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A MULTIDIMENSIONAL INTEGRATION BY PARTS FORMULA FOR THE HENSTOCK-KURZWEIL INTEGRAL

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Abstract. It is shown that if $g$ is of bounded variation in the sense of Hardy-Krause on $\prod_{i=1}^{m} [a_i, b_i]$, then $g\chi_{\prod_{i=1}^{m} (a_i, b_i)}$ is of bounded variation there. As a result, we obtain a simple proof of Kurzweil’s multidimensional integration by parts formula.

Keywords: Henstock-Kurzweil integral, bounded variation in the sense of Hardy-Krause, integration by parts

MSC 2000: 26A39

1. Introduction

It is well known that if $f$ is Henstock-Kurzweil integrable on a compact interval $[a, b] \subset \mathbb{R}$ and $g$ is of bounded variation there, then $fg$ is Henstock-Kurzweil integrable there and the integration by parts formula holds; see, for example, [12] and references therein. Although higher-dimensional analogues of the above-mentioned result have been studied by various authors ([1], [2], [3], [6], [7], [10], [14], [17], [18]), a simpler proof of Kurzweil’s multidimensional integration by parts formula for the Henstock-Kurzweil integral [1, Theorem 2.10] remained elusive. The purpose of this paper is to give a simpler proof of this result.

2. Functions of bounded variation

Let $m \geq 1$ be an integer and let $\mathbb{R}^m$ be the $m$-dimensional Euclidean space equipped with the maximum norm. An interval in $\mathbb{R}^m$ is a set of the form $\prod_{i=1}^{m} [u_i, v_i]$,
where \( u_i, v_i \in \mathbb{R} \) and \( u_i \leq v_i \) for \( i = 1, \ldots, m \). Let \([a, b] := \prod_{i=1}^{m} [a_i, b_i] \) be a fixed non-degenerate compact interval in \( \mathbb{R}^m \), where \( a = (a_1, \ldots, a_m) \) and \( b = (b_1, \ldots, b_m) \), and let \( \mathcal{I}_m([a, b]) \) denote the family of all non-degenerate subintervals of \([a, b] \). For each \( \prod_{i=1}^{m} [u_i, v_i] \in \mathcal{I}_m([a, b]) \), we set \( [u, v] := \prod_{i=1}^{m} [u_i, v_i] \) and \([u, v) := \prod_{i=1}^{m} (u_i, v_i) \), where \( u = (u_1, \ldots, u_m) \) and \( v = (v_1, \ldots, v_m) \).

A division of \([a, b] \) is a finite collection \( \{I_1, \ldots, I_p\} \) of non-overlapping intervals such that \( \bigcup_{i=1}^{p} I_i = [a, b] \). For any given real-valued function \( g \) defined on \([a, b] \), the total variation of \( g \) over \([a, b] \) is defined by

\[
\text{Var}(g, [a, b]) := \sup \left\{ \sum_{[u, v] \in P} |\Delta_g([u, v])| : P \text{ is a division of } [a, b] \right\},
\]

where

\[
\Delta_g([u, v]) := \sum_{t \in [u, v), \forall i \in \{1, \ldots, m\}} g(t) \prod_{i=1}^{m} \text{sgn}(t_i - \frac{u_i + v_i}{2})
\]

for each \([u, v] \in \mathcal{I}_m([a, b]) \).

**Definition 2.1.** A function \( g : [a, b] \to \mathbb{R} \) is said to be of bounded variation (in the sense of Vitali) on \([a, b] \) if \( \text{Var}(g, [a, b]) \) is finite.

The space of functions of bounded variation (in the sense of Vitali) on \([a, b] \) is denoted by \( \text{BV}[a, b] \). Set

\[
\text{BV}_0[a, b] := \{ g \in \text{BV}[a, b] : g(x) = 0 \text{ whenever } x \in [a, b] \setminus \{a, b\} \},
\]

where \((a, b) := \prod_{i=1}^{m} (a_i, b_i) \). The next theorem is an \( m \)-dimensional analogue of [16, Theorem 1].

