be the orthogonal projector of $\mathbb{R}^I$ onto the span of $\mathbf{1}_I$; similarly we define $\mathbf{P}_J$. For the bilinear part the L.S. estimators $\widehat{\boldsymbol{\gamma}}_u$ and $\widehat{\boldsymbol{\delta}}_u$ are the eigenvectors of the matrices $(\mathbf{I}_I - \mathbf{P}_I)\,\mathbf{Y}\,(\mathbf{I}_J - \mathbf{P}_J)\,\mathbf{Y}^T\,(\mathbf{I}_I - \mathbf{P}_I)$ and $(\mathbf{I}_J - \mathbf{P}_J)\,\mathbf{Y}^T\,(\mathbf{I}_I - \mathbf{P}_I)\,\mathbf{Y}\,(\mathbf{I}_J - \mathbf{P}_J)$ such that $\widehat{\boldsymbol{\gamma}}_u^T \widehat{\boldsymbol{\gamma}}_u = \widehat{\boldsymbol{\delta}}_u^T \widehat{\boldsymbol{\delta}}_u = \widehat{C}_u$ where $\widehat{C}_u^2$ is the common uth eigenvalue of these matrices. Note that the directions of $\widehat{\boldsymbol{\gamma}}_u$ and $\widehat{\boldsymbol{\delta}}_u$ must be chosen simultaneously. Classical references for these equations are Eckart and Young [8], Gollob [13], Mandel [16] and Johnson and Graybill [15].

As a consequence of Constraints (1) the estimators of the additive part are independent of $(\mathbf{I}_I - \mathbf{P}_I)\,\mathbf{Y}\,(\mathbf{I}_J - \mathbf{P}_J)$ and consequently of $\widehat{\boldsymbol{\gamma}}_u$ and $\widehat{\boldsymbol{\delta}}_u$.

2.3.3. Asymptotic variances.

The results presented in the next proposition come from Denis and Gower [2, 4, 5]. There are no novelty, merely the formulæ have been translated into our notation.

Proposition 8. *The variance matrix of the first order asymptotic approximation of least squares estimators of Model* (1)*, say* $\widehat{\theta}^{(1)}$*, is*

$$
\operatorname{Var}\left[\widehat{\theta}^{(1)}\right] = \sigma^2
\begin{pmatrix}
 & \underbrace{\mu, \boldsymbol{\alpha}^T, \boldsymbol{\beta}^T} & \underbrace{\boldsymbol{\gamma}_1^T, \boldsymbol{\delta}_1^T} & \underbrace{\boldsymbol{\gamma}_2^T, \boldsymbol{\delta}_2^T} & \cdots & \underbrace{\boldsymbol{\gamma}_r^T, \boldsymbol{\delta}_r^T} \\
\left.\begin{matrix}\mu\\ \boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{matrix}\right\} & \mathbf{A}^V & \mathbf{0}_{p^A \times (I+J)} & \mathbf{0}_{p^A \times (I+J)} & \cdots & \mathbf{0}_{p^A \times (I+J)} \\
\left.\begin{matrix}\boldsymbol{\gamma}_1\\ \boldsymbol{\delta}_1\end{matrix}\right\} & \mathbf{0}_{(I+J)\times p^A} & \mathbf{E}_1 & \mathbf{E}_{12} & \cdots & \mathbf{E}_{1r} \\
\left.\begin{matrix}\boldsymbol{\gamma}_2\\ \boldsymbol{\delta}_2\end{matrix}\right\} & \mathbf{0}_{(I+J)\times p^A} & \mathbf{E}_{21} & \mathbf{E}_2 & \cdots & \mathbf{E}_{2r} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\left.\begin{matrix}\boldsymbol{\gamma}_r\\ \boldsymbol{\delta}_r\end{matrix}\right\} & \mathbf{0}_{(I+J)\times p^A} & \mathbf{E}_{r1} & \mathbf{E}_{r2} & \cdots & \mathbf{E}_r
\end{pmatrix}
$$

where

$$
\mathbf{A}^V = \begin{pmatrix}
(IJ)^{-1} & \mathbf{0}_{1\times I} & \mathbf{0}_{1\times J} \\
\mathbf{0}_{I\times 1} & J^{-1}\mathbf{I}_I - (IJ)^{-1}\,\mathbf{1}_I\mathbf{1}_I^T & \mathbf{0}_{I\times J} \\
\mathbf{0}_{J\times 1} & \mathbf{0}_{J\times I} & I^{-1}\mathbf{I}_J - (IJ)^{-1}\,\mathbf{1}_J\mathbf{1}_J^T
\end{pmatrix}
$$

$$
\mathbf{E}_u = \frac{1}{\overline{C}_u}\begin{pmatrix}\mathbf{I}_I - \mathbf{P}_I & \mathbf{0}_{I\times J} \\ \mathbf{0}_{J\times I} & \mathbf{I}_J - \mathbf{P}_J\end{pmatrix} + \sum_{q=1}^{r}\left(l_{uq}\overline{\boldsymbol{\Delta}}_{qq} + k_{uq}\overline{\boldsymbol{\Gamma}}_{qq}\right),
$$

$$
\mathbf{E}_{uv} = \frac{-1}{\left(\overline{C}_u^2 - \overline{C}_v^2\right)^2}\left((\overline{C}_u^2 + \overline{C}_v^2)\overline{\boldsymbol{\Delta}}_{vu} + 2\overline{C}_u\overline{C}_v\overline{\boldsymbol{\Gamma}}_{vu}\right) \qquad \textit{when } u \neq v
$$

with

$$l_{uu} = \frac{-3}{4\overline{C}_u^2}, \quad k_{uu} = \frac{1}{4\overline{C}_u^2},$$

$$l_{uq} = \frac{\overline{C}_q\big(3\overline{C}_u^2 - \overline{C}_q^2\big)}{\overline{C}_u\big(\overline{C}_u^2 - \overline{C}_q^2\big)^2}, \quad k_{uq} = \frac{2\overline{C}_u^2}{\big(\overline{C}_u^2 - \overline{C}_q^2\big)^2} \quad \text{for } u \neq q,$$

$\boldsymbol{\Delta}_{vu}$, $\boldsymbol{\Gamma}_{vu}$ *are defined in Propositions* 7 *and* 6, $C_u = \boldsymbol{\gamma}_u^T\boldsymbol{\gamma}_u = \boldsymbol{\delta}_u^T\boldsymbol{\delta}_u$, *and bar means the true value of parameters.*

Remark. Notice that $\mathrm{Var}\big(\widehat{\theta}^{(1)}\big) = \mathrm{Var}\big(\Delta^{(1)}\big)$ as used in Pázman and Denis [17], since $\Delta^{(1)} = \widehat{\theta}^{(1)} - \overline{\theta}$.

3. Approximate bias of the parameters

The derivation of the asymptotic bias of the parameter estimator $\widehat{\theta}$ will be done by applying Proposition 2 in [17] which reads

$$b\big(\widehat{\theta}\big) = -\frac{1}{2}\big(M(\overline{\theta}) + L^T(\overline{\theta})L(\overline{\theta})\big)^{-1} \times \big(J^T(\overline{\theta})\,\mathrm{Tr}\{H(\overline{\theta})\,\mathrm{Var}[\widehat{\theta}^{(1)}]\} + L^T(\overline{\theta})\,\mathrm{Tr}\{K(\overline{\theta})\,\mathrm{Var}[\widehat{\theta}^{(1)}]\}\big). \tag{7}$$

The main elements of this formula and the formula itself are calculated in Appendices A.1 and A.2:

- $\big(M(\overline{\theta}) + L^T(\overline{\theta})L(\overline{\theta})\big)^{-1}$ is given by Lemma 18.
- $\mathrm{Tr}\{H(\overline{\theta})\,\mathrm{Var}[\widehat{\theta}^{(1)}]\}$ is given by Lemma 19.
- $\mathrm{Tr}\{K(\overline{\theta})\,\mathrm{Var}[\widehat{\theta}^{(1)}]\}$ is given by Lemma 20.
- $J^T(\overline{\theta})\,\mathrm{Tr}\{H(\overline{\theta})\,\mathrm{Var}[\widehat{\theta}^{(1)}]\} + L^T(\overline{\theta})\,\mathrm{Tr}\{K(\overline{\theta})\,\mathrm{Var}[\widehat{\theta}^{(1)}]\}$ is given by Lemma 21.
- Finally, Expression (7) is obtained in Lemma 22.

