Applications of Mathematics

Niklas Wellander

Homogenization of the Maxwell equations: Case I. Linear theory

Applications of Mathematics, Vol. 46 (2001), No. 1, 29--51

Persistent URL: http://dml.cz/dmlcz/133893

HOMOGENIZATION OF THE MAXWELL EQUATIONS: CASE I. LINEAR THEORY

Niklas Wellander, Linköping

(Received April 16, 1999, in revised version April 19, 2000)

Abstract. The Maxwell equations in a heterogeneous medium are studied. Nguetseng's method of two-scale convergence is applied to homogenize and prove corrector results for the Maxwell equations with inhomogeneous initial conditions. Compactness results, of two-scale type, needed for the homogenization of the Maxwell equations are proved.

Keywords: Maxwell's equations, homogenization, two-scale convergence, corrector results, heterogeneous materials, periodic coefficients, nonperiodic coefficients, compactness result, effective properties, fiber composites

MSC 2000: 35B27, 35Q60, 74Q10, 78A25

1. Introduction

Homogenization results for Maxwell's equations, by the classical method of asymptotic expansions in two scales, are well known (see e.g. [5] and [17]). In this paper we use the quite new method of two-scale convergence, introduced by Nguetseng [13]. It turns out to be a very simple homogenization procedure with straightforward and transparent proofs. We show that the Maxwell equations in a heterogeneous structure possess a unique two-scale limit as the size of the inhomogeneity of the micro-structure tends to zero. We also prove new corrector results for Maxwell's equations.

We consider a material which occupies a bounded open simply connected set Ω in $\mathbb{R}^3$ and assume that the material is ε-periodic in the sense that it can be viewed as the union of a collection of disjoint open identical cubes with side length ε (Y^ε-cells). Further, we assume that the boundary $\partial\Omega$ is regular, i.e., $\partial\Omega$ is a once continuously differentiable two-dimensional manifold. (This assumption can be relaxed, it can be proved that it is enough if $\partial\Omega$ is Lipschitz.)

The Maxwell equations in the ε-periodic material read:

$$\partial_t D^\varepsilon(x,t) + J^\varepsilon(x,t) = \operatorname{rot} H^\varepsilon(x,t) + F^\varepsilon(x,t) \tag{1.1}$$

$$\partial_t B^\varepsilon(x,t) = -\operatorname{rot} E^\varepsilon(x,t) \tag{1.2}$$

$$\operatorname{div} B^\varepsilon(x,t) = 0 \tag{1.3}$$

$$\operatorname{div} D^\varepsilon(x,t) = \varrho^\varepsilon(x,t) \tag{1.4}$$

for $x \in \Omega$ and $t \in]0,T[$, where E^ε, H^ε, D^ε, J^ε and B^ε are the electric, magnetic, induced electric, current density and induced magnetic fields, respectively. Moreover, $\varrho^\varepsilon(x,t)$ is the charge density which is defined by (1.4) and F^ε is a current density source. We assume that F^ε and the time derivative $\partial_t F^\varepsilon$ belong to $L^2(0,T;L^2(\Omega)^3)$, i.e. $F^\varepsilon \in W^{1,2}(0,T;L^2(\Omega)^3)$, and are bounded in the $L^2(0,T;L^2(\Omega)^3)$-norm. Moreover, we assume that $F^\varepsilon \to F$ strongly in $L^2(0,T;L^2(\Omega)^3)$ and that $\operatorname{div} F^\varepsilon$ is bounded in $L^2(\Omega \times]0,T[)$. The system is equipped with the initial conditions

$$E^\varepsilon(x,0) = E_0^\varepsilon(x), \qquad H^\varepsilon(x,0) = H_0^\varepsilon(x), \tag{1.5}$$

which are bounded in $L^2(\Omega)^3$. We assume that $E_0^\varepsilon(x)$ and $H_0^\varepsilon(x)$ two-scale converge to $E^0(x,y)$ and $H^0(x,y)$, respectively. We also have the Neumann boundary condition

$$n \wedge E^\varepsilon(x,t) = 0 \text{ on } \partial\Omega \times]0,T[, \tag{1.6}$$

where n is the outer unit normal to $\partial\Omega$. This boundary condition corresponds to the case when Ω is in contact with an infinitely good conductor. More general boundary conditions can also be treated, see [2] and [6].

There are also three constitutive relations associated to the system (1.1)–(1.4):

$$B_i^\varepsilon(x,t) = \mu_{ij}\left(\frac{x}{\varepsilon}\right) H_j^\varepsilon(x,t), \tag{1.7}$$

$$J_i^\varepsilon(x,t) = \sigma_{ij}\left(\frac{x}{\varepsilon}\right) E_j^\varepsilon(x,t), \tag{1.8}$$

$$D_i^\varepsilon(x,t) = \eta_{ij}\left(\frac{x}{\varepsilon}\right) E_j^\varepsilon(x,t). \tag{1.9}$$

Here μ, η and σ are the magnetic permeability, electric permittivity and conductivity, respectively. Since these are material dependent they are periodic with the same period as the material (ε-periodic). Moreover, μ and η are assumed to be symmetric, i.e., $\mu_{ij} = \mu_{ji}$ and $\eta_{ij} = \eta_{ji}$. Existence of a unique solution

$E^\varepsilon, H^\varepsilon \in W^{1,\infty}(0,T;\mathcal{H}_{\mathrm{rot}}(\Omega), L^2(\Omega)^3)$ was proved in [7]. (For a definition of $\mathcal{H}_{\mathrm{rot}}(\Omega)$, see Section 2.) The material can for example be a carbon fiber composite or an impregnation ore consisting of two different kinds of minerals, one highly conductive distributed periodically in another one with low conductivity. In particular, it can be applied to the Induced Polarization phenomenon (see [19] and the references given there), which is used by geophysicists in the search for ores containing gold, silver and copper among other valuable metals.

The problem to be addressed when modeling the electromagnetic fields is to determine the macroscopic conductivity, electric permittivity and magnetic permeability.

The homogenization problem for Maxwell's system (1.1)–(1.6) with constitutional relations (1.7)–(1.9) is to find the effective constitutive relations, represented by the mappings

$$\begin{aligned} H &\mapsto B, \\ E &\mapsto J, \\ E &\mapsto D, \end{aligned}$$

which need not be of the same type as (1.7)–(1.9). (We will find out that the current density and the induced electric field are obtained by convolutions in time, i.e. the constitutive relations for the homogenized Maxwell's equations possess a memory effect, or in other words, the homogenized permittivity and conductivity are frequency dependent.) The corresponding effective Maxwell's equations read

$$\begin{aligned} \partial_t D + J &= \operatorname{rot} H + F, \\ \partial_t B &= -\operatorname{rot} E, \\ \operatorname{div} B &= 0, \\ \operatorname{div} D &= \varrho, \\ E(x,0) = E_0(x), &\qquad H(x,0) = H_0(x) \end{aligned}$$

for $x \in \Omega$ and $t \in \,]0,T[$, and

$$n \wedge E(x,t) = 0 \text{ on } \partial\Omega \times \,]0,T[.$$

A similar problem has been studied with the use of the classical method of multiple scales expansion technique (see e.g. [16] and [17] for homogeneous initial conditions and unbounded domains, see also [2]–[5], [11], [12], [21] and the references given there). Here we present a new proof based on the two-scale convergence method. We also prove some new corrector results which open the possibility of better numerical modeling of the above problem.

The paper is organized as follows: In Section 2 we present some basic definitions and the concept of two-scale convergence. In Section 3 we present and discuss the announced main homogenization and corrector results. In Sections 4 and 5 we present and prove some compactness results and a priori estimates necessary for the proofs of the main results. The proofs of the main results can be found in Section 6. Section 7 is reserved for some concluding remarks.

2. Preliminaries

In this text we use the Einstein tensor summation convention. Some standard operator symbols will also be used when it simplifies the notation. By C we denote any fixed constant which may take different values on any place it appears in an equation or inequality. The notation for function spaces, not defined below, is standard and can be found in [20].

Let Ω be a bounded open simply connected set in $\mathbb{R}^N$ and $Y =]0,1[^{\mathrm{N}}$ a unit cube in $\mathbb{R}^N$, which we will call a Y-cell. $F_\sharp(Y)$ consists of all functions in $F_{\mathrm{loc}}(\mathbb{R}^N)$ which are periodical repetitions of some function in $F(Y)$. $\mathcal{H}_{\mathrm{rot}}(\Omega) = \{u \in L^2(\Omega)^3\colon \operatorname{rot} u \in L^2(\Omega)^3\}$, with the norm $\|u\|_{\mathcal{H}_{\mathrm{rot}}(\Omega)} = \|u\|_{L^2(\Omega)^3} + \|\operatorname{rot} u\|_{L^2(\Omega)^3}$. $\mathcal{H}_{\mathrm{div}}(\Omega) = \{u \in L^2(\Omega)^3\colon \operatorname{div} u \in L^2(\Omega)\}$ with the norm $\|u\|_{\mathcal{H}_{\mathrm{div}}(\Omega)} = \|u\|_{L^2(\Omega)^3} + \|\operatorname{div} u\|_{L^2(\Omega)}$. $Du = (\partial_{x_i} u)_i = \operatorname{grad} u$ denotes the gradient of u and $n \wedge u$ denotes the vector product $n \in \mathbb{R}^3$ and $u \in \mathbb{R}^3$. $\operatorname{div} u = \partial_{x_i} u_i$ and $\operatorname{rot} u = \operatorname{curl} u = (\partial_{x_2} u_3 - \partial_{x_3} u_2, \partial_{x_3} u_1 - \partial_{x_1} u_3, \partial_{x_1} u_2 - \partial_{x_2} u_1)$ are the usual divergence and curl of vector fields in $\mathbb{R}^N$ and $\mathbb{R}^3$, respectively. We say that a function $u\colon \mathbb{R}^N \to \mathbb{R}$ is *Y-periodic* if $u(x+e_i) = u(x)$ for every $x \in \mathbb{R}^N$ and for every $i \in \{1,2,3,\ldots,N\}$, where $(e_1,\ldots,e_N)$ is the canonical basis of $\mathbb{R}^N$. Sometimes we write σ^ε, η^ε and μ^ε for $\sigma\left(\frac{x}{\varepsilon}\right)$, $\eta\left(\frac{x}{\varepsilon}\right)$ and $\mu\left(\frac{x}{\varepsilon}\right)$, respectively.

