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Rainbow connection in graphs


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1. Introduction

Let $G$ be a nontrivial connected graph on which is defined a coloring $c: E(G) \rightarrow \{1, 2, \ldots, k\}$, $k \in \mathbb{N}$, of the edges of $G$, where adjacent edges may be colored the same. A path $P$ in $G$ is a rainbow path if no two edges of $P$ are colored the same. The graph $G$ is rainbow-connected if $G$ contains a rainbow $u - v$ path for every two vertices $u$ and $v$ of $G$. The minimum $k$ for which there exists such a $k$-edge coloring is the rainbow connection number $rc(G)$ of $G$. A rainbow coloring of $G$ using $rc(G)$ colors is called a minimum rainbow coloring of $G$.

Abstract. Let $G$ be a nontrivial connected graph on which is defined a coloring $c: E(G) \rightarrow \{1, 2, \ldots, k\}$, $k \in \mathbb{N}$, of the edges of $G$, where adjacent edges may be colored the same. A path $P$ in $G$ is a rainbow path if no two edges of $P$ are colored the same. The graph $G$ is rainbow-connected if $G$ contains a rainbow $u - v$ path for every two vertices $u$ and $v$ of $G$. The minimum $k$ for which there exists such a $k$-edge coloring is the rainbow connection number $rc(G)$ of $G$. A rainbow coloring of $G$ using $rc(G)$ colors is called a minimum rainbow coloring of $G$.

Keywords: edge coloring, rainbow coloring, strong rainbow coloring

MSC 2000: 05C15, 05C38, 05C40
Let $c$ be a rainbow coloring of a connected graph $G$. For two vertices $u$ and $v$ of $G$, a **rainbow $u−v$ geodesic** in $G$ is a rainbow $u−v$ path of length $d(u,v)$, where $d(u,v)$ is the distance between $u$ and $v$ (the length of a shortest $u−v$ path in $G$). The graph $G$ is **strongly rainbow-connected** if $G$ contains a rainbow $u−v$ geodesic for every two vertices $u$ and $v$ of $G$. In this case, the coloring $c$ is called a **strong rainbow coloring** of $G$. The minimum $k$ for which there exists a coloring $c: E(G) → \{1,2,\ldots,k\}$ of the edges of $G$ such that $G$ is strongly rainbow-connected is the **strong rainbow connection number** $\text{src}(G)$ of $G$. A strong rainbow coloring of $G$ using $\text{src}(G)$ colors is called a **minimum strong rainbow coloring** of $G$. Thus $\text{rc}(G) \leq \text{src}(G)$ for every connected graph $G$.

Since every coloring that assigns distinct colors to the edges of a connected graph is both a rainbow coloring and a strong rainbow coloring, every connected graph is rainbow-connected and strongly rainbow-connected with respect to some coloring of the edges of $G$. Thus the rainbow connection numbers $\text{rc}(G)$ and $\text{src}(G)$ are defined for every connected graph $G$. Furthermore, if $G$ is a nontrivial connected graph of size $m$ whose diameter (the largest distance between two vertices of $G$) is $\text{diam}(G)$, then

\[
\text{diam}(G) \leq \text{rc}(G) \leq \text{src}(G) \leq m.
\]

To illustrate these concepts, consider the Petersen graph $P$ of Figure 1, where a rainbow 3-coloring of $P$ is also shown. Thus $\text{rc}(P) \leq 3$. On the other hand, if $u$ and $v$ are two nonadjacent vertices of $P$, then $d(u,v) = 2$ and so the length of a $u−v$ path is at least 2. Thus any rainbow coloring of $P$ uses at least two colors and so $\text{rc}(P) \geq 2$. If $P$ has a rainbow 2-coloring $c$, then there exist two adjacent edges of $G$ that are colored the same by $c$, say $e = uv$ and $f = vw$ are colored the same. Since there is exactly one $u−w$ path of length 2 in $P$, there is no rainbow $u−w$ path in $P$, which is a contradiction. Therefore, $\text{rc}(P) = 3$.

