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ON SOME PROPERTIES OF SOLUTIONS OF QUASILINEAR DEGENERATE PARABOLIC EQUATIONS IN $\mathbb{R}^m \times (0, +\infty)$

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Abstract. We study the asymptotic behaviour near infinity of the weak solutions of the Cauchy-problem.

Keywords: weak subsolution, degenerate equation, unbounded domain, asymptotic behaviour

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1. Introduction

We consider the Cauchy problem

\begin{align*}
\frac{\partial u}{\partial t} &= \sum_{i=1}^{m} \frac{\partial}{\partial x_i} a_i(x, t, u, \nabla u) - c_0 u - f(x, t, u, \nabla u) \quad \text{in } Q = \mathbb{R}^m \times (0, +\infty) \\
(1.2) \quad u(x, 0) &= 0 \quad \text{in } \mathbb{R}^m
\end{align*}

assuming degenerate ellipticity condition

\begin{align*}
(1.3) \quad \lambda(|u|) \sum_{i=1}^{m} a_i(x, t, u, p)p_i &\geq \nu(x)\psi(t)|p|^2,
\end{align*}

where $\nu(x)$, $\psi(t)$ and $\lambda(s)$ are nonnegative functions verifying additional conditions to be precised later.

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A model representative of (1.1) is as follows

\[
\frac{\partial u}{\partial t} = \sum_{i=1}^{m} \frac{\partial}{\partial x_i} \left( |x|^{\alpha \beta} \frac{\partial u}{\partial x_i} \right) - \lambda(x,t)u|u|^{p-2},
\]

where \(0 \leq \alpha < 2, \beta > 0, \lambda(x,t) \in L^1(Q)^+\) and \(p \geq 2\).

At present time many results have been established concerning linear and quasilinear degenerate parabolic second or high-order equations. Existence and boundedness of weak solutions of equations of the same class as in the present paper have already been studied, for instance, in [2], [3], [4], [10] and [11]. For regularity results such as Hölder continuity we refer the reader to [12]. Our goal is to study the asymptotic behavior near infinity of any weak solution of the problem (1.1)–(1.2). Analogous result for quasilinear degenerate elliptic equation is contained in [5], while results concerning asymptotic properties of the weak solutions to the parabolic equation

\[
\frac{\partial u}{\partial t} - \sum_{i=1}^{m} \frac{\partial}{\partial x_i} \left( a_{i,j}(x,t) \frac{\partial u}{\partial x_i} \right) + a(x)u|u|^{p-2} = 0,
\]

subject to the Neumann boundary condition, are obtained in [6] via comparison principles.

2. HYPOTHESES AND FORMULATION OF THE MAIN RESULT

Let \(\mathbb{R}^m\) denote the Euclidean \(m\)-space \((m > 2)\) with generic point \(x = (x_1, x_2, \ldots, x_m)\). We denote by \(Q_T\) the cylinder \(\mathbb{R}^m \times ]0, T[\), \(T > 0\).

**Hypothesis 2.1.** Let \(\nu(x)\) be a positive and measurable function defined in \(\Omega\) such that:

\[
\nu(x) \in L^\infty_{\text{loc}}(\mathbb{R}^m), \quad \nu^{-1}(x) \in L^g_{\text{loc}}(\mathbb{R}^m) \quad \left( g > \frac{m}{2} \right).
\]

**Hypothesis 2.2.** Let \(\psi(t)\) be a positive measurable monotone nondecreasing function defined in \([0, +\infty[\).

There exists a positive number \(\tilde{g}\) such that \(1/\psi \in L^{\tilde{g}}(0, T), \forall T > 0\).

Assumptions (2.1), (2.2) are classical in the theory of weighted parabolic equations (see [10] for more details).

