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ON REFLEXIVITY AND HYPERREFLEXIVITY OF SOME SPACES
OF INTERTWINING OPERATORS

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Abstract. Let $T, T'$ be weak contractions (in the sense of Sz.-Nagy and Foiaş), $m, m'$ the minimal functions of their $C_0$ parts and let $d$ be the greatest common inner divisor of $m, m'$. It is proved that the space $I(T, T')$ of all operators intertwining $T, T'$ is reflexive if and only if the model operator $S(d)$ is reflexive. Here $S(d)$ means the compression of the unilateral shift onto the space $H^2 \ominus dH^2$. In particular, in finite-dimensional spaces the space $I(T, T')$ is reflexive if and only if all roots of the greatest common divisor of minimal polynomials of $T, T'$ are simple. The paper is concluded by an example showing that quasisimilarity does not preserve hyperreflexivity of $I(T, T')$.

Keywords: intertwining operator, reflexivity, $C_0$ contraction, weak contraction, hyperreflexivity

MSC 2000: 47A10, 47A15

1. Introduction

Let $H, H'$ be complex separable Hilbert spaces, let $\mathcal{B}(H, H')$ denote the space of all bounded linear operators $H \to H'$. If $H = H'$ then $\mathcal{B}(H, H) = \mathcal{B}(H)$ is the algebra of all bounded linear operators on $H$. By a subspace we mean a closed linear subspace. For a subset $A \subset H$, we denote by $\bigvee A$ the closed linear span of $A$. A subspace $L \subset H$ is called invariant for $T \in \mathcal{B}(H)$ if $TL \subset L$. As usual, $T|L$ means the restriction of the operator $T$ to $L$. If $A \subset \mathcal{B}(H)$ then $\text{Alg} A$ denotes the smallest weakly closed subalgebra of $\mathcal{B}(H)$ containing $A$ and the identity. $\text{Lat} A$ denotes the set of all subspaces of $H$ that are invariant for each $A \in A$. If $\mathcal{L}$ is a set of subspaces

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of $H$, then $\text{Alg} \mathcal{L} = \{T \in \mathcal{B}(H) : \mathcal{L} \subset \text{Lat} T\}$. A (unital weakly closed) subalgebra $\mathcal{A} \subset \mathcal{B}(H)$ is called reflexive if $\mathcal{A} = \text{Alg Lat} \mathcal{A}$. An operator $T \in \mathcal{B}(H)$ is called reflexive if $\text{Alg}\{T\}$ is reflexive.

H. Bercovici, C. Foiaş and B. Sz.-Nagy [3] studied reflexivity of $C_0$ contractions and their commutants. They showed also that if the commutant of a $C_0$ contraction $T$ is reflexive then $T$ is also reflexive. Generally, the reflexivity of $\{T\}'$ does not imply the reflexivity of the operator $T$ [6].

The reflexivity of subalgebras was studied for the first time in [12]. The notion of reflexivity of algebras of operators was generalized to subspaces of operators by V. S. Shul’man [13]:

**Definition 1.1.** Let $\mathcal{M}$ be a subset of $\mathcal{B}(H, H')$. Then the reflexive closure of $\mathcal{M}$ is

$$\text{ref} \mathcal{M} = \bigcap_{x \in H} \{T \in \mathcal{B}(H, H') : Tx \in \bigvee \{Mx : M \in \mathcal{M}\}\}.$$  

A (closed linear) subspace $\mathcal{M} \subset \mathcal{B}(H, H')$ is called reflexive if $\mathcal{M} = \text{ref} \mathcal{M}$.

Clearly, in Definition 1.1 the Hilbert spaces $H, H'$ can be replaced by arbitrary Banach spaces. A stronger concept of hyperreflexivity was introduced for algebras in [1] and extended to subspaces in [10].

**Definition 1.2.** Let $X, X'$ be complex Banach spaces and let $\mathcal{M}$ be a norm-closed subspace of $\mathcal{B}(X, X')$. $\mathcal{M}$ is called hyperreflexive if there exists $c > 0$ such that for all $T \in \mathcal{B}(X, X')$

$$\text{dist}(T, \mathcal{M}) \leq c\alpha(T, \mathcal{M}), \text{ where } \alpha(T, M) = \sup\{\text{dist}(Tx, Mx) : x \in H, \|x\| = 1\}.$$  

$$\inf\{c > 0 : \text{dist}(T, \mathcal{M}) \leq c\alpha(T, \mathcal{M})\}$$ is called the hyperreflexivity constant of $\mathcal{M}$.