Figure 1. A rainbow 3-coloring and a strong rainbow 4-coloring of the Petersen graph
Since \( \text{rc}(P) = 3 \), it follows that \( \text{src}(P) \geq 3 \). Furthermore, since the edge chromatic number of the Petersen graph is known to be 4, any 3-coloring \( c \) of the edges of \( P \) results in two adjacent edges \( uv \) and \( vw \) being assigned the same color. Since \( u, v, w \) is the only \( u - w \) geodesic in \( P \), the coloring \( c \) is not a strong rainbow coloring. Because the 4-coloring of the edges of \( P \) shown in Figure 2 is a strong rainbow coloring, \( \text{src}(P) = 4 \).

As another example, consider the graph \( G \) of Figure 2(a), where a rainbow 4-coloring \( c \) of \( G \) is also shown. In fact, \( c \) is a minimum rainbow coloring of \( G \) and so \( \text{rc}(G) = 4 \), as we now verify.

![Figure 2](image.png)

Figure 2. A graph \( G \) with \( \text{rc}(G) = \text{src}(G) = 4 \)

Since \( \text{diam}(G) \geq 3 \), it follows that \( \text{rc}(G) \geq 3 \). Assume, to the contrary, \( \text{rc}(G) = 3 \). Then there exists a rainbow 3-coloring \( c' \) of \( G \). Since every \( u - v \) path in \( G \) has length 3, at least one of the three \( u - v \) paths in \( G \) is a rainbow \( u - v \) path, say \( u, u_1, v_1, v \) is a rainbow \( u - v \) path. We may assume that \( c'(uu_1) = 1 \), \( c'(u_1v_1) = 2 \), and \( c'(v_1v) = 3 \). (See Figure 2(b).)

If \( x \) and \( y \) are two vertices in \( G \) such that \( d(x, y) = 2 \), then \( G \) contains exactly one \( x - y \) path of length 2, while all other \( x - y \) paths have length 4 or more. This implies that no two adjacent edges can be colored the same. Thus we may assume, without loss of generality, that \( c'(uu_2) = 2 \) and \( c'(uu_3) = 3 \). (See Figure 2(b).) Thus \( \{c'(vv_2), c'(vv_3)\} = \{1, 2\} \). If \( c'(vv_2) = 1 \) and \( c'(vv_3) = 2 \), then \( c'(u_2v_2) = 3 \) and \( c'(u_3v_3) = 1 \). In this case, there is no rainbow \( u_1 - v_3 \) path in \( G \). On the other hand, if \( c'(vv_2) = 2 \) and \( c'(vv_3) = 1 \), then \( c'(u_2v_2) \in \{1, 3\} \) and \( c'(u_3v_3) = 2 \). If \( c'(u_2v_2) = 1 \), then there is no rainbow \( u_2 - v_3 \) path in \( G \); while if \( c'(u_2v_2) = 3 \), there is no rainbow \( u_2 - v_1 \) path in \( G \), a contradiction. Therefore, as claimed, \( \text{rc}(G) = 4 \).

Since \( 4 = \text{rc}(G) \leq \text{src}(G) \) for the graph \( G \) of Figure 2 and the rainbow 4-coloring of \( G \) in Figure 2(a) is also a strong rainbow 4-coloring, \( \text{src}(G) = 4 \) as well.

If \( G \) is a nontrivial connected graph of size \( m \), then we saw in (1) that \( \text{diam}(G) \leq \text{rc}(G) \leq \text{src}(G) \leq m \). In the following result, it is determined which connected graphs \( G \) attain the extreme values 1, 2 or \( m \).
Proposition 1.1. Let $G$ be a nontrivial connected graph of size $m$. Then

(a) $\text{src}(G) = 1$ if and only if $G$ is a complete graph,
(b) $\text{rc}(G) = 2$ if and only if $\text{src}(G) = 2$,
(c) $\text{rc}(G) = m$ if and only if $G$ is a tree.

Proof. We first verify (a). If $G$ is a complete graph, then the coloring that assigns 1 to every edge of $G$ is a strong rainbow 1-coloring of $G$ and so $\text{src}(G) = 1$. On the other hand, if $G$ is not complete, then $G$ contains two nonadjacent vertices $u$ and $v$. Thus each $u - v$ geodesic in $G$ has length at least 2 and so $\text{src}(G) \geq 2$.