Theorem 9. *The approximate bias of the estimator of the parameter vector of Model* (1) *is null for the linear parameters, and is given by*

$$b\big(\widehat{\boldsymbol{\gamma}}_u\big) = \sigma^2\left[\frac{2\,(J-I)-1}{8\overline{C}_u^2} - \sum_{\substack{v=1\\v\neq u}}^{r}\frac{\overline{C}_v^2}{\big(\overline{C}_v^2 - \overline{C}_u^2\big)^2}\right]\overline{\boldsymbol{\gamma}}_u,$$

$$b\big(\widehat{\boldsymbol{\delta}}_u\big) = \sigma^2\left[\frac{2\,(I-J)-1}{8\overline{C}_u^2} - \sum_{\substack{v=1\\v\neq u}}^{r}\frac{\overline{C}_v^2}{\big(\overline{C}_v^2 - \overline{C}_u^2\big)^2}\right]\overline{\boldsymbol{\delta}}_u$$

for the bilinear parameters where $b\big(\widehat{\theta}\big)$ *means the approximate bias of the estimator* $\widehat{\theta}$ *and* $\overline{C}_u = \overline{\boldsymbol{\gamma}}_u^T\overline{\boldsymbol{\gamma}}_u = \overline{\boldsymbol{\delta}}_u^T\overline{\boldsymbol{\delta}}_u$.

A noticeable point of this result is its simplicity: the vectors $b(\widehat{\gamma}_u)$ and $\overline{\gamma}_u$ have the same direction. The same holds for $b(\widehat{\boldsymbol{\delta}}_u)$ and $\overline{\boldsymbol{\delta}}_u$. The vectors $b(\widehat{\gamma}_u)$ and $b(\widehat{\boldsymbol{\delta}}_u)$ respect the symmetry of $\boldsymbol{\gamma}_u$ and $\boldsymbol{\delta}_u$ in Model (1). The only difference is due to the number of levels, I and J, of the two factors.

4. Bias of important functions of the parameters

In this section we give results for a second parameterization of Model (1) as well as for its response.

4.1. Another parametrization

4.1.1. Definition.

In most practical circumstances biadditive models are considered under an equivalent parametrization imposing unit length to vectors $\boldsymbol{\gamma}_u$ and $\boldsymbol{\delta}_u$ and adding r additional parameters ϱ_u, *i.e.*

$$y_{(i,j)} = \mu + \alpha_i + \beta_j + \sum_{u=1}^{r} \varrho_u \widetilde{\gamma}_{iu} \widetilde{\delta}_{ju} + \varepsilon_{(i,j)} \tag{8}$$

with r additional constraints

$$\sum_{i=1}^{I} \widetilde{\gamma}_{iu}^2 = \widetilde{\boldsymbol{\gamma}}_u^T \widetilde{\boldsymbol{\gamma}}_u = 1 \qquad \forall\ u = 1, \ldots, r.$$

The main reason is that the amount of interaction for each multiplicative term is given by ϱ_u^2 while vectors $\widetilde{\boldsymbol{\gamma}}_u$ and $\widetilde{\boldsymbol{\delta}}_u$ develop the contrasts of the interaction. The new parameters can be easily defined as functions of the former ones by

$$\begin{aligned}
\varrho_u &= \boldsymbol{\gamma}_u^T \boldsymbol{\gamma}_u \qquad \forall\ u = 1, \ldots, r \\
\widetilde{\boldsymbol{\gamma}}_u &= \boldsymbol{\gamma}_u \left(\boldsymbol{\gamma}_u^T \boldsymbol{\gamma}_u\right)^{-\frac{1}{2}} \\
\widetilde{\boldsymbol{\delta}}_u &= \boldsymbol{\delta}_u \left(\boldsymbol{\gamma}_u^T \boldsymbol{\gamma}_u\right)^{-\frac{1}{2}}.
\end{aligned}$$

Let us denote the new set of parameters by

$$\widetilde{\theta} = \left(\mu, \boldsymbol{\alpha}^T, \boldsymbol{\beta}^T, \varrho_1, \widetilde{\boldsymbol{\gamma}}_1^T, \widetilde{\boldsymbol{\delta}}_1^T, \varrho_2, \widetilde{\boldsymbol{\gamma}}_2^T, \widetilde{\boldsymbol{\delta}}_2^T, \ldots, \varrho_r, \widetilde{\boldsymbol{\gamma}}_r^T, \widetilde{\boldsymbol{\delta}}_r^T\right)^T$$

and the mapping giving $\widetilde{\theta}$ as a function of θ by τ:

$$\widetilde{\theta} = \tau(\theta). \tag{9}$$

4.1.2. Jacobian and Hessian of the transformation.

Proposition 10. *The Jacobian* $\frac{\partial\tau(\theta)}{\partial\theta^T}$ *of the function* (9) *is the* $((1+r)(1+I+J)) \times (1+(1+r)(I+J))$ *matrix*

$$\begin{pmatrix} \mathbf{I}_{1+I+J} & \mathbf{0} \\ \mathbf{0} & \operatorname{diag}_{u=1,\dots,r}(Z_u) \end{pmatrix},$$

where Z_u *are* $(1+I+J)\times(I+J)$ *matrices given by*

$$Z_u = \begin{pmatrix} 2\boldsymbol{\gamma}_u^T & \mathbf{0}_{1,J} \\ \left(\boldsymbol{\gamma}_u^T\boldsymbol{\gamma}_u\right)^{-\frac{1}{2}}\left[\mathbf{I}_I - \boldsymbol{\gamma}_u\left(\boldsymbol{\gamma}_u^T\boldsymbol{\gamma}_u\right)^{-1}\boldsymbol{\gamma}_u^T\right] & \mathbf{0}_{I,J} \\ -\left(\boldsymbol{\gamma}_u^T\boldsymbol{\gamma}_u\right)^{-\frac{1}{2}}\boldsymbol{\delta}_u\left(\boldsymbol{\gamma}_u^T\boldsymbol{\gamma}_u\right)^{-1}\boldsymbol{\gamma}_u^T & \left(\boldsymbol{\gamma}_u^T\boldsymbol{\gamma}_u\right)^{-\frac{1}{2}}\mathbf{I}_J \end{pmatrix}.$$

It is full column rank.

P r o o f. Straightforward derivations produce the formula. The rank of Z_u is $I+J$ because the two blocks of columns are independent of rank I and J, respectively. Due to the diagonal block structure of $\frac{\partial\tau(\theta)}{\partial\theta^T}$, its rank is the sum of the ranks of the blocks. □

Proposition 11. *The Hessian* $\frac{\partial^2\tau(\theta)}{\partial\theta\partial\theta^T}$ *of the function* (9) *is given by the following series of symmetrical matrices of size* $\left(p^A+p^B\right)\times\left(p^A+p^B\right)$*:*

$$\left\{\frac{\partial^2\tau(\theta)}{\partial\theta\partial\theta^T}\right\}_{(\mu)} = \left\{\frac{\partial^2\tau(\theta)}{\partial\theta\partial\theta^T}\right\}_{(\alpha_i)} = \left\{\frac{\partial^2\tau(\theta)}{\partial\theta\partial\theta^T}\right\}_{(\beta_j)} = \mathbf{0}_{(p^A+p^B)\times(p^A+p^B)}$$

$$\left\{\frac{\partial^2\tau(\theta)}{\partial\theta\partial\theta^T}\right\}_{(\varrho_u)} = \begin{pmatrix} \mathbf{0}_{p^A\times p^A} & \mathbf{0}_{p^A\times p^B} \\ \mathbf{0}_{p^B\times p^A} & \mathbf{g}_u\mathbf{g}_u^T \otimes \begin{pmatrix} 2\mathbf{I}_I & \mathbf{0}_{I\times J} \\ \mathbf{0}_{J\times I} & \mathbf{0}_{J\times J} \end{pmatrix} \end{pmatrix} \quad \forall\, u=1,\dots,r$$