Note that if $\mathcal{M}$ is hyperreflexive then it is reflexive. It is well-known that if both $H$ and $H'$ are finite-dimensional then reflexivity and hyperreflexivity coincide. In [11, Theorem 2.5] V. Müller and M. Ptak have shown that if $X, X'$ are arbitrary Banach spaces and $\mathcal{M}$ is a finite dimensional subspace of $\mathcal{B}(X, X')$ then $\mathcal{M}$ is reflexive if and only if it is hyperreflexive. Clearly, if $\mathcal{M}$ is a subalgebra of $\mathcal{B}(H)$ then $\text{ref} \mathcal{M} = \text{Alg Lat} \mathcal{M}$.

In [13] reflexivity of the space

$$I(T, T') = \{A \in \mathcal{B}(H, H') : AT = T'A\}$$

of operators intertwining $T \in \mathcal{B}(H)$ and $T' \in \mathcal{B}(H')$ was studied and a characterization of reflexive spaces $I(T, T')$ was given in the case of isometries $T, T'$. Moreover, it was stated that if $\text{dim} H < \infty, \text{dim} H' < \infty$ then $I(T, T')$ is reflexive if $T$ or $T'$
is similar to a normal operator. In [5] \( \text{Alg}\{T\}' \) was described if \( \dim H < \infty \) and this showed that \( \{T\}' \) is reflexive if and only if \( T \) is similar to a normal operator or equivalently, if all roots of the minimal polynomial of \( T \) are simple.

In [20] we described (using the Jordan forms of \( T \in \mathcal{B}(H) \), \( T' \in \mathcal{B}(H') \)) \( I(T,T') \) and \( \text{ref } I(T,T') \) in finite-dimensional spaces and we showed that \( I(T,T') \) is reflexive if all roots of the greatest common divisor of the minimal polynomials of \( T \) and \( T' \) are simple. The purpose of this paper is to extend this result to pairs of weak contractions. To prove our results we use the fact that quasi-similarity preserves reflexivity of \( I(T,T') \). We give an example showing that quasi-similarity does not preserve hyperreflexivity of \( I(T,T') \).

2. Compressions of the unilateral shift

We will use the terminology and results of Sz.-Nagy-Foiaş dilation theory [14]. In particular, \( H^2, H^\infty \) mean the Hardy spaces of analytic functions in the unit disc, \( S(\Theta) \) means the compression of the unilateral shift \( S \) onto the space \( H(\Theta) = H^2 \ominus \Theta H^2 \). For \( f, g \in H^\infty \) we write \( f \mid g \) if there exists \( \varphi \in H^\infty \) such that \( g = \varphi f \). The orthogonal projection onto a subspace \( K \) of a Hilbert space \( H \) is denoted by \( P_K \). For \( f_1, f_2 \in H^\infty \) we denote by \( f_1 \wedge f_2 \) the greatest common inner divisor of \( f_1 \) and \( f_2 \).

The following result is an easy consequence of [2, Theorem III.1.16].

**Theorem 2.1.** Let \( v_1, v_2, d \) be inner functions, \( v_1 \wedge v_2 = 1 \). Put \( \Theta_1 = v_1d, \Theta_2 = v_2d \). Then

(i) \( X \in I(S(\Theta_1), S(\Theta_2)) \) if and only if there exists a function \( \varphi \in H^\infty \) such that

\[
X = P_{H(\Theta_2)}u(S)|H(\Theta_1), \quad \text{where} \quad u = v_2\varphi.
\]

Moreover, \( X = 0 \) if and only if \( d \mid \varphi \).

(ii) An operator \( A \in \text{ref } I(S(\Theta_1), S(\Theta_2)) \) if and only if

\[
A|H^2 \ominus dH^2 \in \text{ref } I(S(d), S(\Theta_2)|v_2(H^2 \ominus dH^2)),
\]

and

\[
A|d(H^2 \ominus v_1H^2) = 0.
\]

(iii) \( I(S(\Theta_1), S(\Theta_2)) \) is reflexive if and only if \( S(d) \) is reflexive.