To verify (b), first assume that $\text{rc}(G) = 2$ and so $\text{src}(G) \geq 2$ by (1). Since $\text{rc}(G) = 2$, it follows that $G$ has a rainbow 2-coloring, which implies that every two nonadjacent vertices are connected by a rainbow path of length 2. Because such a path is a geodesic, $\text{src}(G) = 2$. On the other hand, if $\text{src}(G) = 2$, then $\text{rc}(G) \leq 2$ by (1) again. Furthermore, since $\text{src}(G) = 2$, it follows by (a) that $G$ is not complete and so $\text{rc}(G) \geq 2$. Thus $\text{rc}(G) = 2$.

We now verify (c). Suppose first that $G$ is not a tree. Then $G$ contains a cycle $C: v_1, v_2, \ldots, v_k, v_1$, where $k \geq 3$. Then the $(m - 1)$-coloring of the edges of $G$ that assigns 1 to the edges $v_1v_2$ and $v_2v_3$ and assigns the $m - 2$ distinct colors from $\{2, 3, \ldots, m - 1\}$ to the remaining $m - 2$ edges of $G$ is a rainbow coloring. Thus $\text{rc}(G) \leq m - 1$. Next, let $G$ be a tree of size $m$. Assume, to the contrary, that $\text{rc}(G) \leq m - 1$. Let $c$ be a minimum rainbow coloring of $G$. Then there exist edges $e$ and $f$ such that $c(e) = c(f)$. Assume, without loss generality, that $e = uv$ and $f = xy$ and $G$ contains a $u - y$ path $u, v, \ldots, x, y$. Then there is no rainbow $u - y$ path in $G$, which is a contradiction. \qed

Proposition 1.1 also implies that the only connected graphs $G$ for which $\text{rc}(G) = 1$ are the complete graphs and that the only connected graphs $G$ of size $m$ for which $\text{src}(G) = m$ are trees.

2. Some rainbow connection numbers of graphs

In this section, we determine the rainbow connection numbers of some well-known graphs. We refer to the book [1] for graph-theoretical notation and terminology not described in this paper. We begin with cycles of order $n$. Since $\text{diam}(C_n) = \lfloor n/2 \rfloor$, it follows by (1) that $\text{src}(C_n) \geq \text{rc}(C_n) \geq \lfloor n/2 \rfloor$. This lower bound for $\text{rc}(C_n)$ and $\text{src}(C_n)$ is nearly the exact value of these numbers.
Proposition 2.1. For each integer \( n \geq 4 \), \( \text{rc}(C_n) = \text{src}(C_n) = \lceil n/2 \rceil \).

Proof. Let \( C_n : v_1, v_2, \ldots, v_n, v_{n+1} = v_1 \) and for each \( i \) with \( 1 \leq i \leq n \), let \( e_i = v_i v_{i+1} \). We consider two cases, according to whether \( n \) is even or \( n \) is odd.

Case 1. \( n \) is even. Let \( n = 2k \) for some integer \( k \geq 2 \). Thus \( \text{src}(C_n) \geq \text{rc}(C_n) \geq \text{diam}(C_n) = k \). Since the edge coloring \( c_0 \) of \( C_n \) defined by \( c_0(e_i) = i \) for \( 1 \leq i \leq k \) and \( c_0(e_i) = i - k \) if \( k + 1 \leq i \leq n \) is a strong rainbow \( k \)-coloring, it follows that \( \text{rc}(C_n) \leq \text{src}(C_n) \leq k \) and so \( \text{rc}(C_n) = \text{src}(C_n) = k \).

Case 2. \( n \) is odd. Then \( n = 2k + 1 \) for some integer \( k \geq 2 \). First define an edge coloring \( c_1 \) of \( C_n \) by \( c_1(e_i) = i \) for \( 1 \leq i \leq k + 1 \) and \( c_1(e_i) = i - k - 1 \) if \( k + 2 \leq i \leq n \). Since \( c_1 \) is a strong rainbow \((k+1)\)-coloring of \( C_n \), it follows that \( \text{rc}(C_n) \leq \text{src}(C_n) \leq k + 1 \).