$$\left\{\frac{\partial^2\tau(\theta)}{\partial\theta\partial\theta^T}\right\}_{(\tilde{\gamma}_{iu})} = \begin{pmatrix} \mathbf{0}_{p^A\times p^A} & \mathbf{0}_{p^A\times p^B} \\ \mathbf{0}_{p^B\times p^A} & \mathbf{g}_u\mathbf{g}_u^T \otimes \begin{pmatrix} N_{iu} & \mathbf{0}_{I\times J} \\ \mathbf{0}_{J\times I} & \mathbf{0}_{J\times J} \end{pmatrix} \end{pmatrix} \quad \forall\, u=1,\dots,r\, \forall\, i=1,\dots,I$$

$$\left\{\frac{\partial^2\tau(\theta)}{\partial\theta\partial\theta^T}\right\}_{(\tilde{\delta}_{ju})} = \begin{pmatrix} \mathbf{0}_{p^A\times p^A} & \mathbf{0}_{p^A\times p^B} \\ \mathbf{0}_{p^B\times p^A} & \mathbf{g}_u\mathbf{g}_u^T \otimes \begin{pmatrix} U_{ju} & V_{ju}^T \\ V_{ju} & \mathbf{0}_{J\times J} \end{pmatrix} \end{pmatrix} \quad \forall\, u=1,\dots,r\, \forall\, j=1,\dots,J$$

where

$$N_{iu} = \left(\boldsymbol{\gamma}_u^T\boldsymbol{\gamma}_u\right)^{-\frac{3}{2}}\left[3\gamma_{iu}\boldsymbol{\gamma}_u\left(\boldsymbol{\gamma}_u^T\boldsymbol{\gamma}_u\right)^{-1}\boldsymbol{\gamma}_u^T - \left(\mathbf{e}_i\boldsymbol{\gamma}_u^T + \boldsymbol{\gamma}_u\mathbf{e}_i^T + \gamma_{iu}\mathbf{I}_I\right)\right],$$

$$U_{ju} = \delta_{ju}\left(\boldsymbol{\gamma}_u^T\boldsymbol{\gamma}_u\right)^{-\frac{3}{2}}\left[3\left(\boldsymbol{\gamma}_u^T\boldsymbol{\gamma}_u\right)^{-1}\boldsymbol{\gamma}_u\boldsymbol{\gamma}_u^T - \mathbf{I}_I\right],$$

$$V_{ju} = -\left(\boldsymbol{\gamma}_u^T\boldsymbol{\gamma}_u\right)^{-\frac{3}{2}}\mathbf{f}_j\boldsymbol{\gamma}_u^T.$$

P r o o f. Straightforward derivation gives each matrix. □

4.1.3. Asymptotic variance.

Proposition 12. $\operatorname{Var}\big[\widehat{\widetilde{\theta}}^{(1)}\big]$, *the asymptotic variance of the estimators of parameters of Model* (8) *is given by*

$$\sigma^2 \begin{pmatrix} & \underbrace{\mu, \boldsymbol{\alpha}^T, \boldsymbol{\beta}^T} & \underbrace{\varrho_1, \widetilde{\boldsymbol{\gamma}}_1^T, \widetilde{\boldsymbol{\delta}}_1^T} & \underbrace{\varrho_2, \widetilde{\boldsymbol{\gamma}}_2^T, \widetilde{\boldsymbol{\delta}}_2^T} & \cdots & \underbrace{\varrho_r, \widetilde{\boldsymbol{\gamma}}_r^T, \widetilde{\boldsymbol{\delta}}_r^T} \\ \left.\begin{matrix}\mu\\ \boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{matrix}\right\} & \mathbf{A}^V & \mathbf{0}_{p^A\times(1+I+J)} & \mathbf{0}_{p^A\times(1+I+J)} & \cdots & \mathbf{0}_{p^A\times(1+I+J)} \\ \left.\begin{matrix}\varrho_1\\ \widetilde{\boldsymbol{\gamma}}_1\\ \widetilde{\boldsymbol{\delta}}_1\end{matrix}\right\} & \mathbf{0}_{(1+I+J)\times p^A} & \widetilde{\mathbf{E}}_1 & \widetilde{\mathbf{E}}_{12} & \cdots & \widetilde{\mathbf{E}}_{1r} \\ \left.\begin{matrix}\varrho_2\\ \widetilde{\boldsymbol{\gamma}}_2\\ \widetilde{\boldsymbol{\delta}}_2\end{matrix}\right\} & \mathbf{0}_{(1+I+J)\times p^A} & \widetilde{\mathbf{E}}_{21} & \widetilde{\mathbf{E}}_2 & \cdots & \widetilde{\mathbf{E}}_{2r} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \left.\begin{matrix}\varrho_r\\ \widetilde{\boldsymbol{\gamma}}_r\\ \widetilde{\boldsymbol{\delta}}_r\end{matrix}\right\} & \mathbf{0}_{(1+I+J)\times p^A} & \widetilde{\mathbf{E}}_{r1} & \widetilde{\mathbf{E}}_{r2} & \cdots & \widetilde{\mathbf{E}}_r \end{pmatrix}$$

where

$$\widetilde{\mathbf{E}}_u = \begin{pmatrix} 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{\overline{C}_u^2}\mathbf{P}(\gamma) & \sum_{v\neq u} \frac{2\overline{C}_u}{\left(\overline{C}_u^2-\overline{C}_v^2\right)^2}\overline{\boldsymbol{\gamma}}_v\overline{\boldsymbol{\delta}}_v^T \\ \mathbf{0} & \sum_{v\neq u} \frac{2\overline{C}_u}{\left(\overline{C}_u^2-\overline{C}_v^2\right)^2}\overline{\boldsymbol{\delta}}_v\overline{\boldsymbol{\gamma}}_v^T & \frac{1}{\overline{C}_u^2}\mathbf{P}(\delta) \end{pmatrix},$$

$$\widetilde{\mathbf{E}}_{uv} = \frac{-1}{\overline{C}_u^{\frac{1}{2}}\overline{C}_v^{\frac{1}{2}}\left(\overline{C}_u^2-\overline{C}_v^2\right)^2}\left(\left(\overline{C}_u^2+\overline{C}_v^2\right)\begin{pmatrix}\mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \overline{\boldsymbol{\gamma}}_v\overline{\boldsymbol{\gamma}}_u^T & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \overline{\boldsymbol{\delta}}_v\overline{\boldsymbol{\delta}}_u^T\end{pmatrix}\right.$$
$$\left. +2\overline{C}_u\overline{C}_v\begin{pmatrix}\mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \overline{\boldsymbol{\gamma}}_v\overline{\boldsymbol{\delta}}_u^T\\ \mathbf{0} & \overline{\boldsymbol{\delta}}_v\overline{\boldsymbol{\gamma}}_u^T & \mathbf{0}\end{pmatrix}\right)$$

where

$$\mathbf{P}(\gamma) = \left[\mathbf{P}_{\{1,\overline{\gamma}_u\}^\perp} + \sum_{v\neq u}\frac{2\overline{C}_v\left(3\overline{C}_u^2-\overline{C}_v^2\right)}{\left(\overline{C}_u^2-\overline{C}_v^2\right)^2}\overline{\boldsymbol{\gamma}}_v\overline{\boldsymbol{\gamma}}_v^T\right],$$

$$\mathbf{P}(\delta) = \left[\mathbf{P}_{\{1,\overline{\delta}_u\}^\perp} + \sum_{v\neq u}\frac{2\overline{C}_v\left(3\overline{C}_u^2-\overline{C}_v^2\right)}{\left(\overline{C}_u^2-\overline{C}_v^2\right)^2}\overline{\boldsymbol{\delta}}_v\overline{\boldsymbol{\delta}}_v^T\right],$$

$$\mathbf{P}_{\{1,\gamma_u\}^\perp} = \mathbf{I}_I - \frac{1}{I}\mathbf{1}_I\mathbf{1}_I^T - \boldsymbol{\gamma}_u\left(\boldsymbol{\gamma}_u^T\boldsymbol{\gamma}_u\right)^{-1}\boldsymbol{\gamma}_u^T,$$
$$\mathbf{P}_{\{1,\delta_u\}^\perp} = \mathbf{I}_J - \frac{1}{J}\mathbf{1}_J\mathbf{1}_J^T - \boldsymbol{\delta}_u\left(\boldsymbol{\delta}_u^T\boldsymbol{\delta}_u\right)^{-1}\boldsymbol{\delta}_u^T.$$