**Theorem 2.2.** Let \( g : [a, b] \to \mathbb{R} \). Then \( g \in \text{BV}_0[a, b] \) if and only if there exists a sequence \( \{\varphi_n\}_{n=1}^{\infty} \) in \( L^1[a, b] \) such that \( \sup_{n \in \mathbb{N}} \|\varphi_n\|_{L^1[a, b]} \) is finite and

\[
\lim_{n \to \infty} \int_{[a, b]} \varphi_n(t) \, dt = g(x) \text{ for each } x \in [a, b].
\]

The following result of Young [20] is also useful.
Theorem 2.3. Let $x \in [a, b]$ and let $\{x_n\}_{n=1}^\infty$ be a sequence in $[a, b]$ such that $\text{sgn}(x_{k,i} - x_k) = \text{sgn}(x_{k,j} - x_k)$ for all $i, j \in \mathbb{N}$ and $k \in \{1, \ldots, m\}$. If $g \in BV_0[a, b]$ and $\lim_{n \to \infty} x_n = x$, then the limit $\lim_{n \to \infty} g(x_n)$ exists. In particular, $g$ is continuous everywhere on $[a, b]$ except for a countable number of hyperplanes parallel to the coordinate axes.

New proofs of Theorems 2.2 and 2.3 are given in [13].

3. The $m$-dimensional Riemann-Stieltjes integral

The purpose of this section is to recall some useful facts concerning the $m$-dimensional Riemann-Stieltjes integral. In particular, we obtain a useful result (Theorem 3.4) which plays an important role in the proof of Theorem 4.10.

Definition 3.1. Let $F$ and $H$ be two real-valued functions defined on $[a, b]$. $F$ is said to be Riemann-Stieltjes integrable with respect to $H$ on $[a, b]$ if there exists $A \in \mathbb{R}$ with the following property: for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left| \sum_{i=1}^p F(x_i) \Delta_H(I_i) - A \right| < \varepsilon$$

for each division $\{I_1, \ldots, I_p\}$ of $[a, b]$ such that $x_i \in I_i$ and the diameter of $I_i$ is less than $\delta$ for $i = 1, \ldots, p$. In this case, the value of $A$ is uniquely determined and we write $A$ as $\int_{[a,b]} F(x) \text{d}H(x)$.

It is well known that if $F \in C[a, b]$ and $H \in BV[a, b]$, then the Riemann-Stieltjes integral $\int_{[a,b]} F(x) \text{d}H(x)$ exists; in particular, we have the following result.

Theorem 3.2. If $F \in C[a, b]$, $h \in L^1[a, b]$ and $H(x) = \int_{[a,x]} h(t) \text{d}t$ for each $x \in [a, b]$, then the Riemann-Stieltjes integral $\int_{[a,b]} F(x) \text{d}H(x)$ exists, $Fh \in L^1[a, b]$ and

$$\int_{[a,b]} F(x) \text{d}H(x) = \int_{[a,b]} F(x)h(x) \text{d}x.$$

The following convergence theorem is also well known.

Theorem 3.3. Let $F \in C[a, b]$ and suppose that the following assertions hold:

(i) $\{g_n\}_{n=1}^\infty \subset BV[a, b]$ so that $\sup_{n \in \mathbb{N}} \text{Var}(g_n, [a, b])$ is finite.

(ii) $g_n \to g$ pointwise on $[a, b]$. 

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Then the Riemann-Stieltjes integral \( \int_{[a, b]} F(x) \, dg(x) \) exists. Moreover, the limit
\[
\lim_{n \to \infty} \int_{[a, b]} F(x) \, dg_n(x) = \int_{[a, b]} F(x) \, dg(x).
\]

Using Theorems 2.2, 3.2 and 3.3, we obtain the following result.

**Theorem 3.4.** Let \( F \in C[a, b] \) and let \( g \in BV[a, b] \). If \( g(x) = 0 \) for all \( x \in [a, b] \setminus (a, b) \) and there exists \( k \in \{1, \ldots, m\} \) such that \( F \) is independent of \( x_k \), then
\[
\int_{[a, b]} F(x) \, dg(x) = 0.
\]

**Proof.** We may assume that \( k = 1 \) and \( m \geq 2 \). According to Theorem 2.2, there exists a sequence \( \{\varphi_n\}_{n=1}^\infty \) in \( L^1[a, b] \) such that