Since \( \text{rc}(C_n) \geq \text{diam}(C_n) = k \), it follows that \( \text{rc}(C_n) = k \) or \( \text{rc}(C_n) = k + 1 \). We claim that \( \text{rc}(C_n) = k + 1 \). Assume, to the contrary, that \( \text{rc}(C_n) = k \). Let \( c' \) be a rainbow \( k \)-coloring of \( C_n \) and let \( u \) and \( v \) be two antipodal vertices of \( C_n \). Then the \( u - v \) geodesic in \( C_n \) is a rainbow path and the other \( u - v \) path in \( C_n \) is not a rainbow path since it has length \( k + 1 \). Suppose, without loss of generality, that \( c'(v_{k+1}v_{k+2}) = k \).

Consider the vertices \( v_1, v_{k+1}, \) and \( v_{k+2} \). Since the \( v_1 - v_{k+1} \) geodesic \( P: v_1, v_2, \ldots, v_{k+1} \) is a rainbow path and the \( v_1 - v_{k+2} \) geodesic \( Q: v_1, v_n, v_{n-1}, \ldots, v_{k+2} \) is a rainbow path, some edge on \( P \) is colored \( k \) as is some edge on \( Q \). Since the \( v_2 - v_{k+2} \) geodesic \( v_2, v_3, \ldots, v_{k+2} \) is a rainbow path, it follows that \( c'(v_2v_k) = k \). Similarly, the \( v_n - v_{k+1} \) geodesic \( v_n, v_{n-1}, v_{n-2}, \ldots, v_{k+1} \) is a rainbow path and so \( c'(v_nv_1) = k \). Thus \( c'(v_1v_2) = c'(v_nv_1) = k \). This implies that there is no rainbow \( v_2 - v_n \) path in \( G \), producing a contradiction. Thus \( \text{rc}(C_n) = \text{src}(C_n) = k + 1 \). \( \square \)

A well-known class of graphs constructed from cycles are the wheels. For \( n \geq 3 \), the wheel \( W_n \) is defined as \( C_n + K_1 \), the join of \( C_n \) and \( K_1 \), constructed by joining a new vertex to every vertex of \( C_n \). Thus \( W_3 = K_4 \). Next, we determine rainbow connection numbers of wheels.

Proposition 2.2. For \( n \geq 3 \), the rainbow connection number of the wheel \( W_n \) is

\[
\text{rc}(W_n) = \begin{cases} 
1 & \text{if } n = 3, \\
2 & \text{if } 4 \leq n \leq 6, \\
3 & \text{if } n \geq 7.
\end{cases}
\]

Proof. Suppose that \( W_n \) consists of an \( n \)-cycle \( C_n : v_1, v_2, \ldots, v_n, v_{n+1} = v_1 \) and another vertex \( v \) joined to every vertex of \( C_n \). Since \( W_3 = K_4 \), it follows by Proposition 1.1 that \( \text{rc}(W_3) = 1 \). For \( 4 \leq n \leq 6 \), the wheel \( W_n \) is not complete and
so \( \text{rc}(W_n) \geq 2 \). Since the 2-coloring \( c: E(W_n) \to \{1, 2\} \) defined by \( c(v_iv) = 1 \) if \( i \) is odd, \( c(v_iv) = 2 \) if \( i \) is even, and \( c(v_iv_{i+1}) = 1 \) if \( i \) is odd, and \( c(v_iv_{i+1}) = 2 \) if \( i \) is even is a rainbow coloring, it follows that \( \text{rc}(W_n) = 2 \) for \( 4 \leq n \leq 6 \).