P r o o f. This result is obtained by expanding $\frac{\partial\tau(\overline{\theta})}{\partial\theta^T}\operatorname{Var}\big(\widehat{\theta}^{(1)}\big)\frac{\partial\tau^T(\overline{\theta})}{\partial\theta}$ whose terms are given in Propositions 10 and 8. □

4.1.4. Bias.

Proposition 13. *Approximate bias of the second parametrization is*

$$b\big(\widehat{\varrho}_u\big) = \sigma^2\left[\frac{(J+I)-4}{2\overline{C}_u} + \sum_{q\neq u}\frac{\overline{C}_q^2}{\overline{C}_u\left(\overline{C}_u^2-\overline{C}_q^2\right)}\right],$$
$$b\big(\widehat{\overline{\gamma}}_u\big) = \frac{\sigma^2}{2}\overline{C}_u^{-\frac{5}{2}}\left[2 - I - \sum_{q\neq u}\frac{\overline{C}_q^2\left(3\overline{C}_u^2-\overline{C}_q^2\right)}{\left(\overline{C}_u^2-\overline{C}_q^2\right)^2}\right]\overline{\boldsymbol{\gamma}}_u,$$
$$b\big(\widehat{\overline{\boldsymbol{\delta}}}_u\big) = \frac{\sigma^2}{2}\overline{C}_u^{-\frac{5}{2}}\left[2 - J - \sum_{q\neq u}\frac{\overline{C}_q^2\left(3\overline{C}_u^2-\overline{C}_q^2\right)}{\left(\overline{C}_u^2-\overline{C}_q^2\right)^2}\right]\overline{\boldsymbol{\delta}}_u.$$

P r o o f. these formulæ are obtained by applying Proposition 5 in [17] using the previous results on bias (Theorem 9), variance (Proposition 8), Jacobian (Proposition 10) and Hessian (Proposition 11). □

4.2. Estimator of the response function

The estimator of the expectation of the observations is simply given by replacing the estimators of the parameters in the response

$$\eta\big(\widehat{\theta}\big) = (\mathbf{1}_J\otimes\mathbf{1}_I)\,\widehat{\mu} + \mathbf{1}_J\otimes\widehat{\boldsymbol{\alpha}} + \widehat{\boldsymbol{\beta}}\otimes\mathbf{1}_I + \sum_{u=1}^{r}\widehat{\boldsymbol{\delta}}_u\otimes\widehat{\boldsymbol{\gamma}}_u.$$

Proposition 14. *The asymptotic variance of* $\eta\big(\widehat{\theta}\big)$ *is*

$$\sigma^2\Bigg\{\frac{1}{IJ}\left(\mathbf{1}_J\mathbf{1}_J^T\otimes\mathbf{1}_I\mathbf{1}_I^T\right) + \frac{1}{J}\left(\mathbf{1}_J\mathbf{1}_J^T\otimes(\mathbf{I}_I-\mathbf{P}_I)\right) + \frac{1}{I}\left((\mathbf{I}_J-\mathbf{P}_J)\otimes\mathbf{1}_I\mathbf{1}_I^T\right)$$
$$+\sum_{u=1}^{r}\frac{1}{C_u}\left(\boldsymbol{\delta}_u\boldsymbol{\delta}_u^T\otimes(\mathbf{I}_I-\mathbf{P}_I)+(\mathbf{I}_J-\mathbf{P}_J)\otimes\boldsymbol{\gamma}_u\boldsymbol{\gamma}_u^T\right)$$
$$-\sum_{u=1}^{r}\sum_{v=1}^{r}\frac{1}{C_uC_v}\left(\boldsymbol{\delta}_u\boldsymbol{\delta}_u^T\otimes\boldsymbol{\gamma}_v\boldsymbol{\gamma}_v^T\right)\Bigg\}.$$

P r o o f. This result is obtained by expanding $J\left(\overline{\theta}\right)\operatorname{Var}\left(\widehat{\theta}^{(1)}\right)J^T\left(\overline{\theta}\right)$ whose terms are given in Propositions 1 and 8. □

Proposition 15. *The approximate bias of* $\eta\left(\widehat{\theta}\right)$ *is null.*

P r o o f. Application of the general Proposition 5 in [17]. □

R e m a r k. When the maximum number of multiplicative terms is introduced in the model, that is when $r = \min(I-1, J-1)$, Model (1) turns out to be the classical ANOVA interaction model which is a linear model and consequently without bias in the response. Our result is consistent with this fact.

5. Simulations

In order to have an idea about the practical validity of the approximations proposed, we have performed some simulations. Following the investigation made by Chadœuf and Denis [1], we took $I = 8$, $J = 13$, $r = 1$ and a series of values of σ^2 such that their coefficient

$$r(\sigma) = \frac{(I-1)(J-1)\sigma^2}{(I-1)(J-1)\sigma^2 + \varrho_1^2}$$

takes the values $\{0.01, 0.05, 0.1, (0.1), 0.9, 0.95\}$. This coefficient can be interpreted as the ratio of the noise over the sum of the noise plus the signal. In agronomic applications presented or studied by Chadœuf and Denis [1], Gauch [11] and van Eeuwijk [9] its values were (0.02, 0.30, 0.22, 0.17, 0.14, 0.44, 0.59, 0.68), so some practical situations are covered by these computations. For each value of $r(\sigma)$, 1000 simulations were done.

The results are presented in Figure 1 where simulated values and approximations are compared for $\widehat{\gamma}_{11}$, $\widehat{\varrho}_1$ and $\widehat{\eta}_{(1,1)}$. Rather than to give globally the mean square error of the estimators, we thought it useful to look at its two components: the standard deviation and the absolute value of the bias.

Several kinds of comments can be infered from Figure 1. We found them true also for other results investigated but not presented here. In all cases, the two approximations (standard deviation and bias) are quite good until $r(\sigma) = 0.5$ or 0.6. After that point, the approximation underestimates the bias for the response, nevertheless it is still surprisingly good for $\widehat{\varrho}_1$. According to the parameter considered the participation of the bias in the MSE can be the most important ($\widehat{\varrho}_1$) or the smallest ($\widehat{\eta}_{11}$).