(1) \[
\sup_{n \in \mathbb{N}} \|\varphi_n\|_{L^1[a, b]} < \infty
\]

and

(2) \[
\lim_{n \to \infty} \int_{[a, b]} \varphi_n(t) \, dt = g(x) \quad \text{for each } x \in [a, b].
\]

As a consequence of (1), (2), Theorems 3.3 and 3.2, we conclude that

(3) \[
\int_{[a, b]} F(x) \, dg(x) = \lim_{n \to \infty} \int_{[a, b]} F(x) \varphi_n(x) \, dx.
\]

Moreover, it follows from Fubini’s theorem and our assumptions that

(4) \[
\int_{[a, b]} F(x) \varphi_n(x) \, dx = \int_{\prod_{i=2}^m [a_i, b_i]} F(x) \left\{ \int_{[a_1, b_1]} \varphi_n(x) \, dx_1 \right\} d(x_2, \ldots, x_m)
\]

for \( n = 1, 2, \ldots \). In view of (3) and (4), it suffices to prove that

(5) \[
\lim_{n \to \infty} \int_{\prod_{i=2}^m [a_i, b_i]} F(x) \left\{ \int_{[a_1, b_1]} \varphi_n(x) \, dx_1 \right\} d(x_2, \ldots, x_m) = 0.
\]

From (1), we get

(6) \[
\sup_{n \in \mathbb{N}} \int_{\prod_{i=2}^m [a_i, b_i]} \varphi_n(x) \, dx_1 \left| d(x_2, \ldots, x_m) \right| \leq \sup_{n \in \mathbb{N}} \int_{[a, b]} |\varphi_n(x)| \, dx < \infty.
\]
For each \((b_1, x_2, \ldots, x_m) \in [a, b]\), Fubini’s theorem, (2) and our choice of \(g\) yield

\[
(7) \quad \lim_{n \to \infty} \int \prod_{i=2}^{m} [a_i, x_i] \left\{ \int_{[a_1, b_1]} \varphi_n(t) \, dt \right\} \, dt(t_2, \ldots, t_m) = g(b_1, x_2, \ldots, x_m) = 0.
\]

Using an \((m-1)\)-dimensional analogue of Theorem 3.2, (6), (7) and an \((m-1)\)-dimensional analogue of Theorem 3.3, we get (5). The proof is complete.

4. A NEW PROOF OF KURZWEIL’S MULTIDIMENSIONAL INTEGRATION BY PARTS FORMULA

The aim of this section is to give a new proof of the multidimensional integration by parts formula for the Henstock-Kurzweil integral; see Theorem 4.10 for details. Unlike the original proof of [1, Theorem 2.10], our method of proof depends on our simple Theorems 4.8 and 4.5. For the definition, properties and recent results concerning the Henstock-Kurzweil integral, consult for instance [4], [5], [6], [7], [8], [9].

Set \(\Phi_{[a,b],k}(X_k) := \prod_{i=1}^{m} W_i\) where \(W_k = X_k\) and \([a_i, b_i]\) for all \(i \in \{1, \ldots, m\} \setminus \{k\}\).

**Definition 4.1.** A function \(g: [a, b] \to \mathbb{R}\) is said to be of bounded variation (in the sense of Hardy-Krause) on \([a, b]\) if \(g \in BV[a, b]\) and, for each non-empty set \(\Gamma \subset \{1, \ldots, m\}\),

\[
g|_{\bigcap_{k=1}^{m} \Phi_{[a,b],k}\{a_k\}} \in BV\left(\prod_{k=1, k \notin \Gamma}^{m} [a_k, b_k]\right).
\]

The class of functions of bounded variation (in the sense of Hardy-Krause) on \([a, b]\) will be denoted by \(BV_{HK}[a, b]\). As an immediate consequence of Definition 4.1, we have

**Theorem 4.2.** \(BV_0[a, b] \subset BV_{HK}[a, b]\).

Let \(\chi_Y\) denote the characteristic function of a set \(Y\). In order to prove a crucial result for \(BV_{HK}\) functions (cf. Theorem 4.5), we need the following lemmas.