**Proof.** (i) According to [2, Theorem III.1.16], \( X \in I(S(\Theta_1), S(\Theta_2)) \) if and only if there exists an inner function \( u \) such that \( X = P_{H(\Theta_2)}u(S)|H(\Theta_1) \) and \( \Theta_2 \mid u\Theta_1 \). Since \( v_1 \wedge v_2 = 1 \), we have \( v_2d \mid uv_1d \iff v_2 \mid u \) and consequently there exists
ϕ ∈ \(H^\infty\) such that \(u = \varphi v_2\). Moreover, \(X = 0\) if and only if \(\Theta_2 | u\), i.e. if and only if \(d | \varphi\).

(ii) \(H(\Theta_1)\) and \(H(\Theta_2)\) can be written as orthogonal sums

\[H(\Theta_1) = (H^2 \ominus dH^2) \oplus d(H^2 \ominus v_1H^2), \quad H(\Theta_2) = (H_2 \ominus v_2H^2) \oplus v_2(H^2 \ominus dH^2).\]

It is well-known that \(v_2(H^2 \ominus dH^2) \in \text{Lat } S(\Theta_2)\). Using (i) we obtain for any \(f \in H(\Theta_1)\)

\[\bigvee_{X \in I(S(\Theta_1), S(\Theta_2))} Xf = \bigvee_{\varphi \in H^\infty} P_{H(\Theta_2)} v_2 \varphi f \subset v_2(H^2 \ominus dH^2).\]

If \(f \in d(H^2 \ominus v_1H^2)\) then \(v_2 \varphi f \in dv_2H^2 \perp H(\Theta_2)\), consequently

\[\bigvee_{X \in I(S(\Theta_1), S(\Theta_2))} Xf = 0.\]

Herefrom (ii) follows easily.

(iii) \(S(\Theta_2)|v_2H(d)\) is unitarily equivalent to \(S(d)\). So the reflexivity of \(I(S(\Theta_1), S(\Theta_2))\) implies that the commutant of \(S(d)\) is reflexive. Since \(\{S(d)\}' = \text{Alg } S(d)\), this proves (iii).

\[\square\]

3. General \(C_0\) contractions

To prove a characterization of pairs \(T, T'\) of \(C_0\) contractions having reflexive \(I(T, T')\) we need two simple lemmas.

**Lemma 3.1.** Let \(T, X \in \mathcal{B}(H)\), \(T', Y \in \mathcal{B}(H')\) and \(TX = XT, T'Y = YT'\). Put \(T_X = T|(XH)^-\), \(T'_Y = T'|(YH')^-\).

If \(I(T, T')\) is reflexive then \(I(T_X, T'_Y)\) is reflexive as well.

**Proof.** Suppose that \(A \in \text{ref } I(T_X, T'_Y)\). If \(B \in I(T_X, T'_Y)\) then \(BX \in I(T, T')\).

Therefore for all \(h \in H\) we have

\[AXh \in \bigvee_{B \in I(T_X, T'_Y)} BXh \subset \bigvee_{C \in I(T, T')} Ch, \quad \text{i.e.} \quad AX \in \text{ref } I(T, T')\]

and so \(ATX = AXT = T'AX = T'_Y AX\), i.e. \(A \in I(T_X, T'_Y)\). \(\square\)
Lemma 3.2. Let $\vartheta_1, \Theta_1, \vartheta_2, \Theta_2$ be inner functions such that $\vartheta_1 \mid \Theta_1$ and $\vartheta_2 \mid \Theta_2$. If $I(\Theta_1, \Theta_2)$ is reflexive then $I(\vartheta_1, \vartheta_2)$ is reflexive as well.

Proof. Put $\varphi_k = \Theta_k / \vartheta_k$, $k = 1, 2$. Since $S(\vartheta_k)$ is unitarily equivalent to $S(\Theta_k) | (\varphi_k (S(\Theta_k)) H(\Theta_k))^{-}$, Lemma 3.2 is a consequence of Lemma 3.1. □

Now we are ready to state one of our main results.

Theorem 3.3. Let $T \in B(H)$, $T' \in B(H')$ be $C_0$ contractions having minimal functions $m, m'$, respectively. Let $d = m \wedge m'$. Then $I(T, T')$ is reflexive if and only if the operator $S(d)$ is reflexive.