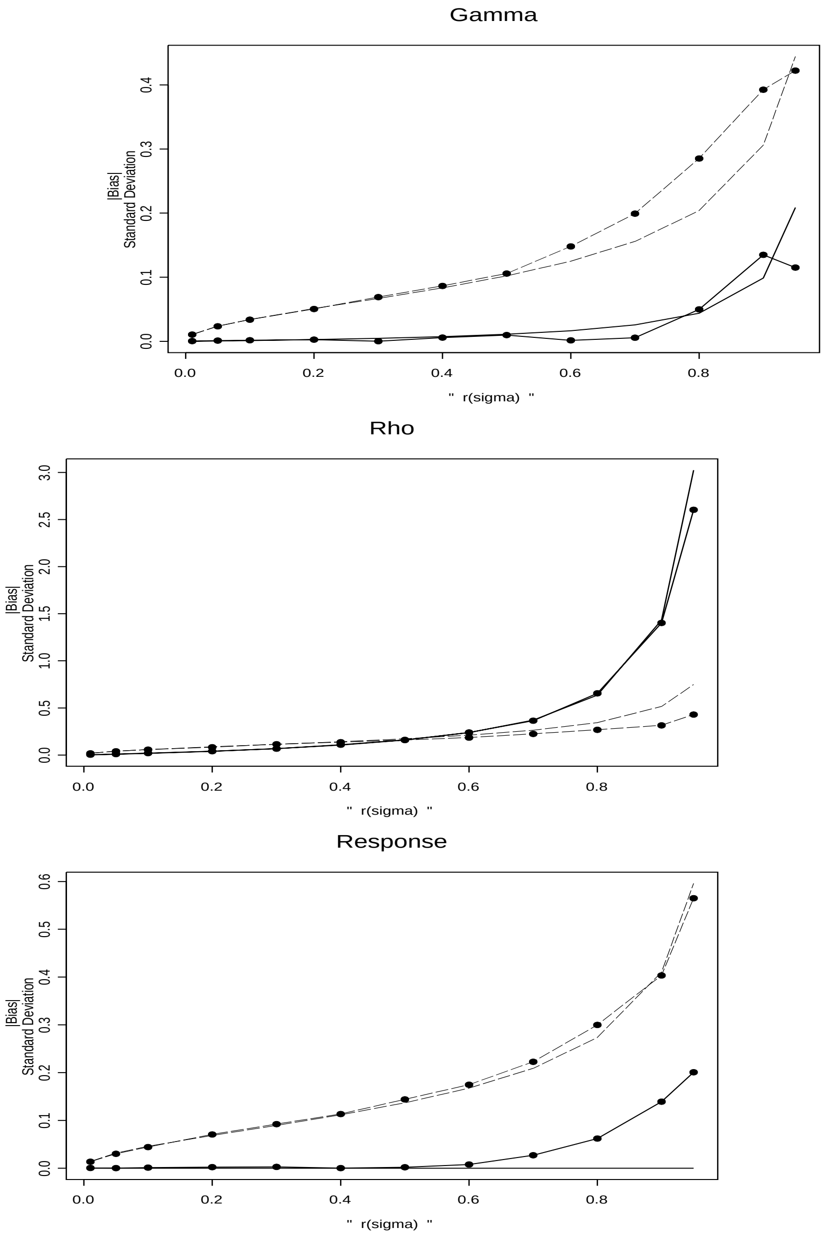

Figure 1. Results of simulations for $\widehat{\gamma}_{11}$, $\widehat{\varrho}_1$ and $\widehat{\eta}_{11}$. The coefficient $r(\sigma)$ is on the x-axis (see text); either the standard deviations (dashed lines), or the absolute values of the bias (solid lines) are on the y-axis. Big dots indicate the simulated values, the other lines give the approximate values.

A.1. Some necessary inverse matrices

Lemma 16. *Let* $\mathbf{A}$ *be the matrix*

$$\begin{pmatrix} IJ & J\mathbf{1}_I^T & I\mathbf{1}_J^T \\ J\mathbf{1}_I & J\mathbf{I}_I + \mathbf{1}_I\mathbf{1}_I^T & \mathbf{1}_I\mathbf{1}_J^T \\ I\mathbf{1}_J & \mathbf{1}_J\mathbf{1}_I^T & I\mathbf{I}_J + \mathbf{1}_J\mathbf{1}_J^T \end{pmatrix}.$$

Its inverse is

$$\begin{pmatrix} (IJ)^{-1} + I^{-2} + J^{-2} & -I^{-2}\mathbf{1}_I^T & -J^{-2}\mathbf{1}_J^T \\ -I^{-2}\mathbf{1}_I & J^{-1}\mathbf{I}_I + \frac{J-I}{JI^2}\mathbf{1}_I\mathbf{1}_I^T & \mathbf{0}_{I\times J} \\ -J^{-2}\mathbf{1}_J & \mathbf{0}_{J\times I} & I^{-1}\mathbf{I}_J + \frac{I-J}{IJ^2}\mathbf{1}_J\mathbf{1}_J^T \end{pmatrix}.$$

Proof. A direct check can be performed by multiplying the two matrices. □

Lemma 17. *Let* $\mathbf{C}$ *be the symmetric square matrix of* $r\times r$ *blocks of size* $(I+J)\times(I+J)$ *defined by*

$$\sum_{u=1}^{r}\left[C_u\mathbf{H}_u + \mathbf{E}_{u1} + \mathbf{E}_{u2} + \sum_{v=1}^{r} C_v\mathbf{D}_{uv} + \sum_{\substack{v=1\\ v\neq u}}^{r}\sqrt{C_uC_v}\mathbf{V}_{uv}\right]$$

where $\mathbf{H}_u = \mathbf{g}_u\mathbf{g}_u^T \otimes \mathbf{I}_{I+J}$, $\mathbf{E}_{u1} = \mathbf{g}_u\mathbf{g}_u^T \otimes \begin{pmatrix} \mathbf{1}_I\mathbf{1}_I^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{pmatrix}$, $\mathbf{E}_{u2} = \mathbf{g}_u\mathbf{g}_u^T \otimes \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_J\mathbf{1}_J^T\end{pmatrix}$, $\mathbf{D}_{uv} = \frac{1}{C_v}\mathbf{g}_u\mathbf{g}_u^T \otimes \mathbf{\Delta}_{vv}$ *and* $\mathbf{V}_{uv} = \frac{1}{\sqrt{C_uC_v}}\mathbf{g}_u\mathbf{g}_v^T \otimes (\mathbf{\Delta}_{vu} + \mathbf{\Gamma}_{vu})$.

Its inverse is

$$\begin{aligned}\mathbf{C}^{-1} = \sum_{u=1}^{r}\Bigg[&\frac{1}{C_u}\mathbf{H}_u - \frac{1}{C_u\,(I+C_u)}\mathbf{E}_{u1} - \frac{1}{C_u\,(J+C_u)}\mathbf{E}_{u2} \\ &- \sum_{v=1}^{r}\frac{C_v}{C_u\,(C_u+C_v)}\mathbf{D}_{uv} - \sum_{\substack{v=1\\ v\neq u}}^{r}\frac{\sqrt{C_uC_v}}{(C_u-C_v)^2}\mathbf{V}_{uv} \\ &+ \sum_{\substack{v=1\\ v\neq u}}^{r}\frac{2C_uC_v}{(C_u+C_v)\,(C_u-C_v)^2}\mathbf{F}_{uv}\Bigg]\end{aligned}$$

where $\mathbf{F}_{uv} = \frac{1}{C_v}\mathbf{g}_u\mathbf{g}_u^T \otimes (\mathbf{\Delta}_{vv} + \mathbf{\Gamma}_{vv})$. *[*$\mathbf{\Delta}_{vu}$ *and* $\mathbf{\Gamma}_{vu}$ *are defined in Propositions* 7 *and* 6.*]*

P r o o f. We will calculate the product $\mathbf{CC}^{-1}$ and check that it gives the identity matrix. To do so, it is convenient to derive the table of products of the auxiliary matrices. Note that the products which are not mentioned in the following table are null. The table is ordered according to the resulting products.

$\mathbf{H}_u \ \mathbf{H}_u = \mathbf{H}_u$	$\mathbf{H}_u \ \mathbf{V}_{uv} = \mathbf{V}_{uv}$
$\mathbf{H}_u \ \mathbf{E}_{u1} = \mathbf{E}_{u1}$	$\mathbf{V}_{uv} \ \mathbf{H}_v = \mathbf{V}_{uv}$
$\mathbf{E}_{u1} \ \mathbf{H}_u = \mathbf{E}_{u1}$	$\mathbf{D}_{uv} \ \mathbf{V}_{uv} = \mathbf{V}_{uv}$
$\mathbf{E}_{u1} \ \mathbf{E}_{u1} = I\mathbf{E}_{u1}$	$\mathbf{V}_{uv} \ \mathbf{D}_{vu} = \mathbf{V}_{uv}$
$\mathbf{H}_u \ \mathbf{E}_{u2} = \mathbf{E}_{u2}$	$\mathbf{F}_{uv} \ \mathbf{V}_{uv} = 2\mathbf{V}_{uv}$
$\mathbf{E}_{u2} \ \mathbf{H}_u = \mathbf{E}_{u2}$	$\mathbf{V}_{uv} \ \mathbf{F}_{vu} = 2\mathbf{V}_{uv}$
$\mathbf{E}_{u2} \ \mathbf{E}_{u2} = J\mathbf{E}_{u2}$	$\mathbf{H}_u \ \mathbf{F}_{uv} = \mathbf{F}_{uv}$
$\mathbf{H}_u \ \mathbf{D}_{uv} = \mathbf{D}_{uv}$	$\mathbf{F}_{uv} \ \mathbf{H}_u = \mathbf{F}_{uv}$
$\mathbf{D}_{uv} \ \mathbf{H}_u = \mathbf{D}_{uv}$	$\mathbf{D}_{uv} \ \mathbf{F}_{uv} = \mathbf{F}_{uv}$
$\mathbf{D}_{uv} \ \mathbf{D}_{uv} = \mathbf{D}_{uv}$	$\mathbf{F}_{uv} \ \mathbf{D}_{uv} = \mathbf{F}_{uv}$
	$\mathbf{V}_{uv} \ \mathbf{V}_{vu} = 2\mathbf{F}_{uv}$
	$\mathbf{F}_{uv} \ \mathbf{F}_{uv} = 2\mathbf{F}_{uv}$