**Lemma 4.3.** Let \(g \in BV_{HK}[a, b]\). If \(T \subset \{1, \ldots, m\}\) is non-empty and \(c_k \in \{a_k, b_k\}\) for all \(k \in \{1, \ldots, m\} \setminus T\), then

\[
g|_{\bigcap_{k=1, k \notin T}^{m} \Phi_{[a,b],k}\{c_k\}} \in BV\left(\prod_{k=1, k \in T}^{m} [a_k, b_k]\right).
\]
Proof. This is an immediate consequence of Definition 4.1. Let
\[ \mathcal{P}_m := \left\{ \prod_{k=1}^{m} Y_k : Y_k \in \{ \{a_k\}, \{b_k\}, [a_k, b_k] \} \right\} \]
and for \( \prod_{k=1}^{m} Y_k \in \mathcal{P}_m \), let
\[ \Gamma\left( \prod_{k=1}^{m} Y_k \right) = \left\{ i \in \{1, \ldots, m\} : Y_i = [a_i, b_i] \right\}. \]

Lemma 4.4. If \( g \in \text{BV}_{\text{HK}}[a, b] \) and \( Y \in \mathcal{P}_m \), then \( g\chi_Y \in \text{BV}[a, b] \).

Proof. Let \( g \in \text{BV}_{\text{HK}}[a, b] \). If \( Y \in \mathcal{P}_m \) and \( \Gamma(Y) \) is empty, then it is clear that \( g\chi_Y \in \text{BV}[a, b] \). On the other hand, for any \( Y \in \mathcal{P}_m \) satisfying \( \Gamma(Y) \neq \emptyset \), it follows from Lemma 4.3 that \( g\chi_Y \in \text{BV}[a, b] \). \( \square \)

Let \( \mu_0 \) denote the counting measure.

Theorem 4.5. If \( g \in \text{BV}_{\text{HK}}[a, b] \), then \( g\chi_{(a, b)} \in \text{BV}_{0}[a, b] \) and
\[ g\chi_{(a, b)} = \sum_{Y \in \mathcal{P}_m} (-1)^{m-\mu_0(\Gamma(Y))} g\chi_Y. \]

Proof. It is clear that (8) holds for any real-valued function \( g \) defined on \([a, b]\). It remains to prove that \( g\chi_{(a, b)} \in \text{BV}_{0}[a, b] \) whenever \( g \in \text{BV}_{\text{HK}}[a, b] \). But this is an immediate consequence of (8) and Lemma 4.4. The proof is complete. \( \square \)

Our next step is to prove Theorem 4.8, which is a special case of Theorem 4.10. We need the following theorems.

Theorem 4.6. If \( f \in L^1[a, b] \) and \( g \in \text{BV}_{0}[a, b] \), then \( fg \in L^1[a, b] \) and
\[ \int_{[a, b]} f(x)g(x) \, dx = \int_{[a, b]} \left\{ \int_{[x, b]} f(t) \, dt \right\} dg(x). \]

Proof. Let \( \{ \varphi_n \}_{n=1}^{\infty} \) be given as in Theorem 2.2. For each \( n \in \mathbb{N} \) we have, by Fubini’s theorem and Theorem 3.2,
\[ \int_{[a, b]} f(x) \left\{ \int_{[a, b]} \varphi_n(t) \, dt \right\} \, dx = \int_{[a, b]} \left\{ \int_{[t, b]} f(x) \, dx \right\} \varphi_n(t) \, dt \]
\[ = \int_{[a, b]} \left\{ \int_{[t, b]} f(x) \, dx \right\} dg_n(t), \]
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where \( g_n(t) := \int_{[a, t]} \varphi_n(x) \, dx \). Therefore Lebesgue’s dominated convergence theorem and Theorem 3.3 yield the desired result.

Let \(|I| := \mu_m(I)\) \((I \in \mathcal{I}_m([a, b]))\), where \(\mu_m\) denotes the \(m\)-dimensional Lebesgue measure.