Proof. If $T_1 \in B(H_1)$ and $T'_1 \in B(H'_1)$ are quasisimilar to $T_2 \in B(H_2)$ and $T'_2 \in B(H'_2)$, respectively, then $I(T_1, T'_1)$ is reflexive if and only if $I(T_2, T'_2)$ is reflexive. This was first stated (without proof which is easy) in [13, Proposition 1]. Since any $C_0$ contraction is quasisimilar to its Jordan model it is enough to prove the theorem for Jordan models

$$T = \bigoplus_{\alpha} S(m_\alpha), \quad T' = \bigoplus_{\beta} S(m'_\beta),$$

where $\oplus$ means the direct orthogonal sum. According to [13, Proposition 2], $I(T, T')$ is reflexive if and only if each of the spaces $I(S(m_\alpha), S(m'_\beta))$ is reflexive. For all indices $\alpha, \beta$, we have $m_\alpha \mid m$, $m'_\beta \mid m'$. Therefore, by Lemma 3.2, $I(T, T')$ is reflexive if and only if $I(S(m), S(m'))$ is reflexive. According to assertion (iii) of Theorem 2.1 this completes the proof. □

Theorem 3.3 generalizes [3, Theorem B]. In finite-dimensional spaces we obtain the following corollary (a generalization of [5, Theorem 3]).

Corollary 3.4. Let $H$, $H'$ be finite-dimensional. Then $I(T, T')$ is reflexive if and only if all roots of the greatest common divisor of the minimal polynomials $m_T$ and $m_{T'}$ of $T$ and $T'$, respectively, are simple.

Proof. Replacing $T$ and $T'$ by $\|T\|^{-1}T$ and $\|T'\|^{-1}T'$ we obtain a pair of contractions the minimal functions $m, m'$ of which are finite Blaschke products whose numerators are $m_T$ and $m_{T'}$, respectively. Then $d$ is also a finite Blaschke product and its numerator is the greatest common inner divisor of the minimal polynomials $m_T$ and $m_{T'}$. It is well-known (see e.g. [7]) that then $S(d)$ is reflexive if and only if all zeroes of $d$ are simple. □

Note that in [20] Corollary 3.4 was proved more directly by describing $I(T, T')$ and $\text{ref} I(T, T')$ for nilpotent $T$ and $T'$. In the case $T = T'$ this was done in [5].
4. Weak contractions

Now, let \( T \in \mathcal{B}(H) \), \( T' \in \mathcal{B}(H') \) be weak contractions. (For the definition of weak contractions and basic results we refer to [14, Chapter VIII]). It is well-known (see, e.g., [18]) that \( T \) and \( T' \) can be splitted into orthogonal sums \( T = T_{ac} \oplus T_{su} \), \( T' = T'_{ac} \oplus T'_{su} \) of their absolutely continuous and singular unitary parts and that

\[
I(T, T') = I(T_{ac}, T'_{ac}) \oplus I(T_{su}, T'_{su}).
\]

It follows that

\[
\text{ref } I(T, T') = \text{ref } I(T_{ac}, T'_{ac}) \oplus \text{ref } I(T_{su}, T'_{su}).
\]

Since for normal operators \( A, B \) the space \( I(A, B) \) is reflexive [13], \( I(T, T') \) is reflexive if and only if so is \( I(T_{ac}, T'_{ac}) \). According to [17, Lemma 3] any absolutely continuous weak contraction \( S \) is similar to a completely non-unitary (c.n.u.) weak contraction \( S' \) and, moreover, the \( C_0 \) parts of \( S \) and \( S' \) coincide. Since similarity (even quasi-similarity [13, Proposition 1]) preserves reflexivity of \( I(T, T') \), it does not restrict generality if we suppose that \( T, T' \) are c.n.u.