In 1989 Nguetseng [13] presented a new concept to homogenize scales of partial differential equations (PDEs), the so called *two-scale convergence method* which was generalized to the $L^p(\Omega)$-case by Holmbom in [10] in the following way:

Definition 2.1. A sequence $\{u^\varepsilon\}$ in $L^p(\Omega)$, $p \in]1,\infty]$, is said to two-scale converge to $u_0 \in L^p(\Omega \times Y)$ if

$$\lim_{\varepsilon\to 0} \int_\Omega u^\varepsilon(x) a\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x = \int_\Omega \int_Y u_0(x,y) a(x,y)\, \mathrm{d}x\, \mathrm{d}y \tag{2.1}$$

for every $a \in D(\Omega; C_\sharp^\infty(Y))$. We will sometimes denote two-scale convergence by $\overset{2-s}{\rightharpoonup}$.

The following useful result is a natural generalization of the corresponding result stated for $L^2(\Omega)$ functions in [13].

Proposition 2.2 (Uniqueness). *Let $\{u^\varepsilon\}$ be a bounded sequence in $L^p(\Omega)$, $p \in]1, \infty]$. Then for a subsequence* (2.1) *holds for a (unique) $u_0 \in L^p(\Omega; L^p_\sharp(Y))$ for all $a \in C_0(\Omega; C_\sharp(Y))$ and for all $a = a_1 \cdot a_2$, $a_1 \in C_0(\Omega)$, $a_2 \in L^q_\sharp(Y)$, $1/p + 1/q = 1$.*

Proof. See [10]. □

In [10] Holmbom also enlarged the class of all test functions for which (2.1) holds to all admissible test functions in the sense defined below.

Definition 2.3. We say that $a \in L^q(\Omega; L^q_\sharp(Y))$ is an admissible test function if $a\left(x, \frac{x}{\varepsilon}\right)$ is measurable and

$$\lim_{\varepsilon \to 0} \left\| a\left(x, \frac{x}{\varepsilon}\right) \right\|_{L^q(\Omega)} = \|a(x,y)\|_{L^q(\Omega \times Y)}.$$

Remark 2.4. Some examples of admissible test functions are $L^q(\Omega; C_\sharp(Y))$ and, for Ω bounded, $L^q_\sharp(Y; C(\overline{\Omega}))$, $q \in [1, \infty[$. Moreover, any a which belongs to either of these spaces for $q = 1$ satisfies (see the proof of Lemma 5.2 in [1])

$$\lim_{\varepsilon \to 0} \int_\Omega a\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x = \int_\Omega \int_Y a(x,y) \,\mathrm{d}x \,\mathrm{d}y.$$

For $p \geqslant 2$ we have the following result:

Theorem 2.5. *Let $\{u^\varepsilon\}$ and $\{a^\varepsilon\}$ be bounded sequences in $L^p(\Omega)$, $p \in [2, \infty[$, and $L^2(\Omega)$, respectively. Let u_0 and a_0 be the two-scale limits of subsequences of the corresponding sequences obtained by the diagonalization procedure. Further, assume that at least for this subsequence*

$$\lim_{\varepsilon \to 0} \|a^\varepsilon(x)\|_{L^2(\Omega)} = \lim_{\varepsilon \to 0} \left\| a\left(x, \frac{x}{\varepsilon}\right) \right\|_{L^2(\Omega)} = \|a(x,y)\|_{L^2(\Omega \times Y)}.$$

Then, for the subsequence selected above,

$$\lim_{\varepsilon \to 0} \int_\Omega u^\varepsilon(x) a^\varepsilon(x) \,\mathrm{d}x = \int_\Omega \int_Y u_0(x,y) a(x,y) \,\mathrm{d}x \,\mathrm{d}y,$$
$$\lim_{\varepsilon \to 0} \left\| a^\varepsilon(x) - a\left(x, \frac{x}{\varepsilon}\right) \right\|_{L^2(\Omega)} = 0.$$

Proof. See [10], where also a nonperiodic case was proved. □

Our next result is a generalization to the L^p-case of the characterization of the two-scale limit in the L^2-case treated in [13].

Proposition 2.6. *Let $\{u^\varepsilon\}$ be a bounded sequence in $L^p(\Omega)$, $p \in \,]1,\infty]$. Then, up to a subsequence, $\{u^\varepsilon\}$ two-scale converges to $u_0(x,y) \in L^p(\Omega \times Y)$ and converges weakly to $u(x) = \int_Y u_0(x,y)\,\mathrm{d}y$ in $L^p(\Omega)$. Furthermore, u_0 is (uniquely) expressible in the form*

$$u_0(x,y) = u(x) + \tilde{u}_0(x,y) \quad \text{with} \quad \int_Y \tilde{u}_0(x,y)\,\mathrm{d}y = 0.$$

Moreover, if $\tilde{u}_0 \neq 0$ on a subset of $\Omega \times Y$ with positive measure, then the sequence $\{u^\varepsilon\}$ will not converge strongly in $L^p(\Omega)$.

Proof. Let u_0 be the two-scale limit of $\{u^\varepsilon\}$ and let a be any function in $L^q(\Omega)$, $1/p + 1/q = 1$. We note that a is an admissible test function and it follows that

$$\lim_{\varepsilon\to 0} \int_\Omega u^\varepsilon(x) a(x)\,\mathrm{d}x = \int_\Omega \int_Y u_0(x,y) a(x)\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega \left(\int_Y u_0(x,y)\,\mathrm{d}y \right) a(x)\,\mathrm{d}x.$$

Clearly $u^\varepsilon \rightharpoonup u(x) := \int_Y u_0(x,y)\,\mathrm{d}y$ weakly in $L^p(\Omega)$. By defining $\tilde{u}_0$ as

$$\tilde{u}_0(x,y) := u_0(x,y) - \int_Y u_0(x,y)\,\mathrm{d}y,$$

we obtain the desired decomposition of $u_0(x,y)$. Moreover, if $\{u^\varepsilon\}$ tends strongly to u in $L^p(\Omega)$, then $\{u^\varepsilon\}$ will also two-scale converge to u (cf. Proposition 2.10 in [8]). It follows that $\tilde{u}_0(x,y) \equiv 0$ except on a subset of $\Omega \times Y$ with measure zero (u_0 is uniquely defined on $\Omega \times Y$ only up to a subset of measure zero). The proposition follows now by contraposition. □

We also need the following result, which will be useful when proving corrector results.

Proposition 2.7. *Assume that $u \in L^p(\Omega \times Y)$, $p \in \,]1,\infty[$ is an admissible test fuction. Then, up to a subsequence, $\left\{u\left(x, \frac{x}{\varepsilon}\right)\right\}$ two-scale converges to u.*

Proof. See [10]. □

We have the following slight generalization of Proposition 2.8 in [8] for coefficients in $L^\infty_\sharp(Y)$. The result is useful for homogenization of PDEs in nonperiodic media.

Lemma 2.8. *Let $\{u^\varepsilon\}$ be a bounded sequence in $L^p(\Omega)$, $p \in \,]1,\infty]$, and let $v \in C_0(\Omega; L^\infty_\sharp(Y))$. Then a subsequence of $\left\{u^\varepsilon(x) v\left(x, \frac{x}{\varepsilon}\right)\right\}$ two-scale converges to $u_0(x,y)v(x,y)$, where u_0 is the two-scale limit of $\{u^\varepsilon\}$.*

Proof. By the boundedness of $\{u^\varepsilon\}$ we find that also the sequence $\{u^\varepsilon(x) \times v\left(x, \frac{x}{\varepsilon}\right)\}$ is bounded in $L^p(\Omega)$. Let the corresponding two-scale limits be $u_0(x,y)$ and

$\chi_0(x,y)$. Choose $a = a_1 \cdot a_2$, $a_1 \in D(\Omega)$, $a_2 \in C_\sharp^\infty(Y)$ as an admissible test function. It follows that also $v \cdot a$ is an admissible test function, i.e.,

$$\lim_{\varepsilon\to 0} \int_\Omega u^\varepsilon(x) v\left(x, \frac{x}{\varepsilon}\right) a\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x = \int_\Omega \int_Y u_0(x,y) v(x,y) a(x,y) \,\mathrm{d}x \,\mathrm{d}y.$$

Since the test functions are chosen such that we have unique limits we conclude that $\chi_0(x,y) = u_0(x,y)v(x,y)$. □

3. The main results

In this section $\mathbb{R}^N = \mathbb{R}^3$. Further, μ, η and σ are bounded, Y-periodic and coercive tensors, i.e., there exist positive constants c_1 and c_2 such that $|\mu_{ij}\xi_j| \leqslant c_1|\xi|$, $\mu_{ij}\xi_j\xi_i \geqslant c_2|\xi|^2$ for all vectors $\xi \neq 0$. Moreover, μ and η are assumed to be symmetric.

First we state the following two-scale convergence result:

Theorem 3.1. *Any sequence $\{E^\varepsilon(x,t)\}, \{H^\varepsilon(x,t)\}$ of solutions to (1.1)–(1.9) two-scale converges in $L^\infty(0,T; L^2(\Omega \times Y)^3)$ to the limits $E(x,t) + D_y\varphi(x,y,t)$ and $H(x,t) + D_y\Phi(x,y,t)$ which are the unique solution of the following two-scale limit Maxwell system:*

$$\begin{aligned}
&\int_Y \big(\eta_{ij}(y)\partial_t + \sigma_{ij}(y)\big)\big(E_j(x,t) + \partial_{y_j}\varphi(x,y,t)\big)\,\mathrm{d}y = \big(\operatorname{rot} H(x,t)\big)_i + F_i(x,t), \\
&\int_Y \mu_{ij}(y)\partial_t\big(H_j(x,t) + \partial_{y_j}\Phi(x,y,t)\big)\,\mathrm{d}y = -\big(\operatorname{rot} E(x,t)\big)_i, \\
&\partial_{x_i}\int_Y \mu_{ij}(y)\big(H_j(x,t) + \partial_{y_j}\Phi(x,y,t)\big)\,\mathrm{d}y = 0, \\
&\partial_{x_i}\int_Y \eta_{ij}(y)\big(E_j(x,t) + \partial_{y_j}\varphi(x,y,t)\big)\,\mathrm{d}y = \varrho(x,t)
\end{aligned} \tag{3.1}$$

a.e. in $\Omega \times]0,T[$, with the boundary condition

$$n \wedge E(x,t) = 0 \quad \text{a.e. on} \quad \partial\Omega \times]0,T[,$$

initial conditions

$$E(x,0) = \int_Y E^0(x,y)\,\mathrm{d}y, \qquad H(x,0) = \int_Y H^0(x,y)\,\mathrm{d}y,$$

and local problems

$$\begin{aligned}
&\int_Y \big(\eta_{ij}(y)\partial_t\big[E_j(x,t) + \partial_{y_j}\varphi(x,y,t)\big] \\
&\qquad + \sigma_{ij}(y)\big[E_j(x,t) + \partial_{y_j}\varphi(x,y,t)\big]\big)\partial_{y_i}v_2(y)\,\mathrm{d}y = 0, \\
&\int_Y \mu_{ij}(y)\big(H_j(x,t) + \partial_{y_j}\Phi(x,y,t)\big)\partial_{y_i}v_2(y)\,\mathrm{d}y = 0
\end{aligned}$$

for all $v_2 \in W^{1,2}_\sharp(Y)/\mathbb{R}$. Here, $E^0(x,y)$ and $H^0(x,y)$ are the two-scale limits of the initial values $E^\varepsilon(x,0)$ and $H^\varepsilon(x,0)$, respectively.