**Theorem 4.7.** If \(f \in \text{HK}[a, b]\) and \(g \in \text{BV}_0[a, b]\), then

\[
\left| \sum_{i=1}^{p} \left\{ f(\xi_i)g(\xi_i)|I_i| - \int_{[a, b]} \left( \text{HK} \int_{[x, b]} f(t)\chi_{I_i}(t) \, dt \right) \, dg(x) \right\} \right|
\leq \sum_{i=1}^{p} |f(\xi_i)| \left| g(\xi_i)|I_i| - \int_{I_i} g(t) \, dt \right|
\leq \sum_{i=1}^{p} \left| f(\xi_i) \int_{I_i} g(t) \, dt - \int_{[a, b]} \left( \text{HK} \int_{[x, b]} f(t)\chi_{I_i}(t) \, dt \right) \, dg(x) \right|
\]

for each partial partition \(\{(I_i, \xi_1), \ldots, (I_p, \xi_p)\}\) of \([a, b]\).

**Proof.** By the triangle inequality,

\[
\left| \sum_{i=1}^{p} \left\{ f(\xi_i)g(\xi_i)|I_i| - \int_{[a, b]} \left( \text{HK} \int_{[x, b]} f(t)\chi_{I_i}(t) \, dt \right) \, dg(x) \right\} \right|
\leq \sum_{i=1}^{p} |f(\xi_i)| \left| g(\xi_i)|I_i| - \int_{I_i} g(t) \, dt \right|
\leq \sum_{i=1}^{p} \left| f(\xi_i) \int_{I_i} g(t) \, dt - \int_{[a, b]} \left( \text{HK} \int_{[x, b]} f(t)\chi_{I_i}(t) \, dt \right) \, dg(x) \right|
\]

It is evident that

\[
\sum_{i=1}^{p} |f(\xi_i)| \left| g(\xi_i)|I_i| - \int_{I_i} g(t) \, dt \right| \leq \sum_{i=1}^{p} |f(\xi_i)| \int_{I_i} |g(\xi_i) - g(t)| \, dt
\]

and, by Theorem 4.6,

\[
\int_{I_i} g(t) \, dt = \int_{[a, b]} \left( \int_{[x, b]} \chi_{I_i}(t) \, dt \right) \, dg(x),
\]

so that

\[
\left| \sum_{i=1}^{p} \left\{ f(\xi_i) \int_{I_i} g(t) \, dt - \int_{[a, b]} \left( \text{HK} \int_{[x, b]} f(t)\chi_{I_i}(t) \, dt \right) \, dg(x) \right\} \right|
= \left| \int_{[a, b]} \left( \text{HK} \int_{[x, b]} \sum_{i=1}^{p} \left\{ f(\xi_i)\chi_{I_i}(t) - f(t)\chi_{I_i}(t) \right\} \, dt \right) \, dg(x) \right|
\leq \sup_{x \in [a, b]} \left| \left( \text{HK} \int_{[x, b]} \sum_{i=1}^{p} \left\{ f(\xi_i)\chi_{I_i}(t) - f(t)\chi_{I_i}(t) \right\} \, dt \right) \, (\text{Var}(g, [a, b])) \right|
\]

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Combining the above estimates proves the theorem.

**Theorem 4.8.** If \( f \in HK[a, b] \) and \( g \in BV_0[a, b] \), then \( fg \in HK[a, b] \) and

\[
(HK) \int_{[a,b]} f(x)g(x) \, dx = \int_{[a,b]} \left\{ (HK) \int_{[x,b]} f(t) \, dt \right\} dg(x).
\]

**Proof.** We may assume that \( \text{Var}(g, [a, b]) < 1 \). According to the Saks-Henstock Lemma, given \( \varepsilon > 0 \) there exists a gauge \( \delta_1 \) on \([a, b]\) such that

\[
\sum_{i=1}^{q} \left| f(\xi_i) |J_i| - (HK) \int_{J_i} f(x) \, dx \right| < \frac{\varepsilon}{2^{m+2}}
\]

for each \( \delta_1 \)-fine partial partition \( \{(J_1, \zeta_1), \ldots, (J_q, \zeta_q)\} \) of \([a, b]\). For each \( x \in [a, b] \), it follows from (10) that

\[
\left| \sum_{i=1}^{q} \left\{ f(\xi_i) \mu_m([x, b] \cap J_i) - (HK) \int_{[a,b]} f(t) \chi_{[x,b] \cap J_i}(t) \, dt \right\} \right| < \frac{2^{m+2}}{2^{m+2}}
\]

for each \( \delta_1 \)-fine partial partition \( \{(J_1, \zeta_1), \ldots, (J_q, \zeta_q)\} \) of \([a, b]\).