We will now perform the calculus of $\mathbf{CC}^{-1}$ using this table of products:

- terms in $\mathbf{H}_u$:

$$\sum_u \frac{C_u}{C_u}\mathbf{H}_u = \sum_u \mathbf{H}_u = \mathbf{I}_{r(I+J)\times r(I+J)}$$

- terms in $\mathbf{E}_{u1}$:

$$\sum_u \left(-\frac{1}{I+C_u} + \frac{1}{C_u} - \frac{I}{C_u\,(I+C_u)}\right)\mathbf{E}_{u1} = \sum_u \left(\frac{-C_u + (I+C_u) - I}{C_u\,(I+C_u)}\right)\mathbf{E}_{u1}$$
$$= \mathbf{0}_{r(I+J)\times r(I+J)}$$

- terms in $\mathbf{E}_{u2}$:

$$\sum_u \left(-\frac{1}{J+C_u} + \frac{1}{C_u} - \frac{I}{C_u\,(J+C_u)}\right)\mathbf{E}_{u2} = \mathbf{0}_{r(I+J)\times r(I+J)}$$

- terms in $\mathbf{D}_{uv}$:

$$\sum_u\sum_v \left(-\frac{C_v}{C_u+C_v} + \frac{C_v}{C_u} - \frac{C_v^2}{C_u\,(C_u+C_v)}\right)\mathbf{D}_{uv}$$
$$= \sum_u\sum_v C_v\left(\frac{-C_u + (C_u+C_v) - C_v}{C_u\,(C_u+C_v)}\right)\mathbf{D}_{uv} = \mathbf{0}_{r(I+J)\times r(I+J)}$$

- terms in $\mathbf{V}_{uv}$:

$$\sum_u \sum_{v\neq u}\Big(\frac{-C_u\sqrt{C_uC_v}}{(C_u-C_v)^2}+\frac{\sqrt{C_uC_v}}{C_v}-\frac{C_v\sqrt{C_uC_v}}{(C_u-C_v)^2}$$
$$-\frac{C_u\sqrt{C_uC_v}}{C_v\left(C_u+C_v\right)}+\frac{4C_uC_v\sqrt{C_uC_v}}{\left(C_u-C_v\right)^2\left(C_u+C_v\right)}\Big)\mathbf{V}_{uv}$$
$$=\sum_u\sum_{v\neq u}\frac{K_{uv}\sqrt{C_uC_v}}{C_v\left(C_u-C_v\right)^2\left(C_u+C_v\right)}\mathbf{V}_{uv}$$

where

$$\begin{aligned}
K_{uv}=&-C_uC_v\left(C_u+C_v\right)+\left(C_u-C_v\right)^2\left(C_u+C_v\right)\\
&-C_v^2\left(C_u+C_v\right)-C_u\left(C_u-C_v\right)^2+4C_uC_v^2\\
=&\left(C_u+C_v\right)\left(-C_uC_v+\left(C_u-C_v\right)^2-C_v^2\right)-C_u\left(\left(C_u-C_v\right)^2+4C_v^2\right)\\
=&\left(C_u+C_v\right)\left(C_u^2-3C_uC_v\right)+C_u\left(-C_u^2+2C_uC_v+3C_v^2\right)=0
\end{aligned}$$

- terms in $\mathbf{F}_{uv}$:

$$\sum_{v\neq u}\left(\frac{C_u2C_uC_v}{\left(C_u+C_v\right)\left(C_u-C_v\right)^2}+\frac{C_v2C_uC_v}{\left(C_u+C_v\right)\left(C_u-C_v\right)^2}-\frac{2C_uC_v}{\left(C_u-C_v\right)^2}\right)\mathbf{F}_{uv}$$
$$=\sum_{v\neq u}\frac{2C_uC_v\left(C_u+C_v\right)-2C_uC_v\left(C_u+C_v\right)}{\left(C_u+C_v\right)\left(C_u-C_v\right)^2}\mathbf{F}_{uv}=0.$$

□

Lemma 18. *The inverse of the matrix $M(\theta)+L^T(\theta)L(\theta)$ of Model (1) is given by*

$$\begin{pmatrix}\mathbf{A}^* & \mathbf{B}^*\\ (\mathbf{B}^*)^T & \mathbf{C}^*\end{pmatrix},$$

where

$$\mathbf{A}^*=\begin{pmatrix}(IJ)^{-1}+I^{-2}+J^{-2} & -I^{-2}\mathbf{1}_I^T & -J^{-2}\mathbf{1}_J^T\\ -I^{-2}\mathbf{1}_I & \begin{matrix}J^{-1}\mathbf{I}_I+\frac{J-I}{JI^2}\mathbf{1}_I\mathbf{1}_I^T\\ +J^{-2}\sum_{u=1}^r\boldsymbol{\gamma}_u\boldsymbol{\gamma}_u^T\end{matrix} & \mathbf{0}_{I\times J}\\ -J^{-2}\mathbf{1}_J & \mathbf{0}_{J\times I} & \begin{matrix}I^{-1}\mathbf{I}_J+\frac{I-J}{IJ^2}\mathbf{1}_J\mathbf{1}_J^T\\ +I^{-2}\sum_{u=1}^r\boldsymbol{\delta}_u\boldsymbol{\delta}_u^T\end{matrix}\end{pmatrix},$$

$$\mathbf{B}^* = -\sum_{u=1}^{r} \mathbf{g}_u^T \otimes \begin{pmatrix} \mathbf{0}_{1\times I} & \mathbf{0}_{1\times J} \\ \mathbf{0}_{I\times I} & J^{-2}\boldsymbol{\gamma}_u \mathbf{1}_J^T \\ I^{-2}\boldsymbol{\delta}_u \mathbf{1}_I^T & \mathbf{0}_{J\times J} \end{pmatrix},$$

$$\mathbf{C}^* = \sum_{u=1}^{r} \left[\frac{1}{C_u}\mathbf{H}_u + \frac{C_u^2 - I^2}{C_u I^2 (I + C_u)}\mathbf{E}_{u1} + \frac{C_u^2 - J^2}{C_u J^2 (J + C_u)}\mathbf{E}_{u2} - \sum_{v=1}^{r} \frac{C_v}{C_u (C_u + C_v)}\mathbf{D}_{uv} - \sum_{\substack{v=1\\v\neq u}}^{r} \frac{\sqrt{C_u C_v}}{(C_u - C_v)^2}\mathbf{V}_{uv} + \sum_{\substack{v=1\\v\neq u}}^{r} \frac{2C_u C_v}{(C_u + C_v)(C_u - C_v)^2}\mathbf{F}_{uv} \right],$$

with the auxiliary matrices $\mathbf{H}_u$, $\mathbf{E}_{u1}$, $\mathbf{E}_{u2}$, $\mathbf{D}_{uv}$ and $\mathbf{F}_{uv}$ defined in Lemma 17.

Proof.

1. $M(\theta) + L^T(\theta) L(\theta) = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{pmatrix}$ where $\mathbf{A}$ is defined in Lemma 16, $\mathbf{C}$ is defined in Lemma 17 and

$$\mathbf{B} = \sum_{u=1}^{r} \mathbf{g}_u^T \otimes \begin{pmatrix} \mathbf{0}_{1\times I} & \mathbf{0}_{1\times J} \\ \mathbf{0}_{I\times I} & \gamma_u \mathbf{1}_J^T \\ \boldsymbol{\delta}_u 1_I^T & \mathbf{0}_{J\times J} \end{pmatrix}.$$

The well known formula of the inverse of such a two blocks by two blocks partitioned square matrix is

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{pmatrix}^{-1} = \begin{pmatrix} \left(\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^T\right)^{-1}, & -\left(\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^T\right)^{-1}\mathbf{B}\mathbf{C}^{-1} \\ -\mathbf{C}^{-1}\mathbf{B}^T\left(\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^T\right)^{-1}, & \mathbf{C}^{-1} + \mathbf{C}^{-1}\mathbf{B}^T\left(\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^T\right)^{-1}\mathbf{B}\mathbf{C}^{-1} \end{pmatrix},$$

we will compute it accordingly.