**Theorem 4.1.** Let \( T \in \mathcal{B}(H) \), \( T' \in \mathcal{B}(H') \) be c.n.u. weak contractions and let \( T_0 \in \mathcal{B}(H_0) \), \( T'_0 \in \mathcal{B}(H'_0) \) be their \( C_0 \) parts and \( T_1 \in \mathcal{B}(H_1) \), \( T'_1 \in \mathcal{B}(H'_1) \) their \( C_{11} \) parts. Then

(i) if \( X \in I(T, T') \) then \( XH_0 \subset H'_0 \) and \( XH_1 \subset H'_1 \);

(ii) if \( A \in \text{ref } I(T, T') \) then its restrictions to subspaces \( H_0, H_1 \) satisfy \( A_0 = A|H_0 \in \text{ref } I(T_0, T'_0) \), \( A_1 = A|H_1 \in \text{ref } I(T_1, T'_1) \);

(iii) \( I(T, T') \) is reflexive if and only if \( I(T_0, T'_0) \) is reflexive.

**Proof.** (i) According to [14, Chapters II.4 and VIII.2]

\[
H_0 = \{ h \in H : T^n h \to 0 \}, \quad H'_0 = \{ h' \in H' : T'^n h' \to 0 \}
\]

and

\[
H'_1 = \{ h \in H : T'^* n h = 0 \}, \quad H''_1 = \{ h' \in H' : T'^* n h' = 0 \}.
\]

\( XT = T'X \) implies \( XT^n = T'^* n X \) for all positive integers \( n \). Therefore \( h_0 \in H_0 \to \lim T^n X h_0 = \lim X T^n h_0 = 0 \), i.e. \( X h_0 \in H'_0 \). By taking adjoints we obtain \( X T = T'X \implies T^*X^* = X^*T'^* \) and so \( X^* H_{11} \subset H_{11} \), which is equivalent to \( XH_1 \subset H'_1 \).

(ii) This is an obvious consequence of (i).

(iii) There are operators \( R, S \in \{ T \}'', R', S' \in \{ T' \}'' \) such that

\[
H_0 = \ker R = (SH)^-, \quad H_1 = (RH)^- = \ker S,
\]

\[
H'_0 = \ker R' = (S'H)^-, \quad H'_1 = (R'H)^- = \ker S'.
\]
(14), (15), (16, Theorem 1). Suppose that \( I(T, T') \) is reflexive. Then, by Lemma 3.1, \( I(T_0, T'_0) \) is reflexive. Conversely, if \( I(T_0, T'_0) \) is reflexive and \( A \in \text{ref} I(T, T') \) then by (ii) \( A|H_0 \in \text{ref} I(T_0, T'_0) \) and \( A|H_1 \in \text{ref} I(T_1, T'_1) \). The operators \( T_1, T'_1 \) are quasi-similar to unitary operators and so \( I(T_1, T'_1) \) is reflexive. Therefore \( A|H_0 \in I(T_0, T'_0) \) and \( A|H_1 \in I(T_1, T'_1) \). Since \( H_0 \vee H_1 = H \), this shows that \( I(T, T') \) is reflexive. □

**Theorem 4.2.** Let \( T, T' \) be weak contractions and let their \( C_0 \) parts \( T_0, T'_0 \) have minimal functions \( m, m' \), respectively. Let \( d = m \wedge m' \) be the greatest common inner divisor of \( m, m' \). Then the space \( I(T, T') \) is reflexive if and only if the operator \( S(d) \) is reflexive.

**Proof.** This is an obvious consequence of Theorems 3.3 and 4.1. □

**Remarks.**
1. Theorems 4.1 and 4.2 are generalizations of [19, Theorem 5.1].
2. Inner functions \( m \) for which \( S(m) \) is a reflexive operator were characterized in [7, Theorem 3.1].

5. **Quasisimilarity does not preserve hyperreflexivity**

First, let us recall the definition of quasisimilarity:

**Definition 5.1.** \( T \in B(H), S \in B(K) \) are quasi-similar (we write \( T^{\text{q.s.}} S \)) if there are quasi-affinities (injective operators with dense range) \( X \in I(T, S), Y \in I(S, T) \).

**Example 5.2.** Put \( H_n = H'_n = C^2, H = H' = \bigoplus_{n=1}^{\infty} H_n, \)
\[
T_n = \frac{1}{n} \begin{pmatrix} 2n & n \\ 0 & 2n + 1 \end{pmatrix}, \quad T'_n = \frac{1}{n} \begin{pmatrix} 2n & 0 \\ -n & 2n + 1 \end{pmatrix},
\]
\[
S_n = S'_n = \frac{1}{n} \begin{pmatrix} 2n + 1 & 0 \\ 0 & 2n \end{pmatrix},
\]
\[
T = \bigoplus_{n=1}^{\infty} T_n, \quad T' = \bigoplus_{n=1}^{\infty} T'_n, \quad S = S' = \bigoplus_{n=1}^{\infty} S_n.
\]

Then, obviously, \( T \in B(H), T' \in B(H'), S = S' \in B(H) \).