Finally, suppose that \( n \geq 7 \). Since the 3-coloring \( c: E(W_n) \to \{1, 2, 3\} \) defined by \( c(v_iv) = 1 \) if \( i \) is odd, \( c(v_iv) = 2 \) if \( i \) is even, and \( c(e) = 3 \) for each \( e \in E(C_n) \) is a rainbow coloring, it follows that \( \text{rc}(W_n) \leq 3 \). It remains to show that \( \text{rc}(W_n) \geq 3 \). Since \( W_n \) is not complete, \( \text{rc}(W_n) \geq 2 \). Assume, to the contrary, that \( \text{rc}(W_n) = 2 \). Let \( c' \) be a rainbow 2-coloring of \( W_n \). Without loss of generality, assume that \( c'(v_1v) = 1 \). For each \( i \) with \( 4 \leq i \leq n - 2 \), \( v_1, v_i, v_i \) is the only \( v_1 - v_i \) path of length 2 in \( W_n \) and so \( c'(v_iv) = 2 \) for \( 4 \leq i \leq n - 2 \). Since \( c(v_4v) = 2 \), it follows that \( c(v_nv) = 1 \). This forces \( c(v_3v) = 2 \), which in turn forces \( c(v_{n-1}v) = 1 \). Similarly, \( c(v_{n-1}v) = 1 \) forces \( c(v_2v) = 2 \). Since \( c(v_2v) = 2 \) and \( c(v_5v) = 2 \), there is no rainbow \( v_2 - v_5 \) path in \( W_n \), which is a contradiction. Therefore, \( \text{rc}(W_n) = 3 \) for \( n \geq 7 \). \( \square \)

**Proposition 2.3.** For \( n \geq 3 \), the strong rainbow connection number of the wheel \( W_n \) is

\[
\text{src}(W_n) = \lceil n/3 \rceil.
\]

**Proof.** Suppose that \( W_n \) consists of an \( n \)-cycle \( C_n: v_1, v_2, \ldots, v_n, v_{n+1} = v_1 \) and another vertex \( v \) joined to every vertex of \( C_n \). Since \( W_3 = K_4 \), it follows by Proposition 1.1 that \( \text{rc}(W_3) = 1 \). If \( 4 \leq n \leq 6 \), then \( \text{rc}(W_n) = 2 \) by Proposition 2.2 and so \( \text{rc}(W_n) = 2 \) by Proposition 1.1. Therefore, \( \text{rc}(W_n) = \lceil n/3 \rceil \) for \( 4 \leq n \leq 6 \).

Thus we may assume \( n \geq 7 \). Then there is an integer \( k \) such that \( 3k - 2 \leq n \leq 3k \).

We first show that \( \text{src}(W_n) \geq k \). Assume, to the contrary, that \( \text{src}(W_n) \leq k - 1 \). Let \( c \) be a strong rainbow \((k - 1)\)-coloring of \( W_n \). Since \( \text{deg} v = n > 3(k - 1) \), there exists \( S \subseteq V(C_n) \) such that \( |S| = 4 \) and all edges in \( \{uv: u \in S\} \) are colored the same. Thus there exist at least two vertices \( u', u'' \in S \) such that \( d_{C_n}(u', u'') = 3 \) and \( d_{W_n}(u', u'') = 2 \). Since \( u', v, u'' \) is the only \( u' - u'' \) geodesic in \( W_n \), it follows that there is no rainbow \( u' - u'' \) geodesic in \( W_n \), which is a contradiction. Thus \( \text{src}(W_n) \geq k \).

To show that \( \text{src}(W_n) \leq k \), we provide a strong rainbow \( k \)-coloring \( c: E(W_n) \to \{1, 2, \ldots, k\} \) of \( W_n \) defined by

\[
c(e) = \begin{cases} 
1 & \text{if } e = v_iv_{i+1} \text{ and } i \text{ is odd}, \\
2 & \text{if } e = v_iv_{i+1} \text{ and } i \text{ is even}, \\
\frac{j+1}{j+2} & \text{if } e = v_iv \text{ if } i \in \{3j + 1, 3j + 2, 3j + 3\} \text{ for } 0 \leq j \leq k - 1.
\end{cases}
\]

Therefore, \( \text{src}(W_n) = k = \lceil n/3 \rceil \) for \( n \geq 7 \) as well. \( \square \)

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We now determine the rainbow connection numbers of all complete multipartite graphs, beginning with the strong connection number of the complete bipartite graph $K_{s,t}$ with $1 \leq s \leq t$.

**Theorem 2.4.** For integers $s$ and $t$ with $1 \leq s \leq t$,

$$\text{src}(K_{s,t}) = \lceil \sqrt{t} \rceil.$$  

**Proof.** Since $\text{src}(K_{1,t}) = t$, the result follows for $s = 1$. So we may assume that $s \geq 2$. Let $\lceil \sqrt{t} \rceil = k$. Hence

$$1 \leq k - 1 < \sqrt{t} \leq k.$$  

Therefore, $(k - 1)^s < t \leq k^s$ and so $(k - 1)^s + 1 \leq t \leq k^s$.