We note that the last equation in (3.1) is a definition of the charge density in the two-scale limit case.

Next we present our main homogenization result.

Theorem 3.2. *Any sequence $\{E^\varepsilon(x,t)\}, \{H^\varepsilon(x,t)\}$ of solutions to (1.1)–(1.9) converges weakly* in $L^\infty(0,T;L^2(\Omega)^3)$ to the limit $E(x,t), H(x,t)$ in $W^{1,\infty}\big(0,T;\mathcal{H}_{\mathrm{rot}}(\Omega), L^2(\Omega)^3\big)$, the unique solution of the following homogenized Maxwell system:*

$$\begin{aligned}
\partial_t D(x,t) + J(x,t) &= \operatorname{rot} H(x,t) + F(x,t) + F^1(x,t),\\
\partial_t B(x,t) &= -\operatorname{rot} E(x,t),\\
\operatorname{div} B(x,t) &= 0,\\
\operatorname{div} D(x,t) &= \varrho(x,t)
\end{aligned} \tag{3.2}$$

almost everywhere in $\Omega\times\,]0,T[$, with the boundary condition

$$n\wedge E(x,t) = 0 \quad \text{a.e. on} \quad \partial\Omega\times\,]0,T[,$$

and initial conditions

$$E(x,0) = \int_Y E^0(x,y)\,\mathrm{d}y, \qquad H(x,0) = \int_Y H^0(x,y)\,\mathrm{d}y.$$

The system (3.2) is equipped with the constitutive relations

$$\begin{aligned}
B_i(x,t) &= \mu^h_{ij} H_j(x,t),\\
D_i(x,t) &= \int_0^t \eta^h_{ij}(t-\tau)E_j(x,\tau)\,\mathrm{d}\tau,\\
J_i(x,t) &= \int_0^t \sigma^h_{ij}(t-\tau)E_j(x,\tau)\,\mathrm{d}\tau,
\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}
\eta^h_{ik}(t) &= \int_Y \eta_{ij}(y)\big[\delta_{jk}\delta(t) - \partial_{y_j}\chi^k(y,t)\big]\,\mathrm{d}y,\\
\sigma^h_{ik}(t) &= \int_Y \sigma_{ij}(y)\big[\delta_{jk}\delta(t) - \partial_{y_j}\chi^k(y,t)\big]\,\mathrm{d}y.
\end{aligned}$$

Here the gradient of $\chi^k(y,t)\in D'(0,\infty;W^{1,2}_\sharp(Y)/\mathbb{R})$ is given by

$$\nabla_y\chi^k(y,t) = \nabla_y\chi^k_\eta(y)\delta(t) + \exp(-\mathrm{A}t)\mathrm{A}\nabla_y\big[\chi^k_\sigma(y) - \chi^k_\eta(y)\big]\Theta(t),$$

where ∇_y is the gradient operator with respect to y and $\Theta(t)$ is the Heaviside step function and $\exp(-\mathrm{A}t)$ is the semigroup which is generated by the infinitesimal generator A defined by $\mathrm{A}_{ij} = (\eta^{-1})_{il}\sigma_{lj}$. χ_η^k and χ_σ^k are the unique solutions in $W_\sharp^{1,2}(Y)/\mathbb{R}$ of the local problems

$$\int_Y \eta_{ij}(y)\big[\delta_{jk} - \partial_{y_j}\chi_\eta^k(y)\big]\partial_{y_i}v_2(y)\,\mathrm{d}y = 0, \text{ a.e. in } \Omega\times]0,T[\ \forall v_2 \in W_\sharp^{1,2}(Y),$$
$$\int_Y \sigma_{ij}(y)\big[\delta_{jk} - \partial_{y_j}\chi_\sigma^k(y)\big]\partial_{y_i}v_2(y)\,\mathrm{d}y = 0, \text{ a.e. in } \Omega\times]0,T[\ \forall v_2 \in W_\sharp^{1,2}(Y),$$

respectively. Furthermore, μ_{ij}^h is given by

$$\mu_{ik}^h = \int_Y \mu_{ij}(y)\big[\delta_{jk} - \partial_{y_j}\chi_\mu^k(y)\big]\,\mathrm{d}y,$$

where χ_μ^k is the unique solution in $W_\sharp^{1,2}(Y)/\mathbb{R}$ of the local problem

$$\int_Y \mu_{ij}(y)\big[\delta_{jk} - \partial_{y_j}\chi_\mu^k(y)\big]\partial_{y_i}v_2(y)\,\mathrm{d}y = 0 \text{ a.e. in } \Omega\times]0,T[\ \forall v_2 \in W_\sharp^{1,2}(Y).$$

The driving term F^1 is given by

$$F^1{}_i(x,t) = -\int_Y \big[\eta_{ij}(y)\partial_t + \sigma_{ij}(y)\big](\exp(-\mathrm{A}t))_{jl}\partial_{y_l}\big[\chi_\eta^k(y)E_k(x,0) + \varphi(x,y,0)\big]\,\mathrm{d}y,$$

where $E(x,0) + \nabla_y\varphi(x,y,0)$ is the two-scale limit of the initial electric field.

We note that in this case, the last equation in (3.2) defines a charge density other than in the previous case. This is because the two-scale limit of the induced electric field is not equal to the induced electric field in the latter case, i.e. $\eta_{ij}(y)\big(E_j(x,t) + \partial_{y_j}\varphi(x,y,t)\big) \neq \int_0^t \eta_{ij}^h(t-\tau)E_j(x,\tau)\,\mathrm{d}\tau$, which is also explaining the new source term introduced in Theorem 3.2.

Finally, we present the corrector results:

Theorem 3.3. *Let the sequences $\{E^\varepsilon(x,t)\}, \{H^\varepsilon(x,t)\}$ of unique solutions to* (1.1)–(1.6) *two-scale converge to*

$$E_j(x,t) + \partial_{y_j}\varphi(x,y,t) \text{ and } \big[\delta_{jk} - \partial_{y_j}\chi_\mu^k(y)\big]H_k(x,t),$$

respectively. Assume that $E^\varepsilon(x,0) = E_0^\varepsilon(x)$ and $H^\varepsilon(x,0) = H_0^\varepsilon(x)$ are admissible test functions. Further, $E_0^\varepsilon(x)$ and $H_0^\varepsilon(x)$ are assumed to two-scale converge to $E(x,0) + D_y\varphi(x,y,0)$ and $H(x,0) - D_y\chi_\mu^k(y)H_k(x,0)$, respectively.

(a) *If, in addition,* $\partial_{y_j}\varphi(x,y,t)$ *and* $\partial_{y_j}\chi_\mu^k(y)H_k(x,t)$ *are admissible test functions, then*

(i) $\lim\limits_{\varepsilon\to 0}\left\|E_j^\varepsilon(x,t)-E_j(x,t)-\partial_{y_j}\varphi\left(x,\dfrac{x}{\varepsilon},t\right)\right\|_{L^2(\Omega\times]0,T[)}=0$

and

(ii) $\lim\limits_{\varepsilon\to 0}\left\|H_j^\varepsilon(x,t)-H_j(x,t)+\partial_{y_j}\chi_\mu^k\left(\dfrac{x}{\varepsilon}\right)H_k(x,t)\right\|_{L^2(\Omega)}=0$

for all $t\in\,]0,T[$.

(b) *Let* $\partial_{y_j}\varphi^\delta\in D(\Omega\times]0,T[;C_\sharp^\infty(Y))$ *be a mollification of* $\partial_{y_j}\varphi\in W^{1,2}\big(0,T;L^2(\Omega;L_\sharp^2(Y))\big)$*, and let* $H_k^\delta\in D(\Omega\times]0,T[)$ *be a mollification of* H_k*. Then,*

(iii) $\lim\limits_{\delta\to 0}\lim\limits_{\varepsilon\to 0}\left\|E_j^\varepsilon(x,t)-E_j(x,t)-\partial_{y_j}\varphi^\delta\left(x,\dfrac{x}{\varepsilon},t\right)\right\|_{L^2(\Omega\times]0,T[)}=0$

and

(iv) $\lim\limits_{\delta\to 0}\lim\limits_{\varepsilon\to 0}\left\|H_j^\varepsilon(x,t)-H_j(x,t)+\partial_{y_j}\chi_\mu^k\left(\dfrac{x}{\varepsilon}\right)H_k^\delta(x,t)\right\|_{L^2(\Omega)}=0$

for all $t\in\,]0,T[$.

4. Compactness results

In this section we give some compactness results needed in the proof of the homogenization theorems. We need the following lemmas.

Lemma 4.1. *Assume that* $\Omega\in\mathbb{R}^3$ *has a regular boundary* $\partial\Omega$ *(i.e.,* $\partial\Omega$ *is a once continuously differentiable two-dimensional manifold). Then* $C^1(\overline{\Omega})^3$ *is dense in the spaces* $\mathcal{H}_{\mathrm{rot}}(\Omega)$ *and* $\mathcal{H}_{\mathrm{div}}(\Omega)$.