As \( f \in BV_0[a, b] \), it follows from Theorem 2.3 that there exists a gauge \( \delta_2 \) on \([a, b]\) such that

\[
\sum_{j=1}^{r} |f(z_j)||\int_{K_i} |g(z_j) - g(t)| \, dt < \frac{\varepsilon}{2^{m+2}}
\]

for each \( \delta_2 \)-fine McShane partial partition \( \{(K_1, z_1), \ldots, (K_r, z_r)\} \) of \([a, b]\).

Define a gauge \( \delta \) on \([a, b]\) by \( \delta(x) = \min\{\delta_1(x), \delta_2(x)\} \). For each \( \delta \)-fine partition \( \{(I_1, \xi_1), \ldots, (I_p, \xi_p)\} \) of \([a, b]\), we infer from Theorem 4.7 and the above estimates that

\[
\left| \sum_{i=1}^{p} f(\xi_i) g(\xi_i) |I_i| - \int_{[a,b]} \left( (HK) \int_{[x,b]} f(t) \, dt \right) dg(x) \right| \\
= \left| \sum_{i=1}^{p} \left\{ f(\xi_i) g(\xi_i) |I_i| - \int_{[a,b]} \left( (HK) \int_{[x,b]} f(t) \chi_{I_i}(t) \, dt \right) dg(x) \right\} \right| \\
\leq \sum_{i=1}^{p} |f(\xi_i)| \int_{I_i} |g(\xi_i) - g(t)| \, dt \\
+ \sup_{x \in [a,b]} \left| (HK) \int_{[x,b]} \sum_{i=1}^{p} \left\{ f(\xi_i) \chi_{I_i}(t) - f(t) \chi_{I_i}(t) \right\} dt \right| \text{Var}(g, [a, b]) \\
< \varepsilon,
\]

thereby completing the proof of the theorem.
Our next aim is to deduce Kurzweil’s multidimensional integration by parts formula [1, Theorem 2.10]. For \( s, t \in [a, b] \), we set

\[
\langle s, t \rangle := \{ (x_1, \ldots, x_m) : \min\{s_i, t_i\} \leq x_i \leq \max\{s_i, t_i\} \text{ for each } i = 1, \ldots, m \}.
\]

For each \( f \in HK[a, b] \) and \( \alpha \in [a, b] \), we define a function \( \tilde{F}_\alpha \) on \([a, b]\) by

\[
\tilde{F}_\alpha(x) = \left( \text{HK} \int_{\langle \alpha, x \rangle} f(t) \, dt \right) \prod_{i=1}^{m} \text{sgn}(x_i - \alpha_i).
\]

It is well known that \( \tilde{F}_\alpha \in C[a, b] \). The next theorem gives our multidimensional integration by parts formula.

**Theorem 4.9.** If \( f \in HK[a, b], \alpha \in [a, b] \) and \( g \in \text{BV}_{HK}[a, b] \), then \( fg \in HK[a, b] \) and

\[
\int_{[a, b]} f(x)g(x) \, dx = \sum_{Y \in P_m} (-1)^{\mu_0(\Gamma(Y))} \left\{ \int_{[a, b]} \tilde{F}_\alpha \, d(g \chi_Y) \right\}.
\]

**Proof.** Let \( g_0 = g \chi_{(a,b)} \). By Theorems 4.5 and 4.8, \( fg_0 \in HK[a, b] \). As \( g = g_0 \) \( \mu_m \)-almost everywhere on \([a, b]\), we see that \( fg \in HK[a, b] \) and

\[
\int_{[a, b]} f(x)g(x) \, dx = \int_{[a, b]} f(x)g_0(x) \, dx.
\]

By Theorem 4.8 again,

\[
\int_{[a, b]} f(x)g_0(x) \, dx = \int_{[a, b]} \left\{ \left( \text{HK} \int_{[x, b]} f(t) \, dt \right) \right\} d g_0(x).
\]