First, we show that $\text{src}(K_{s,t}) \geq k$. Assume, to the contrary, that $\text{src}(K_{s,t}) \leq k - 1$. Then there exists a strong rainbow $(k - 1)$-coloring of $K_{s,t}$. Let $U$ and $W$ be the partite sets of $K_{s,t}$, where $|U| = s$ and $|W| = t$. Suppose that $U = \{u_1, u_2, \ldots, u_s\}$. Let there be given a strong rainbow $(k - 1)$-coloring $c$ of $K_{s,t}$. For each vertex $w \in W$, we can associate an ordered $s$-tuple $\text{code}(w) = (a_1, a_2, \ldots, a_s)$ called the color code of $w$, where $a_i = c(u_iw)$ for $1 \leq i \leq s$. Since $1 \leq a_i \leq k - 1$ for each $i$ ($1 \leq i \leq s$), the number of distinct color codes of the vertices of $W$ is at most $(k - 1)^s$. However, since $t > (k - 1)^s$, there exists at least two distinct vertices $w'$ and $w''$ of $W$ such that $\text{code}(w') = \text{code}(w'')$. Since $c(u_iw') = c(u_iw'')$ for all $i$ ($1 \leq i \leq s$), it follows that $K_{s,t}$ contains no rainbow $w' - w''$ geodesic in $K_{s,t}$, contradicting our assumption that $c$ is a strong rainbow $(k - 1)$-coloring of $K_{s,t}$. Thus, as claimed, $\text{src}(K_{s,t}) \geq k$.

Next, we show that $\text{src}(K_{s,t}) \leq k$, which we establish by providing a strong rainbow $k$-coloring of $K_{s,t}$. Let $A = \{1, 2, \ldots, k\}$ and $B = \{1, 2, \ldots, k - 1\}$. The sets $A^s$ and $B^s$ are Cartesian products of the $s$ sets $A$ and $s$ sets $B$, respectively. Thus $|A^s| = k^s$ and $|B^s| = (k - 1)^s$. Hence $|B^s| < t \leq |A^s|$. Let $W = \{w_1, w_2, \ldots, w_t\}$, where the vertices of $W$ are labeled with $t$ elements of $A^s$ and such that the vertices $w_1, w_2, \ldots, w_{(k - 1)^s}$ are labeled by the $(k - 1)^s$ elements of $B^s$. For each $i$ with $1 \leq i \leq t$, denote the label of $w_i$ by

$$w_i = (w_{i,1}, w_{i,2}, \ldots, w_{i,s}).$$  

For each $i$ with $1 \leq i \leq (k - 1)^s$, we have $1 \leq w_{i,j} \leq k - 1$ for $1 \leq j \leq s$. We now define a coloring $c: E(K_{s,t}) \to \{1, 2, \ldots, k\}$ of the edges of $K_{s,t}$ by

$$c(w_{i,j}u_j) = w_{i,j}$$  

where $1 \leq i \leq t$ and $1 \leq j \leq s$.  

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Thus for $1 \leq i \leq t$, the color code $\text{code}(w_i)$ of $w_i$ provided by the coloring $c$ is in fact $w_i$, as described in (2). Hence distinct vertices in $W$ have distinct color codes.

We show that $c$ is a strong rainbow $k$-coloring of $K_{s,t}$. Certainly, for $w_i \in W$ and $u_j \in U$, the $w_i - u_j$ path $w_i, u_j$ is a rainbow geodesic. Let $w_a$ and $w_b$ be two vertices of $W$. Since these vertices have distinct color codes, there exists some $l$ with $1 \leq l \leq s$ such that $\text{code}(w_a)$ and $\text{code}(w_b)$ have different $l$-th coordinates. Thus $c(w_a, u_l) \neq c(w_b, u_l)$ and $w_a, u_l, w_b$ is a rainbow $w_a - w_b$ geodesic in $K_{s,t}$. We now consider two vertices $u_p$ and $u_q$ in $U$, where $1 \leq p < q \leq s$. Since there exists a vertex $w_i \in W$ with $1 \leq i \leq (k-1)s$ such that $w_i,p \neq w_i,q$, it follows that $u_p, w_i, u_q$ is a rainbow $u_p - u_q$ geodesic in $K_{s,t}$. Thus, as claimed, $c$ is a strong rainbow $k$-coloring of $K_{s,t}$ and so $\text{src}(K_{s,t}) \leq k$.