2. Here we will derive $\mathbf{B}\mathbf{C}^{-1}$. From the definition of these matrices (see Lemmas 17 and 18) it can be checked that

$$\mathbf{B}\mathbf{H}_u = \mathbf{g}_u^T \otimes \begin{pmatrix} \mathbf{0}_{1\times I} & \mathbf{0}_{1\times J} \\ \mathbf{0}_{I\times I} & \gamma_u \mathbf{1}_J^T \\ \boldsymbol{\delta}_u \mathbf{1}_I^T & \mathbf{0}_{J\times J} \end{pmatrix},$$

$$\mathbf{B}\mathbf{E}_{u1} = \mathbf{g}_u^T \otimes \begin{pmatrix} \mathbf{0}_{1\times I} & \mathbf{0}_{1\times J} \\ \mathbf{0}_{I\times I} & \mathbf{0}_{I\times J} \\ I\boldsymbol{\delta}_u \mathbf{1}_I^T & \mathbf{0}_{J\times J} \end{pmatrix}; \quad \mathbf{B}\mathbf{E}_{u2} = \mathbf{g}_u^T \otimes \begin{pmatrix} \mathbf{0}_{1\times I} & \mathbf{0}_{1\times J} \\ \mathbf{0}_{I\times I} & J\boldsymbol{\gamma}_u \mathbf{1}_J^T \\ \mathbf{0}_{J\times I} & \mathbf{0}_{J\times J} \end{pmatrix},$$

$$\mathbf{B}\mathbf{D}_{uv} = \mathbf{B}\mathbf{V}_{uv} = \mathbf{B}\mathbf{F}_{uv} = \mathbf{0}_{(1+I+J)\times r(I+J)}.$$

It follows that

$$\mathbf{BC}^{-1} = \sum_{u=1}^{r} \mathbf{g}_u^T \otimes \begin{pmatrix} \mathbf{0}_{1\times I} & \mathbf{0}_{1\times J} \\ \mathbf{0}_{I\times I} & \frac{1}{J+C_u}\gamma_u \mathbf{1}_J^T \\ \frac{1}{I+C_u}\boldsymbol{\delta}_u \mathbf{1}_I^T & \mathbf{0}_{J\times J} \end{pmatrix}.$$

3. Here we will derive $\mathbf{BC}^{-1}\mathbf{B}^T$. A straightforward matrix product from the preceding expression gives

$$\mathbf{BC}^{-1}\mathbf{B}^T = \sum_{u=1}^{r} \begin{pmatrix} 0 & \mathbf{0}_{1\times I} & \mathbf{0}_{1\times J} \\ \mathbf{0}_{I\times 1} & \frac{J}{J+C_u}\gamma_u\gamma_u^T & \mathbf{0}_{I\times J} \\ \mathbf{0}_{J\times 1} & \mathbf{0}_{J\times I} & \frac{I}{I+C_u}\boldsymbol{\delta}_u\boldsymbol{\delta}_u^T \end{pmatrix}.$$

4. Here we will derive $\left(\mathbf{A} - \mathbf{BC}^{-1}\mathbf{B}^T\right)^{-1}$. It can be checked that it is given by the expression

$$\mathbf{A}^{-1} + \sum_{u=1}^{r} \begin{pmatrix} 0 & \mathbf{0}_{1\times I} & \mathbf{0}_{1\times J} \\ \mathbf{0}_{I\times 1} & J^{-2}\gamma_u\gamma_u^T & \mathbf{0}_{I\times J} \\ \mathbf{0}_{J\times 1} & \mathbf{0}_{J\times I} & I^{-2}\boldsymbol{\delta}_u\boldsymbol{\delta}_u^T \end{pmatrix}.$$

5. Here we will derive $\left(\mathbf{A} - \mathbf{BC}^{-1}\mathbf{B}^T\right)^{-1}\mathbf{BC}^{-1}$ from the previous results. It turns out to be equal to

$$\sum_{u=1}^{r} \mathbf{g}_u^T \otimes \left[\left(\mathbf{A} - \mathbf{BC}^{-1}\mathbf{B}^T\right)^{-1} \begin{pmatrix} \mathbf{0}_{1\times I} & \mathbf{0}_{1\times J} \\ \mathbf{0}_{I\times I} & \frac{1}{J+C_u}\gamma_u \mathbf{1}_J^T \\ \frac{1}{I+C_u}\boldsymbol{\delta}_u \mathbf{1}_I^T & \mathbf{0}_{J\times J} \end{pmatrix}\right]$$
$$= \sum_{u=1}^{r} \mathbf{g}_u^T \otimes \begin{pmatrix} \mathbf{0}_{1\times I} & \mathbf{0}_{1\times J} \\ \mathbf{0}_{I\times I} & J^{-2}\gamma_u \mathbf{1}_J^T \\ I^{-2}\boldsymbol{\delta}_u \mathbf{1}_I^T & \mathbf{0}_{J\times J} \end{pmatrix}.$$

6. Here we will derive $\mathbf{C}^{-1}\mathbf{B}^T\left(\mathbf{A} - \mathbf{BC}^{-1}\mathbf{B}^T\right)^{-1}\mathbf{BC}^{-1}$ by multiplying $\mathbf{C}^{-1}\mathbf{B}^T$ and $\left(\mathbf{A} - \mathbf{BC}^{-1}\mathbf{B}^T\right)^{-1}\mathbf{BC}^{-1}$ previously obtained:

$$\left[\sum_{u=1}^{r} \mathbf{g}_u \otimes \begin{pmatrix} \mathbf{0}_{I\times 1} & \mathbf{0}_{I\times I} & \frac{1}{I+C_u}\mathbf{1}_I\boldsymbol{\delta}_u^T \\ \mathbf{0}_{J\times 1} & \frac{1}{J+C_u}\mathbf{1}_J\gamma_u^T & \mathbf{0}_{J\times J} \end{pmatrix}\right]\left[\sum_{u=1}^{r} \mathbf{g}_u^T \otimes \begin{pmatrix} \mathbf{0}_{1\times I} & \mathbf{0}_{1\times J} \\ \mathbf{0}_{I\times I} & J^{-2}\gamma_u \mathbf{1}_J^T \\ I^{-2}\boldsymbol{\delta}_u \mathbf{1}_I^T & \mathbf{0}_{J\times J} \end{pmatrix}\right]$$
$$= \sum_{u=1}^{r} \left(\mathbf{g}_u\mathbf{g}_u^T\right) \otimes \begin{pmatrix} \frac{C_u}{I^2(I+C_u)}\mathbf{1}_I\mathbf{1}_I^T & \mathbf{0}_{I\times J} \\ \mathbf{0}_{J\times I} & \frac{C_u}{J^2(J+C_u)}\mathbf{1}_J\mathbf{1}_J^T \end{pmatrix}.$$

From here, there is no difficulty in obtaining $\mathbf{C}^*$ because $\mathbf{C}^{-1}$ is already given in Lemma 17.

□

A.2. Calculations for the bias formula

The aim of this section is to calculate for the parameters of the biadditive Model (1) according to Proposition 2 in [17].

Lemma 19. *For the biadditive Model* (1) *we have*

$$\operatorname{Tr}\left\{H(\theta)\operatorname{Var}\left[\widehat{\theta}^{(1)}\right]\right\} = \sigma^2 \sum_{u=1}^{r} \left\{ \left[\sum_{\substack{v=1\\ v\neq u}}^{r} \frac{4C_v^2}{\left(C_v^2 - C_u^2\right)^2} \right] + \frac{1}{2C_u^2} \right\} \boldsymbol{\delta}_u \otimes \boldsymbol{\gamma}_u$$

where $C_u = \boldsymbol{\gamma}_u^T \boldsymbol{\gamma}_u = \boldsymbol{\delta}_u^T \boldsymbol{\delta}_u$.