Proof. See [7]. □

Lemma 4.2. *Assume that* Ω *has a regular boundary* $\partial\Omega$ *with the normal* n *directed towards the exterior of* Ω*. Then the mappings*

(i) $C^1(\overline{\Omega})^3\to C^1(\partial\Omega)^3$, $u\mapsto n\wedge u|_{\partial\Omega}$

and

(ii) $C^1(\overline{\Omega})^3\to C^1(\partial\Omega)$, $\quad u\mapsto n\cdot u|_{\partial\Omega}$

can be extended by continuity to linear and continuous mappings

$$\mathcal{H}_{\mathrm{rot}}(\Omega)\to H^{-1/2}(\partial\Omega)^3 \text{ and } \mathcal{H}_{\mathrm{div}}(\Omega)\to H^{-1/2}(\partial\Omega),$$

respectively.

Proof. See [7]. □

We are now ready to present and prove the announced compactness results.

Proposition 4.3. *Let $\{u^\varepsilon\}$ be a bounded sequence in $\mathcal{H}_{\text{rot}}(\Omega)$. Then $\{u^\varepsilon\}$ has a subsequence which two-scale converges to $u_0(x,y) = u(x) + D_y\varphi(x,y)$, where φ is a scalar-valued function satisfying $\int_Y D_y\varphi(x,y)\,\mathrm{d}y = 0$. Moreover,* $\operatorname{rot} u^\varepsilon \rightharpoonup \operatorname{rot} u$ *weakly in $L^2(\Omega)^3$.*

P r o o f. Since $\{u^\varepsilon\}$ is bounded in $L^2(\Omega)^3$ it, up to a subsequence, two-scale converges to $u_0(x,y)$ in $L^2(\Omega\times Y)^3$. That is, for any $a(x,y) \in D(\Omega; C_\sharp^\infty(Y))^3$, we have

$$\lim_{\varepsilon\to 0}\int_\Omega u_i^\varepsilon(x) a_i\left(x,\frac{x}{\varepsilon}\right)\,\mathrm{d}x = \int_\Omega\int_Y u_{0_i}(x,y)a_i(x,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{4.1}$$

By using Lemma 4.2, Green's identity and the compact support of $a\left(x,\frac{x}{\varepsilon}\right)$ we obtain

$$\begin{aligned}\int_\Omega &\left[(\operatorname{rot} u^\varepsilon(x))_i\, a_i\left(x,\frac{x}{\varepsilon}\right) - u_i^\varepsilon(x)\left(\operatorname{rot} a\left(x,\frac{x}{\varepsilon}\right)\right)_i\right]\,\mathrm{d}x\\ &= -\int_{\partial\Omega}(n\wedge u^\varepsilon(x))_i\, a_i\left(x,\frac{x}{\varepsilon}\right)\,\mathrm{d}x = 0.\end{aligned}$$

Here n denotes the exterior normal to the boundary of Ω. Integration by parts and the use of the chain rule yields

$$\begin{aligned}\varepsilon\int_\Omega(\operatorname{rot} u^\varepsilon(x))_i\, a_i\left(x,\frac{x}{\varepsilon}\right)\,\mathrm{d}x &= \varepsilon\int_\Omega u_i^\varepsilon(x)\left(\operatorname{rot} a\left(x,\frac{x}{\varepsilon}\right)\right)_i\,\mathrm{d}x\\ = \varepsilon\int_\Omega u_i^\varepsilon(x)\left(\operatorname{rot}_x a\left(x,\frac{x}{\varepsilon}\right)\right)_i\,\mathrm{d}x &+ \int_\Omega u_i^\varepsilon(x)\left(\operatorname{rot}_y a\left(x,\frac{x}{\varepsilon}\right)\right)_i\,\mathrm{d}x.\end{aligned} \tag{4.2}$$

Sending $\varepsilon\to 0$ in (4.2) and using (4.1) gives

$$\int_\Omega\int_Y u_{0_i}(x,y)(\operatorname{rot}_y a(x,y))_i\,\mathrm{d}x\,\mathrm{d}y = 0,$$

which implies that $\operatorname{rot}_y u_0(x,y) = 0$ a.e. in $\Omega\times Y$. Thus we conclude that $u_0(x,y)$ is a gradient with respect to the variable y for some scalar valued function $\varphi_1(x,y)$, i.e. $u_0(x,y) = D_y\varphi_1(x,y)$. According to Proposition 2.6 this can be written as $u_0(x,y) = u(x) + D_y\varphi(x,y)$, where $u(x) = \int_Y u_0(x,y)\,\mathrm{d}y$, for some scalar-valued function $\varphi(x,y)$. Next, choose an admissible test function $a(x)\in D(\Omega)^3$. Integration by parts yields

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_\Omega(\operatorname{rot} u^\varepsilon(x))_i\, a_i(x)\,\mathrm{d}x &= \lim_{\varepsilon\to 0}\int_\Omega u_i^\varepsilon(x)(\operatorname{rot} a(x))_i\,\mathrm{d}x\\ = \int_\Omega\int_Y u_{0_i}(x,y)\,\mathrm{d}y(\operatorname{rot} a(x))_i\,\mathrm{d}x &= \int_\Omega(\operatorname{rot} u(x))_i\, a_i(x)\,\mathrm{d}x.\end{aligned}$$

The proof follows by using Proposition 2.6. $\square$

R e m a r k 4.4. If $D_y\varphi(x,y)\neq 0$, then, by Proposition 2.6, the sequence $\{u^\varepsilon\}$ will not converge strongly in $L^2(\Omega)^3$. Moreover, according to the above proof it clearly follows that $\varphi(x,\cdot)$ is Y-periodic.

We also have the following, similar compactness result, which is proved by similar arguments as in the proof of Proposition 4.3.

Proposition 4.5. *Let $\{u^\varepsilon\}$ be a bounded sequence in $\mathcal{H}_{\mathrm{div}}(\Omega)$. Then, up to a subsequence,*

$$u^\varepsilon(x) \overset{2-s}{\rightharpoonup} u_0(x,y),$$
$$\varepsilon \operatorname{div} u^\varepsilon(x) \overset{2-s}{\rightharpoonup} \operatorname{div}_y u_0(x,y),$$

and

$$\operatorname{div} u^\varepsilon(x) \rightharpoonup \operatorname{div} u(x) = \operatorname{div}\int_Y u_0(x,y)\,\mathrm{d}y \text{ weakly in } L^2(\Omega).$$

We need the following lemma, which is proved in [9]:

Lemma 4.6. *Let $u^\varepsilon(x,t)$ and $\partial_t u^\varepsilon$ be bounded uniformly in $L^\infty(0,T;L^2(\Omega)^3)$ and let $u_0(x,y,t)$ be the two-scale limit of u^ε in $L^\infty(0,T;L^2(\Omega\times Y)^3)$. Then $\partial_t u_0$ is the two-scale limit of $\partial_t u^\varepsilon$.*

5. A priori estimates

Let $\mathcal{H} = L^2(\Omega)^6 = L^2(\Omega)^3 \times L^2(\Omega)^3$, which is equipped with the usual scalar product. Let the operator $\mathcal{A}$ be defined by

$$\mathcal{A}\Phi = \{-\operatorname{rot}\psi, \operatorname{rot}\varphi\}$$

for any $\Phi\in D(\mathcal{A})$, where

$$D(\mathcal{A}) = \{\Phi=\{\varphi,\psi\}\in\mathcal{H}\colon\ \operatorname{rot}\varphi\in L^2(\Omega)^3,\ \operatorname{rot}\psi\in L^2(\Omega)^3,\ n\wedge\varphi|_{\partial\Omega}=0\}.$$

We can now state the following lemma:

Lemma 5.1. *The domain $D(\mathcal{A})$ is dense in $\mathcal{H}$ and $\mathcal{A}$ is closed. Further,*

$$\mathcal{A}^* = -\mathcal{A} \quad \text{and} \quad D(\mathcal{A}^*) = D(\mathcal{A}).$$

As usual, $\mathcal{A}^*$ denotes the adjoint operator of $\mathcal{A}$.

P r o o f. The proof follows easily after some minor changes in the proof of a similar result in [7]. □

According to Lemma 5.1 the following corollary clearly follows

Corollary 5.2. $(\mathcal{A}\Phi, \Phi)_{\mathcal{H}} = 0$ *for all* $\Phi \in D(\mathcal{A})$.

For $\Phi = \{\varphi, \theta\}$ let $\mathcal{M}^\varepsilon \Phi = \{\sigma^\varepsilon \varphi, 0\}$ define a linear operator $\mathcal{M}^\varepsilon \colon \mathcal{H} \to \mathcal{H}$. We also define a linear operator $\mathcal{N}^\varepsilon \colon \mathcal{H} \to \mathcal{H}$ by $\mathcal{N}^\varepsilon \Phi = \{\eta^\varepsilon \varphi, \mu^\varepsilon \theta\}$.

We are now in position to present the following useful a priori estimates:

Proposition 5.3. *If* $F^\varepsilon \in W^{1,2}(0,T;L^2(\Omega)^3)$ *has a bounded norm, then the sequences* E^ε, H^ε, $\partial_t H^\varepsilon$, $\partial_t E^\varepsilon$, $\operatorname{rot} H^\varepsilon$ *and* $\operatorname{rot} E^\varepsilon$ *are all bounded in* $L^\infty(0,T;L^2(\Omega)^3)$.