Using the additivity of the indefinite HK-integral of \( f \) over \([a, b]\) and [1, Lemma 1.3], we see that

\[
\int_{[x, b]} f(t) \, dt = \Delta \tilde{F}_\alpha([x, b])
\]

for each \( x \in [a, b] \). Thus it follows from [1, (1.9), (1.8)], the linearity of the Riemann-Stieltjes integral and Theorem 3.4 that

\[
\int_{[a, b]} \Delta \tilde{F}_\alpha([x, b]) \, d g_0(x) = \int_{[a, b]} (-1)^m \tilde{F}_\alpha(x) \, d g_0(x).
\]
Now the linearity of the Riemann-Stieltjes integral and Theorem 4.5 imply that
\[
\int_{[a,b]} (-1)^m \tilde{F}_\alpha(x) \, dg_0(x) = \sum_{Y \in P_m} (-1)^{\mu_0(\Gamma(Y))} \left\{ \int_{[a,b]} \tilde{F}_\alpha \, d(g|_Y) \right\}. 
\]

Combining the above equalities yields (12). The proof is complete.

Let \( \sigma\left(\prod_{k=1}^{m} Y_k\right) := \{i \in \{1, \ldots, m\} : Y_i = \{a_i\}\} \). It remains to show that Theorem 4.9 is equivalent to the following Kurzweil’s multidimensional integration by parts formula [1, Theorem 2.10].

**Theorem 4.10.** If \( f \in HK[a, b] \), \( g \in BV_{HK}[a, b] \) and \( \alpha \in [a, b] \), then \( fg \in HK[a, b] \) and

\[
(HK) \int_{[a,b]} f(x)g(x) \, dx = \sum_{\{e\} \in P_m} (-1)^{\sigma(\{e\})} \tilde{F}_\alpha(e)g(e)
\]

\[
+ \sum_{k=1}^{m} \sum_{Y \in P_m} (-1)^{k} \int_{[a,b]} (-1)^{\sigma(Y)} \tilde{F}_\alpha \big|_Y \, d(g|_Y). 
\]

**Proof.** If \( Y \in P_m \) and \( \mu_0(\Gamma(Y)) = 0 \), then there exists a vertex \( c \) of \( [a, b] \) such that

\[
\int_{[a,b]} \tilde{F}_\alpha \, d(g|_Y) = (-1)^{\sigma(\{c\})} \tilde{F}_\alpha(c)g(c).
\]

A similar argument shows that if \( Y \in P_m \) and \( \mu_0(\Gamma(Y)) > 0 \), then

\[
\int_{[a,b]} \tilde{F}_\alpha \, d(g|_Y) = \int_{\prod_{j=1}^{m}} [a_j,b_j] (-1)^{\sigma(Y)} \tilde{F}_\alpha \big|_Y \, d(g|_Y).
\]

Hence, as a consequence of Theorem 4.9, we get the desired result:

\[
(HK) \int_{[a,b]} f(x)g(x) \, dx = \sum_{Y \in P_m} (-1)^{\mu_0(\Gamma(Y))} \left\{ \int_{[a,b]} \tilde{F}_\alpha \, d(g|_Y) \right\}
\]

\[
= \sum_{Y \in P_m} (-1)^{\mu_0(\Gamma(Y))} \left\{ \int_{[a,b]} \tilde{F}_\alpha \, d(g|_Y) \right\} + \sum_{k=1}^{m} \sum_{Y \in P_m} (-1)^{k} \left\{ \int_{[a,b]} \tilde{F}_\alpha \, d(g|_Y) \right\}
\]

\[
= \sum_{\{e\} \in P_m} (-1)^{\sigma(\{e\})} \tilde{F}_\alpha(e)g(e)
\]

\[
+ \sum_{k=1}^{m} \sum_{Y \in P_m} (-1)^{k} \int_{[a,b]} (-1)^{\sigma(Y)} \tilde{F}_\alpha \big|_Y \, d(g|_Y).
\]
The proof of Theorem 4.10 depends heavily on (11), which is also true for some other generalized Riemann integrals; more precisely, we have

Remark 4.11. Theorem 4.10 also holds if the Henstock-Kurzweil integral is replaced by any of the following generalized Riemann integrals:

(i) the Lebesgue integral (see also [19], [21]);
(ii) the Cauchy-Lebesgue integral;
(iii) the strong ρ-integral in [6];
(iv) the R-integral in [10].

References


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