The symbol \(W^{1,0}(\nu \psi, Q)\) stands for the set of all real valued functions \(u \in L^2(Q)\) such that their derivatives (in the sense of distributions), with respect to \(x_i\), are functions which have the following property

\[
\sqrt{\nu \psi} \frac{\partial u}{\partial x_i} \in L^2(Q), \quad i = 1, 2, \ldots, m.
\]
\( W^{1,0}(\nu \psi, Q) \) is a Hilbert space with respect to the norm
\[
\|u\|_{1,0} = \left( \int_Q \left( |u|^2 + \sum_{i=1}^m \nu \psi \left| \frac{\partial u}{\partial x_i} \right|^2 \right) \, dx \, dt \right)^{\frac{1}{2}}.
\]

\( W^{1,1}(\nu \psi, Q) \) is the subset of \( W^{1,0}(\nu \psi, Q) \) of all functions \( u \) such that \( \partial u/\partial t \) (in the sense of distributions) belongs to \( L^2(Q) \). We can suppose that any function of \( W^{1,1}(\nu \psi, Q) \) is continuous in \([0, +\infty[\) with respect to values in \( L^2(\mathbb{R}^m) \).

**Hypothesis 2.3.** The functions \( f(x,t,u,p), \ a_i(x,t,u,p) \ (i = 1, 2, \ldots, m) \) are Carathéodory functions in \( Q \times \mathbb{R} \times \mathbb{R}^m \), i.e. measurable with respect to \((x,t)\) for any \((u,p) \in \mathbb{R} \times \mathbb{R}^m\), continuous with respect to \((u,p)\) for a.e. \((x,t)\) in \( Q \). \( \lambda: [0, +\infty[ \rightarrow [1, +\infty[ \) is monotone nondecreasing.

**Hypothesis 2.4.** There exists a function \( f^*(x,t) \in L^1(Q) \) such that
\[
|f(x,t,u,p)| \leq \lambda(|u|) \left[ f^*(x,t) + \nu(x)\psi(t)|p|^2 \right]
\]
holds for almost every \((x,t) \in Q\) and for all real numbers \( u, p_1, p_2, \ldots, p_m \).

**Hypothesis 2.5.** There exist a function \( f_0(x,t) \in L^1(Q) \cap L^\infty(Q) \) and a non-negative real number \( c_1 < c_0 \) such that for almost every \((x,t) \in Q\) and for all real numbers \( u, p_1, p_2, \ldots, p_m \) the inequality
\[
uf(x,t,u,p) + c_1^2 + \lambda(|u|)\nu(x)\psi(t)|p|^2 + f_0(x,t) \geq 0
\]
holds.

**Hypothesis 2.6.** There exists a function \( a^*(x,t) \in L^2(Q) \) such that, for almost every \((x,t) \in Q\), we have
\[
\frac{|a_i(x,t,u,p)|}{\sqrt{\nu \psi}} \leq \lambda(|u|)[a^*(x,t) + \sqrt{\nu \psi}|p|]
\]
for any real numbers \( u, p_1, p_2, \ldots, p_m \).

**Hypothesis 2.7.** The condition (1.3) is satisfied for almost every \((x,t) \in Q\) and for all real numbers \( u, p_1, p_2, \ldots, p_m \).

**Hypothesis 2.8.** For almost every \((x,t) \in Q\) we have
\[
\sum_{i=1}^m [a_i(x,t,u,p) - a_i(x,t,u,q)] (p_i - q_i) \geq 0
\]
for any real numbers $u, p_1, p_2, \ldots, p_m, q_1, q_2, \ldots, q_m$; the inequality holds if and only if $p \neq q$.

Now, we are in position to give the definition of weak solution of the equation (1.1).

**Definition 1.** A weak solution of the problem (1.1)-(1.2) in $Q = \mathbb{R}^m \times [0, +\infty[$ is a function $u(x, t) \in W^{1,0}(\nu \psi, Q) \cap L^\infty(Q)$ such that the equality

$$
\int_0^{+\infty} \int_{\mathbb{R}^m} \left\{ \sum_{i=1}^m a_i(x, t, u, \nabla u) \frac{\partial w}{\partial x_i} + c_0 uw + f(x, t, u, \nabla u)w - u \frac{\partial w}{\partial t} \right\} \, dx \, dt = 0
$$

holds for any $w \in W^{1,1}(\nu \psi, Q) \cap L^\infty(Q)$.