P r o o f. First we note that by summing up (1.1) and (1.2) we get

$$\mathcal{N}^\varepsilon \partial_t U^\varepsilon(x,t) + \mathcal{A} U^\varepsilon(x,t) + \mathcal{M}^\varepsilon U^\varepsilon(x,t) = G^\varepsilon(x,t), \tag{5.1}$$

where $U^\varepsilon = \{E^\varepsilon, H^\varepsilon\}$ and $G^\varepsilon = \{F^\varepsilon, 0\}$. Taking the scalar product in $\mathcal{H}$ with U^ε we obtain

$$\big(\mathcal{N}^\varepsilon \partial_t U^\varepsilon(t), U^\varepsilon(t)\big) + \big(\mathcal{A} U^\varepsilon(t), U^\varepsilon(t)\big) + \big(\mathcal{M} U^\varepsilon(t), U^\varepsilon(t)\big) = \big(G^\varepsilon(t), U^\varepsilon(t)\big).$$

Moreover, by using Corollary 5.2, the symmetry of η and μ, the coerciveness of σ, the definition of $\mathcal{N}^\varepsilon$, $\mathcal{M}^\varepsilon$ and the scalar product in $\mathcal{H}$, we get

$$\frac{1}{2}\partial_t\big(E^\varepsilon(t), D^\varepsilon(t)\big) + \frac{1}{2}\partial_t\big(H^\varepsilon(t), B^\varepsilon(t)\big) + C\|E^\varepsilon(t)\|^2 \leqslant \big(F^\varepsilon(t), E^\varepsilon(t)\big).$$

By integrating with respect to t and using the initial conditions and Hölder's inequality, we find that, for any $T_1 \in \,]0,T[$,

$$\begin{aligned}
\frac{1}{2}\big(E^\varepsilon(T_1), D^\varepsilon(T_1)\big) + \frac{1}{2}\big(H^\varepsilon(T_1), B^\varepsilon(T_1)\big) + C\int_0^{T_1} \|E^\varepsilon(t)\|^2 \,\mathrm{d}t \\
\leqslant \Big(\int_0^{T_1} \|F^\varepsilon(t)\|^2 \,\mathrm{d}t\Big)^{1/2} \Big(\int_0^{T_1} \|E^\varepsilon(t)\|^2 \,\mathrm{d}t\Big)^{1/2} \\
+ \frac{1}{2}\Big(\big(E^\varepsilon(0), D^\varepsilon(0)\big) + \big(H^\varepsilon(0), B^\varepsilon(0)\big)\Big).
\end{aligned}$$

The boundedness of F^ε and Gronwall's inequality yield

$$\int_0^T \|E^\varepsilon(t)\|^2 \,\mathrm{d}t \leqslant C,$$

i.e., $E^\varepsilon \in L^2(0,T;L^2(\Omega)^3)$. This implies that $\big(E^\varepsilon(t), D^\varepsilon(t)\big) \leqslant C$ and $\big(H^\varepsilon(t), B^\varepsilon(t)\big) \leqslant C$. By considering all $t \in \,]0,T[$ and the coerciveness of η and μ we arrive at

$\max\limits_{0\leqslant t\leqslant T} \|E^\varepsilon(t)\|_{L^2(\Omega)^3} \leqslant C$ and $\max\limits_{0\leqslant t\leqslant T} \|H^\varepsilon(t)\|_{L^2(\Omega)^3} \leqslant C$. Thus $E^\varepsilon, H^\varepsilon$ are bounded in $L^\infty(0,T;L^2(\Omega)^3)$.

Next, we shall derive an estimate, uniformly in ε, for the time derivatives $\partial_t E^\varepsilon$ and $\partial_t H^\varepsilon$. We start by making a partition P_N, $N \in \mathbb{N}$, of the interval $[0,T]$, i.e., $P_N := \{0 = t_0, t_1, \dots, t_N = T\colon\ t_{i-1} < t_i\}$. We consider the subintervals $I_{n(N)} = (t_{n-1}, t_n)$ and the local time steps $k_n = t_n - t_{n-1}$. Let $U_n^\varepsilon = U^\varepsilon(t_n)$ and $(U_n^\varepsilon)' = \partial_{t_n} U^\varepsilon(t_n)$ which satisfy the Maxwell equations at the time $t = t_n$. By using two sets of unique solutions at $t = t_{n-1}$ and $t = t_n$, respectively, we can write

$$\Big(\mathcal{N}^\varepsilon \frac{(U_n^\varepsilon)' - (U_{n-1}^\varepsilon)'}{k_n}, \frac{U_n^\varepsilon - U_{n-1}^\varepsilon}{k_n}\Big)_{\mathcal{H}} + \Big(\mathcal{A}\,\frac{U_n^\varepsilon - U_{n-1}^\varepsilon}{k_n}, \frac{U_n^\varepsilon - U_{n-1}^\varepsilon}{k_n}\Big)_{\mathcal{H}}$$
$$+ \Big(\mathcal{M}^\varepsilon \frac{U_n^\varepsilon - U_{n-1}^\varepsilon}{k_n}, \frac{U_n^\varepsilon - U_{n-1}^\varepsilon}{k_n}\Big)_{\mathcal{H}} = \Big(\frac{G_n^\varepsilon - G_{n-1}^\varepsilon}{k_n}, \frac{U_n^\varepsilon - U_{n-1}^\varepsilon}{k_n}\Big)_{\mathcal{H}}.$$

By using similar argument as above and letting $k_n \to 0$ we obtain

$$\frac{1}{2}\partial_t\big(\partial_t E^\varepsilon(t), \partial_t D^\varepsilon(t)\big) + \frac{1}{2}\partial_t\big(\partial_t H^\varepsilon(t), \partial_t B^\varepsilon(t)\big) + C\|\partial_t E^\varepsilon(t)\|^2 \leqslant \big(\partial_t F^\varepsilon(t), \partial_t E^\varepsilon(t)\big). \tag{5.2}$$

Moreover, from (5.1) and Lemmas 4.1 and 5.1 it follows that

$$(\mathcal{N}^\varepsilon \partial_t U^\varepsilon(0), v) = -(\mathcal{A}U_0, v) - (\mathcal{M}^\varepsilon U_0, v) + (G^\varepsilon(0), v)$$

for all $v \in C_0^\infty(\overline{\Omega})^6$. This implies that $\mathcal{N}^\varepsilon \partial_t U^\varepsilon(0) = -\mathcal{A}U_0 - \mathcal{M}^\varepsilon U_0 - G^\varepsilon(0)$ almost everywhere in Ω, i.e.,

$$\|\mathcal{N}^\varepsilon \partial_t U^\varepsilon(0)\|_{\mathcal{H}} = \|G^\varepsilon(0) - \mathcal{A}U_0 - \mathcal{M}^\varepsilon U_0\|_{\mathcal{H}} = C_0$$

for some positive constant C_0. By integrating (5.2) with respect to t and using the initial conditions and Hölder's inequality, we obtain that, for any $T_1 \in\,]0, T[$,

$$\frac{1}{2}\big(\partial_t E^\varepsilon(T_1), \partial_t D^\varepsilon(T_1)\big) + \frac{1}{2}\big(\partial_t H^\varepsilon(T_1), \partial_t B^\varepsilon(T_1)\big) + C\int_0^{T_1} \|\partial_t E^\varepsilon(t)\|^2\, \mathrm{d}t$$
$$\leqslant \Big(\int_0^{T_1} \|\partial_t F^\varepsilon(t)\|^2\, \mathrm{d}t\Big)^{1/2}\Big(\int_0^{T_1} \|\partial_t E^\varepsilon(t)\|^2\, \mathrm{d}t\Big)^{1/2} + C_0.$$

The boundedness of $\partial_t F^\varepsilon$ and Gronwall's inequality yield

$$\int_0^T \|\partial_t E^\varepsilon(t)\|^2\, \mathrm{d}t \leqslant C.$$

Similar arguments as above give

$$\max_{0\leqslant t\leqslant T} \|\partial_t E^\varepsilon(t)\|_{L^2(\Omega)^3} \leqslant C, \qquad \max_{0\leqslant t\leqslant T} \|\partial_t H^\varepsilon(t)\|_{L^2(\Omega)^3} \leqslant C.$$

Therefore $\partial_t E^\varepsilon, \partial_t H^\varepsilon$ are bounded in $L^\infty(0,T;L^2(\Omega)^3)$. Moreover, it follows from the Maxwell equations that $\operatorname{rot} H^\varepsilon = \eta^\varepsilon \partial_t E^\varepsilon + \sigma^\varepsilon E^\varepsilon - F^\varepsilon$ and $\operatorname{rot} E^\varepsilon = -\mu^\varepsilon \partial_t H^\varepsilon$ are bounded in $L^\infty(0,T;L^2(\Omega)^3)$. The proof is complete. □

6. Proofs of the main results

Proof of Theorem 3.1. By taking the divergence of (1.1), using equation (1.3) and the constitutional laws (1.7)–(1.9) we obtain

$$\partial_{x_i}\left(\eta_{ij}\left(\frac{x}{\varepsilon}\right)\partial_t E_j^\varepsilon + \sigma_{ij}\left(\frac{x}{\varepsilon}\right)E_j^\varepsilon\right) = \partial_{x_i} F_i^\varepsilon, \tag{6.1}$$

$$\partial_{x_i}\left(\mu_{ij}\left(\frac{x}{\varepsilon}\right)H_j^\varepsilon\right) = 0. \tag{6.2}$$

(The quantity $\partial_{x_i} F_i^\varepsilon$ is an external source of charges per time and volume unit.) Note that, by assumption, $\partial_{x_i} F_i^\varepsilon$ is bounded for a.e. $t \in]0,T[$, in $L^2(\Omega)$. We obtain a weak formulation of (6.1) and (6.2) by multiplying with $\varepsilon v_1 v_2$, $v_1 \in D(\Omega)$, $v_2 \in W_\sharp^{1,2}(Y)$ and integrating over Ω. Moreover, integrating by parts and using Lemmas 2.8, 4.6 and Propositions 4.3 and 4.5, we arrive at the following two-scale limit system when $\varepsilon \to 0$:

$$\begin{aligned}\int_\Omega \int_Y \big(&\eta_{ij}(y)\partial_t\big[E_j(x,t) + \partial_{y_j}\varphi(x,y,t)\big] \\ &+ \sigma_{ij}(y)\big[E_j(x,t) + \partial_{y_j}\varphi(x,y,t)\big]\big) v_1(x)\partial_{y_i} v_2(y)\,\mathrm{d}x\,\mathrm{d}y = 0,\end{aligned} \tag{6.3}$$

$$\int_\Omega \int_Y \mu_{ij}(y)\big(H_j(x,t) + \partial_{y_j}\Phi(x,y,t)\big) v_1(x)\partial_{y_i} v_2(y)\,\mathrm{d}y\,\mathrm{d}x = 0. \tag{6.4}$$