P r o o f. $H(\theta)$ is an $IJ \times \left(p^A + p^B\right) \times \left(p^A + p^B\right)$ structure given by Proposition 2 and $\operatorname{Var}\left[\widehat{\theta}^{(1)}\right]$ is a $\left(p^A + p^B\right) \times \left(p^A + p^B\right)$ matrix detailed in Proposition 8. This means that we must obtain an IJ vector whose (i,j) component is

$$\operatorname{Tr}\left\{H_{**}^{(i,j)}(\theta)\operatorname{Var}\left[\widehat{\theta}^{(1)}\right]\right\}.$$

Since $H_{**}^{(i,j)} = \begin{pmatrix} \mathbf{0}_{p^A \times p^A} & \mathbf{0}_{p^A \times p^B} \\ \mathbf{0}_{p^B \times p^A} & \mathbf{I}_r \otimes \begin{pmatrix} \mathbf{0} & \mathbf{e}_i \mathbf{f}_j^T \\ \mathbf{f}_j \mathbf{e}_i^T & \mathbf{0} \end{pmatrix} \end{pmatrix}$, so only the codiagonal blocks of the terms $\mathbf{E}_{uu}$ have to be taken into account for the trace. This gives

$$\begin{aligned} &\sigma^2 \sum_{u=1}^{r} 2 \sum_{q=1}^{r} k_{uq} \gamma_{iq} \delta_{jq} \\ &= \sigma^2 \sum_{q=1}^{r} \gamma_{iq}\delta_{jq} \left\{ 2 \left[\sum_{\substack{u=1\\ u\neq q}}^{r} \frac{1}{C_u C_q} \frac{2 r_{qu}}{\left(r_{qu} - r_{uq}\right)^2} \right] + \frac{1}{2C_q^2} \right\}. \end{aligned}$$

□

Lemma 20. *For the biadditive Model* (1) *we have*

$$\operatorname{Tr}\left\{K(\theta)\operatorname{Var}\left[\widehat{\theta}^{(1)}\right]\right\} = \sigma^2 \Big(\mathbf{0}_{1\times(2+2r)}, \Big(\frac{I-J}{C_1}, \frac{I-J}{C_2}, \dots, \frac{I-J}{C_r}\Big), \mathbf{0}_{1\times\left(r(r-1)\right)} \Big)^T$$

where $C_u = \boldsymbol{\gamma}_u^T \boldsymbol{\gamma}_u = \boldsymbol{\delta}_u^T \boldsymbol{\delta}_u$.

P r o o f. $K(\theta)$ is a $\left(3 + 2r + r^2\right) \times \left(p^A + p^B\right) \times \left(p^A + p^B\right)$ structure given by Proposition 5 and $\operatorname{Var}\left[\widehat{\theta}^{(1)}\right]$ is a $\left(p^A + p^B\right) \times \left(p^A + p^B\right)$ matrix detailed in Proposition 8. This means that we must obtain a $\left(3 + 2r + r^2\right)$ vector whose components will be calculated according to the seven types of constraints presented in the definition of Model (1):

- components associated with the four centering constraints are null because the corresponding matrices of the Hessian are null;
- components associated with the normalization constraints reduce to

$$\sigma^2\left[\sum_{q=1}^{r} l_{uq}\left(\boldsymbol{\gamma}_q^T\boldsymbol{\gamma}_q-\boldsymbol{\delta}_q^T\boldsymbol{\delta}_q\right)+\frac{1}{C_u}\left(\mathrm{Tr}\left(\mathbf{I}_I-\mathbf{P}_I\right)-\mathrm{Tr}\left(\mathbf{I}_J-\mathbf{P}_J\right)\right)\right]=\sigma^2\frac{I-J}{C_u};$$

- components associated with the orthogonalization constraints vanish because the result is a linear combination of $\mathrm{Tr}\left(\boldsymbol{\gamma}_u\boldsymbol{\gamma}_v^T\right)=\boldsymbol{\gamma}_v^T\boldsymbol{\gamma}_u=0$ and $\mathrm{Tr}\left(\boldsymbol{\delta}_u\boldsymbol{\delta}_v^T\right)=\boldsymbol{\delta}_v^T\boldsymbol{\delta}_u=0$ because $u\neq v$.

□

Lemma 21. *For the biadditive Model* (1) *we have*

$$\begin{aligned}&J^T(\theta)\,\mathrm{Tr}\left\{H(\theta)\,\mathrm{Var}\left[\widehat{\theta}^{(1)}\right]\right\}+L^T(\theta)\,\mathrm{Tr}\left\{K(\theta)\,\mathrm{Var}\left[\widehat{\theta}^{(1)}\right]\right\}\\&\quad=\sigma^2\left(\mathbf{0}_{1\times p^A},Q_1\boldsymbol{\gamma}_1^T,R_1\boldsymbol{\delta}_1^T,\ldots,Q_r\boldsymbol{\gamma}_r^T,R_r\boldsymbol{\delta}_r^T\right)^T\end{aligned}$$

where

$$Q_u=\frac{1+2\left(I-J\right)}{2C_u}+C_u\sum_{\substack{v=1\\v\neq u}}^{r}\frac{4C_v^2}{\left(C_v^2-C_u^2\right)^2},$$

$$R_u=\frac{1+2\left(J-I\right)}{2C_u}+C_u\sum_{\substack{v=1\\v\neq u}}^{r}\frac{4C_v^2}{\left(C_v^2-C_u^2\right)^2}$$

and $C_u=\boldsymbol{\gamma}_u^T\boldsymbol{\gamma}_u=\boldsymbol{\delta}_u^T\boldsymbol{\delta}_u$.

P r o o f. $J(\theta)$ is given by Proposition 1, $\mathrm{Tr}\left\{H(\theta)\,\mathrm{Var}\left[\widehat{\theta}^{(1)}\right]\right\}$ by Lemma 19, $L(\theta)$ by Proposition 3 and $\mathrm{Tr}\left\{K(\theta)\,\mathrm{Var}\left[\widehat{\theta}^{(1)}\right]\right\}$ by Lemma 20. The result is obtained by multiplying matrices and summing vectors. □

Lemma 22. *For the biadditive Model* (1) *we have*

$$\begin{aligned}&-\frac{1}{2}\left(M(\theta)+L^T(\theta)\,L(\theta)\right)^{-1}\\&\qquad\times\left(J^T(\theta)\,\mathrm{Tr}\left\{H(\theta)\,\mathrm{Var}\left[\widehat{\theta}^{(1)}\right]\right\}+L^T(\theta)\,\mathrm{Tr}\left\{K(\theta)\,\mathrm{Var}\left[\widehat{\theta}^{(1)}\right]\right\}\right)\\&\quad=\sigma^2\left(\mathbf{0}_{1\times p^A},S_1\boldsymbol{\gamma}_1^T,T_1\boldsymbol{\delta}_1^T,\ldots,S_r\boldsymbol{\gamma}_r^T,T_r\boldsymbol{\delta}_r^T\right)^T,\end{aligned}$$

where

$$S_u = \frac{2\,(J-I)-1}{8C_u^2} - \sum_{\substack{v=1\\v\neq u}}^{r} \frac{C_v^2}{\left(C_v^2-C_u^2\right)^2},$$

$$T_u = \frac{2\,(I-J)-1}{8C_u^2} - \sum_{\substack{v=1\\v\neq u}}^{r} \frac{C_v^2}{\left(C_v^2-C_u^2\right)^2}$$

and $C_u = \boldsymbol{\gamma}_u^T\boldsymbol{\gamma}_u = \boldsymbol{\delta}_u^T\boldsymbol{\delta}_u$.

P r o o f. $\left(M(\theta)+L^T(\theta)\,L(\theta)\right)^{-1}$ is given by Lemma 18 and

$$J^T(\theta)\,\mathrm{Tr}\left\{H(\theta)\,\mathrm{Var}\left[\widehat{\theta}^{(1)}\right]\right\}+L^T(\theta)\,\mathrm{Tr}\left\{K(\theta)\,\mathrm{Var}\left[\widehat{\theta}^{(1)}\right]\right\}$$

by Lemma 21. The result is obtained by a simple matrix multiplication. □

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