Under Hypotheses 2.1–2.8 the existence of a weak solution $u$ of the equation (1.1) follows from the results of [3], [4], [7], [8] and [9].

The following theorem states the asymptotic behavior of the solutions near infinity.

**Theorem 2.1.** Let Hypotheses 2.1–2.8 be satisfied and let $R_0$ be a positive real number such that

$$
\text{supp } a^*(x, t), \text{supp } f_0(x, t), \text{supp } f^*(x, t) \subseteq \{ x \in \mathbb{R}^m ; |x| \leq R_0 \} \times [0, +\infty[.
$$

Take a function $u(x, t) \in W^{1,0}(\nu \psi, Q) \cap L^\infty(Q)$ which satisfies (2.5) for all $w \in W^{1,1}(\nu \psi, Q) \cap L^\infty(Q)$. Then for any $T > 0$ there exist two positive constants $\beta$ and $\tilde{\gamma}$, depending on $L = \text{ess sup}_Q |u(x, t)|$, such that

$$
H_R(T) \leq \beta \left\{ \|f_0\|_{L^1(Q_T)} + \|f^*\|_{L^1(Q_T)} \right\} e^{-\frac{\tilde{\gamma}(R - R_0)^2}{(R(R+3\nu T))^2}} \quad \forall R > R_0,
$$

where

$$
H_R(T) = \int_{|x| > R} u^2(x, T) \, dx + \int_0^T \int_{|x| > R} \nu \psi |\nabla u|^2 \, dx \, dt,
$$

and

$$
\eta(R) = \sup_{R < |x| < 2R} \nu(x).
$$
3. Proof of Theorem 2.1

Let $R > R_0$, $0 < \varrho \leq R$ and

$$
\xi(x) = \xi(|x|) = \begin{cases} 
0 & \text{if } |x| \leq R \\
\frac{|x| - R}{\varrho} & \text{if } R < |x| < R + \varrho \\
1 & \text{if } |x| \geq R + \varrho.
\end{cases}
$$

For fixed $T > 0$, we extend $u(x, t)$ by zero in $\mathbb{R}^m \times ]-\infty, +\infty[$ and for any $n, s \in \mathbb{N}$, we define

$$
\Theta_n(t) = \begin{cases} 
0 & \text{if } t \leq 0 \\
nt & \text{if } 0 < t \leq \frac{1}{n} \\
1 & \text{if } \frac{1}{n} < t \leq T \\
1 + n(T - t) & \text{if } T < t \leq T + \frac{1}{n} \\
0 & \text{if } t \geq T + \frac{1}{n},
\end{cases}
$$

$$
v_n^s(x, t) = s\Theta_n(t) \int_t^{t+1/s} u(x, \lambda)|u(x, \lambda)|^{\gamma} \Theta_n^{\gamma+1}(\lambda) \, d\lambda
$$

where $\gamma > 0$ will be chosen later.

Taking $\xi^2(x)v_n^s(x, t)$ as test function in (2.5) we obtain

$$
\int_0^{+\infty} \int_{\mathbb{R}^m} \left\{ \sum_{i=1}^m a_i(x, t, u, \nabla u) \frac{\partial}{\partial x_i} (\xi^2(x)v_n^s) + c_0 u\xi^2(x)v_n^s \right\} \, dx \, dt = 0.
$$