We will now study the two-scale limit of the Maxwell system (1.1)–(1.3). Using $v(x,t) = v_1(x)b(t)$, where $v_1 \in D(\Omega)$ and $b \in D(0,T)$, as test functions, we get the following weak formulation of (1.1)–(1.3):

$$\begin{aligned}\int_0^T \int_\Omega &\left[\partial_t\left[\eta_{ij}\left(\frac{x}{\varepsilon}\right)E_j^\varepsilon(x,t)\right] + \sigma_{ij}\left(\frac{x}{\varepsilon}\right)E_j^\varepsilon(x,t)\right] v_1(x)b(t)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T \int_\Omega [\operatorname{rot} H^\varepsilon(x,t) + F^\varepsilon(x,t)]_i\, v_1(x)b(t)\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

$$\int_0^T \int_\Omega \partial_t\left[\mu_{ij}\left(\frac{x}{\varepsilon}\right)H_j^\varepsilon(x,t)\right] v_1(x)b(t)\,\mathrm{d}x\,\mathrm{d}t = -\int_0^T \int_\Omega [\operatorname{rot} E^\varepsilon(x,t)]_i\, v_1(x)b(t)\,\mathrm{d}x\,\mathrm{d}t,$$

$$\int_0^T \int_\Omega \partial_{x_i}\mu_{ij}\left(\frac{x}{\varepsilon}\right)H_j^\varepsilon(x,t)v_1(x)b(t)\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Passing $\varepsilon \to 0$, while considering that $\eta_{ij}\big(\frac{x}{\varepsilon}\big)v_1(x)b(t)$, $\mu_{ij}\big(\frac{x}{\varepsilon}\big)v_1(x)b(t)$, $\sigma_{ij}\big(\frac{x}{\varepsilon}\big) \times v_1(x)b(t)$ and $v_1(x)b(t)$ are admissible test functions, we obtain, by using Lemmas 2.8 and 4.6 and Propositions 4.3 and 4.5, the following two-scale limit of the Maxwell equations:

$$\begin{aligned}
&\int_0^T \int_\Omega \int_Y \big(\eta_{ij}(y)\partial_t\big[E_j(x,t) + \partial_{y_j}\varphi(x,y,t)\big] \\
&\qquad + \sigma_{ij}(y)\big[E_j(x,t) + \partial_{y_j}\varphi(x,y,t)\big]v_1(x)b(t)\big)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\qquad = \int_0^T \int_\Omega (\operatorname{rot} H(x,t))_i\, v_1(x)b(t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega F_i(x,t)v_1(x)b(t)\,\mathrm{d}x\,\mathrm{d}t, \\
&\int_0^T \int_\Omega \int_Y \mu_{ij}(y)\partial_t\big[H_j(x,t) + \partial_{y_j}\Phi(x,y,t)\big]v_1(x)b(t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\qquad = -\int_0^T \int_\Omega (\operatorname{rot} E(x,t))_i\, v_1(x)b(t)\,\mathrm{d}x\,\mathrm{d}t, \\
&\int_0^T \int_\Omega \int_Y \partial_{x_i}\mu_{ij}(y)\big[H_j(x,t) + \partial_{y_j}\Phi(x,y,t)\big]v_1(x)b(t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = 0.
\end{aligned}$$

The charge density is defined by

$$\partial_{x_i}\int_Y \eta_{ij}(y)\big[E_j(x,t) + \partial_{y_j}\varphi(x,y,t)\big]\,\mathrm{d}y = \varrho.$$

The proof is complete. □

The two-scale limit system can be proved to have a unique solution by using similar arguments as in the ε-dependent problem (cf. [7]). The next results will be used in the proof of Theorem 3.2.

For $\nabla_y\varphi, \nabla_y\theta \in L^2_\sharp(Y)^3$ we define the scalar product

$$\langle \nabla_y\varphi, \nabla_y\theta\rangle = \int_Y \eta_{ij}(y)\partial_{y_j}\varphi(y)\partial_{y_i}\theta(y)\,\mathrm{d}y. \tag{6.5}$$

We note that this is a different scalar product compared with the one used in [17]. For fixed $\nabla_y\varphi$ (and σ) we consider the map

$$\partial_{y_i}\theta \mapsto \int_Y \sigma_{ij}(y)\partial_{y_j}\varphi(y)\partial_{y_i}\theta(y)\,\mathrm{d}y$$

which is a bounded linear functional on the Hilbert space $L^2_\sharp(Y)^3$. Thus, by the Riesz representation theorem, there exists an element $\mathrm{A}\nabla_y\varphi \in L^2_\sharp(Y)^3$ such that

$$\langle \mathrm{A}\nabla_y\varphi, \nabla_y\theta\rangle = \int_Y \sigma_{ij}(y)\partial_{y_j}\varphi(y)\partial_{y_i}\theta(y)\,\mathrm{d}y,$$

which defines an operator A. We find that $\mathrm{A}_{ij} = (\eta^{-1})_{il}\sigma_{lj}$. Obviously, A is a positive definite operator and $-\mathrm{A}$ generates a uniformly continuous semigroup, $\exp(-\mathrm{A}t)$.

R e m a r k 6.1. It follows from the Lummer-Phillips Theorem that $-\mathrm{A}$ is a dissipative operator and that $-\mathrm{A}$ is the infinitesimal generator of a C_0 semigroup of contraction, $\exp(-\mathrm{A}t)$, on $L^2_\sharp(Y)^3$ (see e.g. [14]). It follows that $\|\exp(-\mathrm{A}t)\| \leqslant 1$ for $t \geqslant 0$.

P r o o f of Theorem 3.2. We will in particular use Theorem 3.1 and some arguments in the proof of this theorem. We start with a separation of the variables in (6.4) by assuming that $\Phi(x,y,t) = -H_k(x,t)\chi^k_\mu(y)$. It follows from the Lax-Milgram Lemma that χ^k_μ is the unique solution in $W^{1,2}_\sharp(Y)/\mathbb{R}$ to the local elliptic problem

$$\int_Y \mu_{ij}(y)\big[\delta_{jk} - \partial_{y_j}\chi^k_\mu(y)\big]\partial_{y_i} v_2(y)\,\mathrm{d}y = 0 \tag{6.6}$$

for all $v_2 \in W^{1,2}_\sharp(Y)$.

We also introduce functions $\chi^k_\eta, \chi^k_\sigma \in W^{1,2}_\sharp(Y)/\mathbb{R}$, which solve the local problems

$$\begin{aligned}
&\int_Y \eta_{ij}(y)\big[\delta_{jk} - \partial_{y_j}\chi^k_\eta(y)\big]\partial_{y_i} v_2(y)\,\mathrm{d}y = 0 \ \ \forall v_2 \in W^{1,2}_\sharp(Y)/\mathbb{R},\\
&\int_Y \sigma_{ij}(y)\big[\delta_{jk} - \partial_{y_j}\chi^k_\sigma(y)\big]\partial_{y_i} v_2(y)\,\mathrm{d}y = 0 \ \ \forall v_2 \in W^{1,2}_\sharp(Y)/\mathbb{R}
\end{aligned} \tag{6.7}$$

respectively. The point of doing this is that we can identify $\eta_{ij}E_j$ and $\sigma_{ij}E_j$ with the functions $\partial_{y_i}\chi^k_\eta E_k$ and $\mathrm{A}_{ij}\partial_{y_j}\chi^k_\sigma E_k$, respectively, when they are multiplied with test functions in the space $L^2_\sharp(Y)^3$ equipped with the scalar product (6.5) in the local problem (6.3). To be explicit, by using (6.7) we get

$$\begin{aligned}
\int_Y \eta_{ij}(y)E_j(x,t)\partial_{y_i}v_2(y)\,\mathrm{d}y &= \int_Y \eta_{ij}(y)\delta_{jk}E_k(x,t)\partial_{y_i}v_2(y)\,\mathrm{d}y\\
&= \int_Y \eta_{ij}(y)\partial_{y_j}\chi^k_\eta E_k(x,t)\partial_{y_i}v_2(y)\,\mathrm{d}y \ \ \forall v_2 \in W^{1,2}_\sharp(Y)/\mathbb{R}.
\end{aligned}$$

Thus, by using the described identification, (6.3) can be written as

$$\begin{aligned}
\partial_t\langle \nabla_y\chi^k_\eta E_k(x,t) + \nabla_y\varphi(x,t), \nabla_y v_2\rangle &+ \langle \mathrm{A}\nabla_y\big(\chi^k_\eta E_k(x,t) + \varphi(x,t)\big), \nabla_y v_2\rangle\\
&= \langle \mathrm{A}\nabla_y\big(\chi^k_\eta - \chi^k_\sigma\big)E_k(x,t), \nabla_y v_2\rangle
\end{aligned}$$

for every v_2 in $W^{1,2}_\sharp(Y)/\mathbb{R}$. This is an ordinary differential equation with a solution

$$\begin{aligned}
&\nabla_y\chi^k_\eta(y)E_k(x,t) + \nabla_y\varphi(x,y,t)\\
&\qquad = \nabla_y\big[\chi^k_\eta(y)E_k(x,0) + \varphi(x,y,0)\big]\exp(-\mathrm{A}t)\\
&\qquad\quad + \int_0^t \exp(-\mathrm{A}[t-\tau])\mathrm{A}\nabla_y\big(\chi^k_\eta(y) - \chi^k_\sigma(y)\big)E_k(x,\tau)\,\mathrm{d}\tau.
\end{aligned} \tag{6.8}$$

By solving (6.8) for $\nabla_y\varphi$ the two-scale limit of the electric field can be written

$$
\begin{aligned}
&E_j(x,t)+\partial_{y_j}\varphi(x,y,t)\\
&\quad=\big[\delta_{jk}-\partial_{y_j}\chi_\eta^k(y)\big]E_k(x,t)\\
&\qquad+\int_0^t(\exp(-\mathrm{A}[t-\tau]))_{jl}\mathrm{A}_{li}\partial_{y_i}\big(\chi_\eta^k(y)-\chi_\sigma^k(y)\big)E_k(x,\tau)\,\mathrm{d}\tau\\
&\qquad+\big(\exp(-At)\big)_{jl}\partial_{y_l}\big[\chi_\eta^k(y)E_k(x,0)+\varphi(x,y,0)\big].
\end{aligned}
$$

This proves existence of a solution of the local problem.