The following assertions hold.
(a) \( T^{\text{q.s.}} S = S^{\text{q.s.}} T' \),
(b) all \( I(T_m, T'_n) \) are hyperreflexive,
(c) \( I(T, T') \) is not hyperreflexive,
(d) \( I(S, S') \) is hyperreflexive.
Proof. The common minimal polynomial \((\lambda - 2n)(\lambda - 2n - 1)\) of \(T_n, T'_n, S_n\) has simple roots, which implies that all \(I(T_n, T'_n)\) are reflexive. In finite dimension this implies that they are also hyperreflexive and this proves (b).

Putting \(A_n = \begin{pmatrix} 0 & n \\ 0 & 1 \end{pmatrix}, B_n = \begin{pmatrix} 0 & 0 \\ n & 1 \end{pmatrix}, C_n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\) we obtain

\[
T_n = \frac{1}{n}(2nI + A_n), \quad T'_n = \frac{1}{n}(2nI + B_n), \quad S_n = S'_n = \frac{1}{n}(2nI + C_n)
\]

and if \(P_n = \begin{pmatrix} n & 1 \\ 1 & 0 \end{pmatrix}\) then \(P_n^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -n \end{pmatrix}\) and \(A_n = P_nC_nP_n^{-1}\).

Hence \(A_nP_n = P_nC_n, P_n^{-1}A_n = C_nP_n^{-1}\) and after perturbation by \(2nI, T_nP_n = P_nS_n, P_n^{-1}T_n = S_nP_n^{-1}\).

Now, it is easy to compute \(\|P_n\|\) and \(\|P_n^{-1}\|\):

\[
P_n^\top = P_n \implies \|P_n\| = \varrho(P_n) = \frac{n + \sqrt{n^2 + 4}}{2} = \varrho(P_n^{-1}) = \|P_n^{-1}\|.
\]

Putting \(Y = \bigoplus_{n=1}^\infty n^{-1}P_n, X = \bigoplus_{n=1}^\infty n^{-1}P_n^{-1}\) we obtain quasiaffinities \(X \in I(T, S), Y \in I(S, T)\), i.e., \(T^n \approx S\). Similarly, it can be proved that \(T^n \approx S\). This completes the proof of (a).

(c): \(m \neq n \implies I(T_n, T'_n) = \{0\}\) because their minimal polynomials are relatively prime. Therefore \(I(T, T') = \bigoplus_{n=1}^\infty I(T_n, T'_n)\) and similarly \(I(S, S') = \bigoplus_{n=1}^\infty I(S_n, S'_n)\). By a simple direct computation we obtain \(X_n \in I(T_n, T'_n) = I(A_n, B_n)\) if and only if \(X_n = \begin{pmatrix} 0 & \alpha \\ \beta & -n(\alpha + \beta) \end{pmatrix}\) for some \(\alpha, \beta \in C\). So \(I(T_n, T'_n) = S_n\) from an example due to Kraus and Larson [9] (see also [4, Example 58.9]) who proved that \(S_n\) is hyperreflexive with \(\kappa S_n \geq \frac{1}{\sqrt{3}}\). So \(I(T, T') = \bigoplus_{n=1}^\infty S_n\) is not hyperreflexive.

(d): Observe that \(I(S_n, S'_n) = I \left( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right)\) for all \(n\), i.e. its hyperreflexivity constant does not depend on \(n\). Using a recent result of K. Kliš and M. Ptak [8, Theorem 5.1] we obtain that \(I(S, S')\) is hyperreflexive.

It easy to show that if \(T = \bigoplus_{n=1}^\infty T_n, \ T' = \bigoplus_{n=1}^\infty T'_n\) and \(I(T, T')\) is hyperreflexive, then all \(I(T_n, T'_n)\) are hyperreflexive. From Example 5.2 it follows that the converse implication does not hold.
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