We note that

$$
\frac{\partial v_n^s}{\partial x_i} = s\Theta_n(t) \int_t^{t+1/s} \frac{\partial}{\partial x_i} (u(x, \lambda)|u(x, \lambda)|^{\gamma}) \Theta_n^{\gamma+1}(\lambda) \, d\lambda,
$$

so, according to the Hypothesis 2.2, we have

$$
\nu(x)\psi(t) \left\| \frac{\partial v_n^s}{\partial x_i} \right\|^2 \leq s\nu(x) \int_t^{t+1/s} \psi(\lambda) \left\| \frac{\partial}{\partial x_i} (u(x, \lambda)|u(x, \lambda)|^{\gamma}) \right\|^2 \, d\lambda.
$$
Moreover, it follows that
\[
\int_0^{+\infty} \int_{\mathbb{R}^m} u \xi^2(x) \Theta_n(t) \frac{\partial}{\partial t} \left( s \int_t^{t+1/s} u(x, \lambda) |u(x, \lambda)|^{\gamma} \Theta_n^{\gamma+1}(\lambda) \, d\lambda \right) \, dx \, dt
\]
\[
= s \int_{-\infty}^{+\infty} \int_{\mathbb{R}^m} u(x, t) \xi^2(x) \Theta_n(t) \left[ u \left(x, t + \frac{1}{s}\right) \right]^{\gamma} \times \Theta_n^{\gamma+1} \left(t + \frac{1}{s}\right) - u(x, t) |u(x, t)|^{\gamma} \Theta_n^{\gamma+1}(t) \, dx \, dt
\]
\[
\leq s \left( \int_{-\infty}^{+\infty} \int_{\mathbb{R}^m} |u(x, t)|^{\gamma+2} \xi^2(x) \Theta_n^{\gamma+2}(t) \, dx \, dt \right)^{1/(\gamma+2)} \left( \int_{-\infty}^{+\infty} \int_{\mathbb{R}^m} |u \left(x, t + \frac{1}{s}\right)|^{\gamma+2} \, dx \, dt \right)^{\frac{\gamma+2}{\gamma+1}}
\]
\[\times \xi^2(x) \Theta_n^{\gamma+2} \left(t + \frac{1}{s}\right) \, dx \, dt \right) - s \int_{-\infty}^{+\infty} \int_{\mathbb{R}^m} |u(x, t)|^{\gamma+2} \xi^2(x) \Theta_n^{\gamma+2}(t) \, dx \, dt = 0.
\]

Then, from (3.1), letting \( s \to +\infty \), we get
\[
\int_0^{+\infty} \int_{\mathbb{R}^m} \left\{ \sum_{i=1}^m a_i(x, t, u, \nabla u) \frac{\partial}{\partial x_i} (u|u|^{\gamma} \xi^2(x)) \Theta_n^{\gamma+2}(t) \right.
\]
\[+ c_0 |u|^{\gamma+2} \xi^2(x) \Theta_n^{\gamma+2}(t) + f(x, t, u, \nabla u) u|u|^{\gamma} \xi^2(x) \Theta_n^{\gamma+2}(t)
\]
\[\left. - |u|^{\gamma+2} \xi^2(x) \Theta_n^{\gamma+1}(t) \Theta_n^{\gamma+1}(t) \right\} \, dx \, dt \leq 0.
\]

On the other hand, for \( \sigma \in ]0, 1[, \) we have
\[
\int_0^{+\infty} \int_{\mathbb{R}^m} |u|^{\gamma+2} \xi^2(x) \Theta_n^{\gamma+1}(t) \, dx \, dt
\]
\[
\geq - n \int_0^{1/n} \int_{\mathbb{R}^m} |u|^{\gamma+2} \xi^2(x) \, dx \, dt + n(1 - \sigma)^{\gamma+1} \int_T^{T+\sigma/n} \int_{\mathbb{R}^m} |u|^{\gamma+2} \xi^2(x) \, dx \, dt.
\]