Consider the solutions of the local equations (6.3) and (6.4). By defining

$$
\nabla_y\chi^k(y,t)=\nabla_y\chi_\eta^k(y)\delta(t)+\exp(-\mathrm{A}t)\mathrm{A}\nabla_y\big[\chi_\sigma^k(y)-\chi_\eta^k(y)\big]\Theta(t),
$$

the two-scale limit of the electric field can be written as

$$
\begin{aligned}
E_j(x,t)+\partial_{y_j}\varphi(x,y,t)=&\int_0^t\big(\delta_{jk}\delta(t-\tau)-\partial_{y_j}\chi^k(y,t-\tau)\big)E_k(x,\tau)\,\mathrm{d}\tau\\
&+(\exp(-\mathrm{A}t))_{jl}\partial_{y_l}\big[\chi_\eta^k(y)E_k(x,0)+\varphi(x,y,0)\big].
\end{aligned}
$$

Then, by inserting it in (3.1), the homogenized system can be written as

$$
\begin{aligned}
&\partial_t\int_0^t\eta_{ij}^h(t-\tau)E_j(x,\tau)\,\mathrm{d}\tau+\int_0^t\sigma_{ij}^h(t-\tau)E_j(x,\tau)\,\mathrm{d}\tau\\
&\quad=(\operatorname{rot}H(x,t))_i+F_i(x,t)\\
&\qquad-\int_Y\big(\eta_{ij}(y)\partial_t(\exp(-\mathrm{A}t))_{jl}\partial_{y_l}\big(\chi_\eta^k(y)E_k(x,0)+\varphi(x,y,0)\big)\\
&\qquad-\sigma_{ij}(y)(\exp(-\mathrm{A}t))_{jl}\partial_{y_l}\big(\chi_\eta^k(y)E_k(x,0)+\varphi(x,y,0)\big)\big)\,\mathrm{d}y,\\
&\mu_{ij}^h\partial_tH_j(x,t)=-(\operatorname{rot}E(x,t))_i,\\
&\partial_{x_i}\mu_{ij}^hH_j(x,t)\,\mathrm{d}x\,\mathrm{d}t=0,\\
&\partial_{x_i}\int_0^t\eta_{ij}^h(t-\tau)E_j(x,\tau)\,\mathrm{d}\tau=\varrho(x,t)
\end{aligned}
\tag{6.9}
$$

a.e. in $\Omega\times\,]0,T[$,

$$
n\wedge E(x,\tau)=0 \text{ a.e. on } \partial\Omega\times\,]0,T[,
$$

where the homogenized coefficients $\eta_{ik}^h(t)$, $\sigma_{ik}^h(t)$ and μ_{ik}^h are given by

$$
\begin{aligned}
\eta_{ik}^h(t)&=\int_Y\eta_{ij}(y)\big[\delta_{jk}\delta(t)-\partial_{y_j}\chi^k(y,t)\big]\,\mathrm{d}y,\\
\sigma_{ik}^h(t)&=\int_Y\sigma_{ij}(y)\big[\delta_{jk}\delta(t)-\partial_{y_j}\chi^k(y,t)\big]\,\mathrm{d}y,\\
\mu_{ik}^h&=\int_Y\mu_{ij}(y)\big[\delta_{jk}-\partial_{y_j}\chi_\mu^k(y)\big]\,\mathrm{d}y.
\end{aligned}
$$

We note that the last equation in (6.9) defines the charge density in the homogenized problem. The proof is complete. $\square$

We will now continue with the proof of the corrector results.

P r o o f of Theorem 3.3. Let $v^\varepsilon(x,t) = E(x,t) + \partial_y \varphi\big(x, \frac{x}{\varepsilon}, t\big)$, $u^\varepsilon(x,t) = H(x,t) - \partial_y \chi_\mu^k\big(\frac{x}{\varepsilon}\big) H_k(x,t)$, $r^\varepsilon(x,t) = E^\varepsilon(x,t) - v^\varepsilon(x,t)$ and $p^\varepsilon(x,t) = H^\varepsilon(x,t) - u^\varepsilon(x,t)$. By the coerciveness assumptions of σ_{ij}^ε, η_{ij}^ε, μ_{ij}^ε and the symmetry of η_{ij}^ε and μ_{ij}^ε we get

$$
\begin{aligned}
(6.10)\qquad 0 &\leqslant C\big(\|E^\varepsilon - v^\varepsilon\|^2_{L^2(]0,T[\times\Omega)^3} + \|H^\varepsilon(T) - u^\varepsilon(T)\|^2_{L^2(\Omega)^3}\big) \\
&\leqslant \int_0^T \int_\Omega \sigma_{ij}^\varepsilon r_j^\varepsilon(x,t) r_i^\varepsilon(x,t)\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_\Omega \eta_{ij}^\varepsilon r_j^\varepsilon(x,T) r_i^\varepsilon(x,T)\,\mathrm{d}x \\
&\quad + \frac{1}{2}\int_\Omega \mu_{ij}^\varepsilon p_j^\varepsilon(x,T) p_i^\varepsilon(x,T)\,\mathrm{d}x \\
&= \int_0^T \int_\Omega \sigma_{ij}^\varepsilon r_j^\varepsilon(x,t) r_i^\varepsilon(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \frac{1}{2}\int_0^T \partial_t \int_\Omega \big(\eta_{ij}^\varepsilon r_j^\varepsilon(x,t) r_i^\varepsilon(x,t) + \mu_{ij}^\varepsilon p_j^\varepsilon(x,t) p_i^\varepsilon(x,t)\,\mathrm{d}x\,\mathrm{d}t\big) \\
&\quad + \frac{1}{2}\int_\Omega \eta_{ij}^\varepsilon r_j^\varepsilon(x,0) r_i^\varepsilon(x,0)\,\mathrm{d}x + \frac{1}{2}\int_\Omega \mu_{ij}^\varepsilon p_j^\varepsilon(x,0) p_i^\varepsilon(x,0)\,\mathrm{d}x.
\end{aligned}
$$

In (6.10) there is a set of different limits which have to be evaluated. By using Corollary 5.2 and the Maxwell equations (1.1)– (1.2) and the fact that $F_i^\varepsilon \to F_i$ strongly in $L^2(\Omega)$ for any $t \in\,]0,T[$ we find that

$$
\begin{aligned}
&\int_0^T \int_\Omega J_i^\varepsilon(x,t) E_i^\varepsilon(x,t)\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_0^T \partial_t \int_\Omega D_i^\varepsilon(x,t) E_i^\varepsilon(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&+\frac{1}{2}\int_0^T \partial_t \int_\Omega B_i^\varepsilon(x,t) H_i^\varepsilon(x,t)\,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_\Omega F_i(x,t) E_i(x,t)\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
$$

as $\varepsilon \to 0$. We note that E_j and $\partial_t E_j$ are bounded in $L^2(\Omega \times]0,T[)$ so they can be considered as admissible test functions. Further, by assumption, $\partial_{y_j}\varphi\big(x, \frac{x}{\varepsilon}, t\big)$ is an admissible test function and according to Lemma 2.8, also $\sigma_{ij}^\varepsilon E_j$, $\eta_{ij}^\varepsilon \partial_t E_j$, $\eta_{ij}^\varepsilon \partial_{y_j}\varphi\big(x, \frac{x}{\varepsilon}, t\big)$ and $\sigma_{ij}^\varepsilon \partial_{y_j}\varphi\big(x, \frac{x}{\varepsilon}, t\big)$ are admissible test functions.

Integrating by parts with respect to time and taking Proposition 4.3, Lemma 4.6, the local problem (6.3) and Theorem 3.2 into account we obtain that

$$
\begin{aligned}
&-\int_0^T \int_\Omega \sigma_{ij}^\varepsilon \left(E_j(x,t) + \partial_{y_j}\varphi\left(x, \frac{x}{\varepsilon}, t\right)\right) E_i^\varepsilon(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&-\int_0^T \int_\Omega \eta_{ij}^\varepsilon \partial_t \left(E_j(x,t) + \partial_{y_j}\varphi\left(x, \frac{x}{\varepsilon}, t\right)\right) E_i^\varepsilon(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
\to &-\int_0^T \int_\Omega \int_Y \sigma_{ij}(y)(E_j(x,t) + \partial_{y_j}\varphi(x,y,t))(E_i(x,t) + \partial_{y_i}\varphi(x,y,t))\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\
&-\int_0^T \int_\Omega \int_Y \eta_{ij}(y)\partial_t\big(E_j(x,t) + \partial_{y_j}\varphi(x,y,t)\big)\big(E_i(x,t) + \partial_{y_i}\varphi(x,y,t)\big)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_0^T \int_\Omega J_i(x,t)E_i(x,t)\,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_\Omega \partial_t D_i(x,t)E_i(x,t)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Using similar arguments we find that

$$
\begin{aligned}
&-\int_0^T \int_\Omega \big(J_i^\varepsilon(x,t) + \partial_t D_i^\varepsilon(x,t)\big)\left(E_i(x,t) + \partial_{y_i}\varphi\left(x, \frac{x}{\varepsilon}, t\right)\right)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad \to -\int_0^T \int_\Omega \big(J_i(x,t) + \partial_t D_i(x,t)\big)E_i(x,t)\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^T \int_\Omega \sigma_{ij}^\varepsilon \left(E_j(x,t) + \partial_{y_j}\varphi\left(x, \frac{x}{\varepsilon}, t\right)\right)\left(E_i(x,t) + \partial_{y_i}\varphi\left(x, \frac{x}{\varepsilon}, t\right)\right)\,\mathrm{d}x\,\mathrm{d}t \\
&+\frac{1}{2}\int_0^T \partial_t \int_\Omega \eta_{ij}^\varepsilon \left(E_j(x,t) + \partial_{y_j}\varphi\left(x, \frac{x}{\varepsilon}, t\right)\right)\left(E_i(x,t) + \partial_{y_i}\varphi\left(x, \frac{x}{\varepsilon}, t\right)\right)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad \to \int_0^T \int_\Omega \big(J_i(x,t) + \partial_t D_i(x,t)\big)E_i(x,t)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Moreover, Proposition 4.3, the local problem (6.6) and the same argument as above yield

$$
\begin{aligned}
&-\frac{1}{2}\int_0^T \partial_t \int_\Omega \mu_{ij}^\varepsilon \left[\delta_{jk} - \partial_{y_j}\chi_\mu^k\left(\frac{x}{\varepsilon}\right)\right] H_k(x,t)H_i^\varepsilon(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
\to &-\int_0^T \int_\Omega \partial_t B_i(x,t)H_i(x,t)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega [\mathrm{rot}\, E(x,t)]_i H_i(x,t)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

The admissibility of $E^\varepsilon(x,0)$ and $H^\varepsilon(x,0)$ yields

$$
\frac{1}{2}\int_\Omega \eta_{ij}^\varepsilon r_j^\varepsilon(x,0)r_i^\varepsilon(x,0)\,\mathrm{d}x + \frac{1}{2}\int_\Omega \mu_{ij}^\varepsilon p_j^\varepsilon(x,0)p_i^\varepsilon(x,0)\,\mathrm{d}x \to 0.
$$

Then, by using Corollary 5.2, we conclude that the limit of (6.10) is bounded by

$$\int_0^T \int_\Omega (F_i - J_i - \partial_t D_i)E_i + (\operatorname{rot} E)_i H_i \, \mathrm{d}x \, \mathrm{d}t$$
$$= \int_0^T \int_\Omega [-(\operatorname{rot} H)_i E_i + (\operatorname{rot} E)_i H_i] \, \mathrm{d}x \, \mathrm{d}t = 0.$$

This proves (i). Integrating with respect to time (t) to any $T_1 \in [0, T]$ and using the same argument proves (ii). The proof of (a) is complete.