Combining (3.2) and (3.3), for \( n \to +\infty \), we obtain
\[
\int_0^T \int_{\mathbb{R}^m} \left\{ \sum_{i=1}^m a_i(x, t, u, \nabla u) \frac{\partial}{\partial x_i} (u|u|^{\gamma} \xi^2(x)) + c_0 |u|^{\gamma+2} \xi^2(x)
\]
\[+ f(x, t, u, \nabla u) u|u|^{\gamma} \xi^2(x) \right\} \, dx \, dt
\]
\[+ (1 - \sigma)^{\gamma+1} \sigma \int_{\mathbb{R}^m} |u(x, T)|^{\gamma+2} \xi^2(x) \, dx \leq 0.
\]

Let us prove, for instance, that
\[
\lim_{n \to +\infty} \int_0^{+\infty} \int_{\mathbb{R}^m} \sum_{i=1}^m a_i(x, t, u, \nabla u) \frac{\partial}{\partial x_i} (u|u|^{\gamma} \xi^2(x)) \Theta_n^{\gamma+2}(t) \, dx \, dt
\]
\[= \int_0^T \int_{\mathbb{R}^m} \sum_{i=1}^m a_i(x, t, u, \nabla u) \frac{\partial}{\partial x_i} (u|u|^{\gamma} \xi^2(x)) \, dx \, dt.
\]
In fact
\[
\Theta_n(t) \rightarrow \begin{cases} 
0 & \text{if } t \leq 0 \\
1 & \text{if } 0 < t < T \\
0 & \text{if } t \geq T;
\end{cases}
\]

moreover, the integration of the first term of the previous relation can be evaluated in \( \mathbb{R}^m \times [0, T + 1\) where a.e. we have
\[
\left| \sum_{i=1}^{m} a_i(x, t, u, \nabla u) \frac{\partial}{\partial x_i} \left( u|u|^{\gamma \nu \psi} \Theta_n^{\gamma + 1}(t) \right) \right| \\
\leq \sum_{i=1}^{m} \left| a_i(x, t, u, \nabla u) \right| \left| \frac{\partial}{\partial x_i} \left( u|u|^{\gamma \xi^2}(x) \right) \right| \quad \forall n \in \mathbb{N}.
\]

Then with respect to these facts, our assertion is true due to the Lebesgue theorem.

Next, choosing \( \sigma = (\gamma + 2)^{-1} \) and using the growth conditions in the right-hand side of the inequality (3.4) we have
\[
\int_0^T \int_{\mathbb{R}^m} (\gamma + 1)|u|^\gamma \frac{\nu \psi}{\lambda(L)} |\nabla u|^2 \xi^2(x) \, dx \, dt + (c_0 - c_1) \int_0^T \int_{\mathbb{R}^m} |u|^{\gamma + 2} \xi^2(x) \, dx \, dt \\
- \lambda(L) \int_0^T \int_{\mathbb{R}^m} |u|^{\gamma + 2} \xi^2(x) \, dx \, dt + \frac{1}{e(\gamma + 2)} \int_0^T \int_{\mathbb{R}^m} |u(x, T)|^{\gamma + 2} \xi^2(x) \, dx \\
\leq \int_0^T \int_{\mathbb{R}^m} |f_0||u|^\gamma \xi^2(x) \, dx \, dt - 2 \int_0^T \int_{\mathbb{R}^m} \sum_{i=1}^{m} a_i(x, t, u, \nabla u) \frac{\partial \xi}{\partial x_i} u|u|^\gamma \xi(x) \, dx \, dt,
\]

and from this, for \( \gamma \) such that \( (\gamma + 1)/\lambda(L) - \lambda(L) > 1 \) \( (\gamma > 1) \),
\[
\int_0^T \int_{\mathbb{R}^m} |u|^\gamma \nu \psi |\nabla u|^2 \xi^2(x) \, dx \, dt + \frac{1}{e(\gamma + 2)} \int_0^T \int_{\mathbb{R}^m} |u(x, T)|^{\gamma + 2} \xi^2(x) \, dx \\
\leq \int_0^T \int_{\mathbb{R}^m} |f_0||u|^\gamma \xi^2(x) \, dx \, dt \\
+ 2 \int_0^T \int_{\mathbb{R}^m} \sum_{i=1}^{m} |a_i(x, t, u, \nabla u)| \left| \frac{\partial \xi}{\partial x_i} \right| |u|^\gamma + 1 \xi(x) \, dx \, dt.
\]