(b) Let $v^{\varepsilon,\delta}(x,t) = E(x,t) + \partial_y \varphi^\delta\left(x, \frac{x}{\varepsilon}, t\right)$, $u^{\varepsilon,\delta}(x,t) = H(x,t) - \partial_y \chi_\mu^k\left(\frac{x}{\varepsilon}\right) H_k^\delta(x,t)$, $r^{\varepsilon,\delta}(x,t) = E^\varepsilon(x,t) - v^{\varepsilon,\delta}(x,t)$ and $p^{\varepsilon,\delta}(x,t) = H^\varepsilon(x,t) - u^{\varepsilon,\delta}(x,t)$. Similar argument as in (a) gives

$$\begin{aligned} 0 &\leqslant C \lim_{\varepsilon \to 0} \left(\|E^\varepsilon - v^{\varepsilon,\delta}\|^2_{L^2(]0,T[\times\Omega)^3} + \|H^\varepsilon(T) - u^{\varepsilon,\delta}(T)\|^2_{L^2(\Omega)^3} \right) \\ &\leqslant \int_0^T \int_\Omega [-(\operatorname{rot} H)_i E_i + (\operatorname{rot} E)_i H_i] \, \mathrm{d}x \, \mathrm{d}t \\ &\quad + \left(\|\sigma_{ij}\left(E_j + \partial_{y_j}\varphi^\delta\right)\|_{L^2(\Omega\times Y\times]0,T[)} \right. \\ &\quad \left. + \|\eta_{ij}\partial_t\left(E_j + \partial_{y_j}\varphi^\delta\right)\|_{L^2(\Omega\times Y\times]0,T[)} \right) \|\partial_{y_i}\varphi^\delta - \partial_{y_i}\varphi\|_{L^2(\Omega\times Y\times]0,T[)} \\ &\quad + \|\mu_{ij}\partial_t\left(H_j + \partial_{y_j}\chi_\mu^k H_k^\delta\right)\|_{L^2(\Omega\times Y\times]0,T[)} \|\partial_{y_j}\chi_\mu^l\left(H_l - H_l^\delta\right)\|_{L^2(\Omega\times Y\times]0,T[)} \to 0 \end{aligned}$$

as $\delta \to 0$ by assumption and Corollary 5.2. This proves (iii). Integrating with respect to time (t) to any $T_1 \in [0, T]$ and using the same argument proves (iv). The proof is complete. □

7. Concluding remarks

Remark 7.1. Existence and uniqueness of solutions to (3.2)–(3.3) supplied with homogeneous initial conditions are given in [15]. Their proof also holds in the case with non-homogeneous initial conditions.

Remark 7.2. The memory effect indicates that the homogenized material possesses frequency dependent properties. In the special case when the conductivity is proportional to the permittivity, i.e. $\sigma(y) = \text{constant} * \eta(y)$ for almost every $y \in Y$, the memory effect vanishes and the local equations boil down to two elliptic equations (see also [15], [16] and [17]). This corresponds to the case where we have no surface charge distributions on the boundaries between the two different materials in the domain (e.g. see [18]). We conclude that the memory effect in the homogenized system is caused by surface charges on the internal boundaries between different materials. It is remarkable that this effect remains even in the case when the conductivity and permittivity functions are continuously distributed without any discontinuity surfaces in the medium. We also get pure elliptic local problems when the displacement current can be neglected or when $J = 0$ (see for instance [21]).

R e m a r k 7.3. We also remark that the solution of the local equation generates an extra driving term in the homogenized Maxwell system. This effect vanishes if the initial electric field has a vanishing two-scale limit.

R e m a r k 7.4. By using Lemma 2.8 we find that the main Theorems 3.1–3.3 hold true also in the case when $\eta_{ij}, \mu_{ij}, \sigma_{ij} \in C_0(\Omega; L^\infty_\sharp(Y))$. This is, for example, the case when we have a nonperiodic material with piecewise constant properties.

R e m a r k 7.5. A final remark is that the charge density in the original ε-problem, i.e. ϱ^ε, cannot be bounded in $L^2(\Omega)$. That would give a contradiction to our local problem for the electric field. This can also be understood from the heuristic point of view. The charge density can be expected to grow as $1/\varepsilon$, due to the growing number of discontinuity surfaces in the material and the corresponding surface charges.

Acknowledgments. This work has in part been carried out when the author was visiting the Department of Mathematics at UCSB in Santa Barbara during the academic year 1996–97. The author would like to thank the Department of Mathematics for the hospitality he experienced during this stay. The author wants to thank Assoc. Prof. Nils Svanstedt, Chalmers University of Technology, Professor Lars Erik Persson, Luleå University of Technology, Professor Björn Birnir, University of California at Santa Barbara and Dr. Eero Sakksman, University of Helsinki, for valuable discussions and suggestions during the preparation of this paper. Finally, the University of Gävle-Sandviken, Luleå University of Technology, the Arne S. Lundbergs Fund for Applied Geosciences, the Swedish Research Council for Engineering Sciences (TFR) and the Swedish Institute are acknowledged for their financial support.

References

[1] *G. Allaire*: Homogenization and two-scale convergence. SIAM J. Math. Anal. *23* (1992), 1482–2518.

[2] *M. Artola*: Homogenization and electromagnetic wave propagation in composite media with high conductivity inclusions. In: Proceedings of the Second Workshop Composite Media and Homogenization Theory (G. Dal Maso and G. Dell'Antonio, eds.). World Scientific Publishing Company, Singapore-New York-London, 1995.

[3] *M. Artola, M. Cessenat*: Un probléme raide avec homogénéisation en électromagnétisme. C. R. Acad. Sci. Paris, Sér. I Math. *310* (1990), 9–14.

[4] *M. Artola, M. Cessenat*: Diffraction d'une onde électromagnetique par un obstacle borné à permittivité et perméabilité élevées. C. R. Acad. Sci. Paris, Sér. I Math. *314* (1992), 349–354.

[5] *A. Bensoussan, J. L. Lions, G. Papanicolaou*: Asymptotic Analysis for Periodic Structures. Studies in Mathematics and its Applications. North-Holland Publishing Company, Amsterdam-New York-Oxford, 1978.
[6] *M. Cessenat*: Mathematical Methods in Electromagnetism. Linear Theory and Applications. Series on Advances in Mathematics for Applied Sciences, Vol 41. World Scientific Publishing Company, Singapore-New York-London, 1996.
[7] *G. Duvaut, J. L. Lions*: Inequalities in Mechanics and Physics. Springer-Verlag, Berlin-Heidelberg-New York, 1976.
[8] *A. Holmbom*: Homogenization of parabolic equations: an alternative approach and some corrector-type results. Appl. Math. *42* (1997), 321–343.
[9] *A. Holmbom*: The concept of parabolic two-scale convergence, a new compactness result and its application to homogenization of evolution partial differential equations. Research report 1994–18. Mid-Sweden University, Östersund.
[10] *A. Holmbom*: Some Modes of Convergence and Their Application to Homogenization and Optimal Composites Design. Ph.D. thesis, Luleå University of Technology, 1996.
[11] *P. A. Markowich, F. Poupaud*: The Maxwell equation in a periodic medium: Homogenization of the energy density. Ann. Sc. Norm. Sup. Pisa Cl. Sci. *23* (1996), 301–324.
[12] *A. Negro*: Some problems of homogenization in quasistationary Maxwell equations. In: Applications of Multiple Scaling in Mechanics. Proc. Int. Conf., Ecole Normale Superieure, Paris 1986, Rech. Math. Appl. 4. Masson, Paris, 1987, pp. 246–258.
[13] *G. Nguetseng*: A general convergence result for a functional related to the theory of homogenization. SIAM J. Math. Anal. *20* (1989), 608–623.
[14] *A. Pazy*: Semigroups of Linear Operators and Applications to Partial Differential Equations. Applied Mathematical Sciences 44. Springer-Verlag, New York, 1983.
[15] *J. Sanchez-Hubert*: Étude de certaines équations intégrodifférentielles issues de la théorie de l'homogénéisation. Boll. Un. Mat. Ital. B 5 *16* (1979), 857–875.
[16] *J. Sanchez-Hubert, E. Sanchez-Palencia*: Sur certain problémes physiques d'homogénéisation donnant lieu à des phénomènes de relaxation. C. R. Acad. Sci. Paris, Sér. A *286* (1978), 903–906.
[17] *E. Sanchez-Palencia*: Non-homogeneous Media and Vibration Theory Lecture Notes in Physics 127. Springer-Verlag, Berlin-Heidelberg-New York, 1980.
[18] *J. Vanderlinde*: Classical Electromagnetic Theory. John Wiley & Sons, New York, 1993.
[19] *J. Wyller, N. Wellander, F. Larsson, D. S. Parasnis*: Burger's equation as a model for the IP phenomenon. Geophysical Prospecting *40* (1992), 325–341.
[20] *E. Zeidler*: Nonlinear Functional Analysis and its Applications, Volumes IIA and IIB. Springer-Verlag, Berlin, 1990.
[21] *V. V. Zhikov, S. M. Kozlov and O. A. Oleinik*: Homogenization of Differential Operators and Integral Functionals. Springer-Verlag, Berlin, 1994.

Author's address: *Niklas Wellander*, Defence Research Establishment (FOA), Department of Microwave Technology, P.O. Box 1165, SE-581 11 Linköping, Sweden, email: `niklas@lin.foa.se`, and Department of Mathematics, Luleå University of Technology, SE-971 87 Luleå, Sweden.