By Hypothesis 2.6 and the Young inequality it results
\[
\int_0^T \int_{\mathbb{R}^m} \sum_{i=1}^{m} |a_i(x, t, u, \nabla u)| \left| \frac{\partial \xi}{\partial x_i} \right| |u|^\gamma + 1 \xi(x) \, dx \, dt \\
\leq \beta_1 \left\{ \int_0^T \int_{\mathbb{R}^m} a^* \sqrt{\nu \psi} |u|^\gamma \xi |\nabla \xi| \, dx \, dt + \frac{\varepsilon}{2} \int_0^T \int_{\mathbb{R}^m} |u|^{\gamma + 2} \nu \psi |\nabla u|^2 \xi^2(x) \, dx \, dt \\
+ \frac{1}{2\varepsilon} \int_0^T \int_{\mathbb{R}^m} \nu \psi |u|^{\gamma + 2} |\nabla \xi|^2 \, dx \, dt \right\}.
\]
Hence, taking into account that \( \text{supp } a^\ast(x, t) \) and \( \text{supp } f_0(x, t) \) are subsets of \( \{ x \in \mathbb{R}^m ; |x| \leq R_0 \} \times [0, +\infty[ \), from (3.5) for \( \varepsilon > 0 \) sufficiently small, we obtain

\[
(3.6) \quad \int_0^T \int_{\mathbb{R}^m} |u|^\gamma \nu \psi |\nabla u|^2 \xi^2(x) \, dx \, dt \leq \beta_2 \int_0^T \int_{\mathbb{R}^m} \nu \psi |u|^2 |\nabla \xi|^2 \, dx \, dt.
\]

On the other hand, from (3.4) for \( \gamma = 0 \) and \( \sigma = \frac{1}{2} \), we get

\[
\int_0^T \int_{\mathbb{R}^m} \left\{ \sum_{i=1}^m a_i(x, t, u, \nabla u) \frac{\partial}{\partial x_i} (u \xi^2(x)) + c_0 u^2 \xi^2(x) + f(x, t, u, \nabla u) u \xi^2(x) \right\} \, dx \, dt
\]

\[
+ 120 \quad \frac{1}{2e} \int_{\mathbb{R}^m} |u(x, T)|^2 \xi^2(x) \, dx \leq 0.
\]

From this, according to Hypotheses 2.4, 2.6 and 2.7, we have

\[
\lambda(L) \int_0^T \int_{\mathbb{R}^m} \nu \psi |\nabla u|^2 \xi^2(x) \, dx \, dt + \frac{1}{2e} \int_{\mathbb{R}^m} |u(x, T)|^2 \xi^2(x) \, dx
\]

\[
\leq \lambda(L) \left( \int_0^T \int_{\mathbb{R}^m} |u|^\gamma \nu \psi |\nabla u|^2 \xi^2(x) \, dx \, dt \right)^{\frac{1}{\gamma}} \left( \int_0^T \int_{\mathbb{R}^m} \nu \psi |\nabla u|^2 \xi^2(x) \, dx \, dt \right)^{\frac{2-\gamma}{2}}
\]

\[
+ 2\lambda(L) \int_0^T \int_{\mathbb{R}^m} \nu \psi |\nabla u| |u| \xi(x) |\nabla \xi| \, dx \, dt
\]

and, after a simple calculation,

\[
\int_0^T \int_{\mathbb{R}^m} \nu \psi |\nabla u|^2 \xi^2(x) \, dx \, dt + \int_{\mathbb{R}^m} |u(x, T)|^2 \xi^2(x) \, dx
\]

\[
\leq \beta_3 \left( \int_0^T \int_{\mathbb{R}^m} |u|^\gamma \nu \psi |\nabla u|^2 \xi^2(x) \, dx \, dt \right) + \beta_4 \int_0^T \int_{\mathbb{R}^m} \nu \psi |u|^2 |\nabla \xi|^2 \, dx \, dt.
\]

The above inequality and (3.6) give

\[
\int_0^T \int_{\mathbb{R}^m} \nu \psi |\nabla u|^2 \xi^2(x) \, dx \, dt + \int_{\mathbb{R}^m} |u(x, T)|^2 \xi^2(x) \, dx
\]

\[
\leq \beta_5 \int_0^T \int_{\mathbb{R}^m} \nu \psi |u|^2 |\nabla \xi|^2 \, dx \, dt;
\]

in this way, by the definition of \( \xi(x) \), we get

\[
(3.7) \quad H_{R+\varepsilon}(T) \leq \frac{\beta_5}{\varepsilon^2} \varpi(R, R + \varepsilon) \int_0^T \psi(\tau) H_R(\tau) \, d\tau,
\]

where if \( R_1 < R_2 \), \( \varpi(R_1, R_2) = \sup_{R_1 < |x| < R_2} \nu(x) \).
Let us prove, by induction, the following inequality

\[(3.8) \quad H_{R_0+k\varrho}(T) \leq \beta^k \{ \| f_0 \|_{L^1(Q_T)} + \| f^* \|_{L^1(Q_T)} \} \left( \frac{(T\psi(T))^k}{\varrho^{2k}k!} \right) \nu \left[ (R + \varrho, R_0 + k\varrho) \right] \, . \]

Our next claim is that to prove (3.8) where \( k = 0 \).

Choosing \( \nu^*(x, t) \) as test function in (2.5) and proceeding analogously to the proof of (3.4), we get

\[(3.9) \quad \int_0^T \int_{\mathbb{R}^m} (\gamma + 1) \sum_{i=1}^m a_i(x, t, u, \nabla u) \frac{\partial u}{\partial x_i} |u|^\gamma \, dx \, dt + c_0 \int_0^T \int_{\mathbb{R}^m} |u|^\gamma + 2 \, dx \, dt
\]

\[+ \int_0^T \int_{\mathbb{R}^m} f(x, t, u, \nabla u) u |u|^\gamma \, dx \, dt + \frac{1}{e(\gamma + 2)} \int_{\mathbb{R}^m} |u(x, T)|^\gamma + 2 \, dx \leq 0 \]

and from this for \( \gamma > \lambda^2(L) + \lambda(L) - 1 \)

\[(3.10) \quad \int_0^T \int_{\mathbb{R}^m} \nu |\nabla u|^2 \, dx \, dt + \frac{1}{e(\gamma + 2)} \int_{\mathbb{R}^m} |u(x, T)|^\gamma + 2 \, dx
\]

\[\leq \left( \text{ess sup}_{\mathbb{R}^m \times [0, +\infty[} |u| \right) \int_0^T \int_{\mathbb{R}^m} |f_0(x, t)| \, dx \, dt . \]

On the other hand if we write (3.9) for \( \gamma = 0 \) we obtain

\[\frac{1}{\lambda(L)} \int_0^T \int_{\mathbb{R}^m} \nu |\nabla u|^2 \, dx \, dt + \frac{1}{2e} \int_{\mathbb{R}^m} |u(x, T)|^2 \, dx
\]

\[\leq \lambda(L) \int_0^T \int_{\mathbb{R}^m} |f^*(x, t)| + \nu |\nabla u|^2 |u| \, dx \, dt \leq L\lambda(L) \int_0^T \int_{\mathbb{R}^m} |f^*(x, t)| \, dx \, dt
\]

\[+ \lambda(L) \left( \int_0^T \int_{\mathbb{R}^m} |u|^\gamma \nu \psi |\nabla u|^2 \, dx \, dt \right)^{\frac{1}{2}} \left( \int_0^T \int_{\mathbb{R}^m} \nu \psi |\nabla u|^2 \, dx \, dt \right)^{\frac{\gamma - 1}{2}} . \]

Hence using the Young inequality we conclude that

\[\int_0^T \int_{\mathbb{R}^m} \nu |\nabla u|^2 \, dx \, dt + \frac{1}{2e} \int_{\mathbb{R}^m} |u(x, T)|^2 \, dx
\]

\[\leq \beta_6 \int_0^T \int_{\mathbb{R}^m} |f^*(x, t)| \, dx \, dt + \beta_7 \left( \int_0^T \int_{\mathbb{R}^m} |u|^\gamma \nu \psi |\nabla u|^2 \, dx \, dt \right)
\]

and finally, according to (3.10), that

\[(3.11) \quad H_{R_0}(T) \leq \beta \left\{ \| f_0 \|_{L^1(Q_T)} + \| f^* \|_{L^1(Q_T)} \right\} . \]
Let us assume that the inequality (3.8) holds for some integer \( k > 0 \). Due to (3.7) and (3.8), we obtain

\[
H_{R_0 + (k+1)\varrho}(T) \leq \frac{\beta}{\varrho^2} \mathcal{P}(R_0 + k\varrho, R_0 + (k+1)\varrho) \int_0^T \psi(\tau) H_{R_0 + k\varrho}(\tau)
\]

\[
\leq \frac{\beta}{\varrho^2} \mathcal{P}(R_0 + k\varrho, R_0 + (k+1)\varrho) \int_0^T \psi(\tau) \beta^k \left\{ \| f_0 \|_{L^1(Q_T)} + \| f^* \|_{L^1(Q_T)} \right\}
\]

\[
\times \frac{(\tau \psi(\tau))^k}{\beta^{2k}k!} \left[ \mathcal{P}(R_0 + k\varrho, R_0 + (k+1)\varrho) \right]^k d\tau
\]

\[
= \frac{\beta^{k+1}}{\varrho^{2(k+1)}k!} \left[ \mathcal{P}(R_0 + k\varrho, R_0 + (k+1)\varrho) \right]^{k+1} \left\{ \| f_0 \|_{L^1(Q_T)} + \| f^* \|_{L^1(Q_T)} \right\}
\]

\[
\times \int_0^T \tau^k [\psi(\tau)]^{k+1} d\tau.
\]

According Hypothesis 2.1, taking into account that

\[
\bar{\nu}(R_0 + k\varrho, R_0 + (k+1)\varrho) \leq \sup_{R+\varrho<|x|<R_0+(k+1)\varrho} \nu(x),
\]

the last inequality implies (3.8) for \( k + 1 \). Let \( k \geq 1 \). Choosing in (3.8) \( \varrho = (R - R_0)/k \) we obtain

\[
H_R(T) \leq \beta^k \left\{ \| f_0 \|_{L^1(Q_T)} + \| f^* \|_{L^1(Q_T)} \right\} \frac{(T \psi(T))^k}{(R - R_0)^{2k}k!} \left[ \eta(R) \right]^k.
\]

From this inequality, using also Stirling’s formula, it follows that

(3.12) \[
H_R(T) \leq \left\{ \| f_0 \|_{L^1(Q_T)} + \| f^* \|_{L^1(Q_T)} \right\} e^{-k \log \frac{\beta(R-R_0)^2}{e^2 \eta(R) T \psi(T)}}.
\]

Now, if

\[
\frac{\beta(R-R_0)^2}{e^2 \eta(R) T \psi(T)} \leq e
\]

the estimate (2.6) easily follows from (3.11). Otherwise, we can obtain (2.6) from (3.12) taking as \( k \) the integer part of

\[
\left\lfloor \frac{\beta(R-R_0)^2}{e^2 \eta(R) T \psi(T)} \right\rfloor.
\]
References


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