With the aid of Theorem 2.4, we are now able to determine the strong rainbow connection numbers of all complete multipartite graphs.

**Theorem 2.5.** Let $G = K_{n_1, n_2, \ldots, n_k}$ be a complete $k$-partite graph, where $k \geq 3$ and $n_1 \leq n_2 \leq \ldots \leq n_k$ such that $s = \sum_{i=1}^{k-1} n_i$ and $t = n_k$. Then

$$\text{src}(G) = \begin{cases} 1 & \text{if } n_k = 1, \\ 2 & \text{if } n_k \geq 2 \text{ and } s > t, \\ \lceil \sqrt{t} \rceil & \text{if } s \leq t. \end{cases}$$

**Proof.** Let $n = \sum_{i=1}^{k} n_i$. If $n_k = 1$, then $G = K_n$ and by Proposition 1.1, $\text{src}(G) = 1$. Suppose next that $n_k \geq 2$ and $s > t$. Since $n_k \geq 2$, it follows that $G \neq K_n$ and so $\text{src}(G) \geq 2$ by Proposition 1.1. It remains to show that $\text{src}(G) \leq 2$ in this case.

Partition the multiset $S = \{n_1, n_2, \ldots, n_k\}$ into two multisets

$$A = \{a_1, a_2, \ldots, a_p\} \text{ and } B = \{b_1, b_2, \ldots, b_q\},$$

where then $p + q = k$, such that

$$a = \sum_{i=1}^{p} a_i \leq \sum_{j=1}^{q} b_j = b$$

and $b - a$ is the minimum nonnegative integer among all such partitions of $S$. Hence $K_{a,b}$ is a spanning subgraph of $G$. Since $\text{diam}(K_{a,b}) = 2$, for every two nonadjacent
vertices $u$ and $v$ of $K_{a,b}$, a path $P$ is a $u-v$ geodesic in $K_{a,b}$ if and only if $P$ is a $u-v$ geodesic in $G$. Thus, from Theorem 2.4,

$$\text{src}(G) \leq \text{src}(K_{a,b}) = \lceil \sqrt[3]{b} \rceil.$$  

We claim that $b \leq 2^a$. Assume, to the contrary, that $b > 2^a$. Since $s > t$, it follows that $q \geq 2$. We consider two cases, according to $a \leq 3$ or $a \geq 4$. If $G$ is a complete $k$-partite graph with $a \leq 3$, then the only ordered pairs $(a,b)$ for $K_{a,b}$ are: $(2,3)$, $(2,4)$, $(3,3)$, $(3,4)$, $(3,5)$, $(3,6)$. In all cases, $\text{src}(G) \leq \text{src}(K_{a,b}) = \lceil \sqrt[3]{b} \rceil = 2$. Hence we may assume that $a \geq 4$. Let $b_1$ be the smallest element of $B$. Hence $a + b_1 > b - b_1$. Because $a \geq 4$, it follows that

$$b_1 > \frac{b - a}{2} > \frac{2^a - a}{2} > \frac{3a - a}{2} = a.$$  

Let $A' = \{b_1\}$ and let the multiset $B' = S - \{b_1\}$. Since $b_2 \in B'$, $b_1 \leq b_2$, and $a < b_1$, this contradicts the defining properties of the sets $A$ and $B$. Hence, as claimed, $b \leq 2^a$. Thus

$$\text{src}(G) \leq \lceil \sqrt[3]{b} \rceil \leq \lceil \sqrt[3]{2^a} \rceil = 2,$$

giving us the desired result.

Next, suppose that $s \leq t$. Let $W$ be the unique independent set of $n_k = t$ vertices of $G$. Since $K_{s,t}$ is a connected spanning subgraph of $G$, it follows again, since $\text{diam}(G) = 2$, that

$$\text{src}(G) \leq \text{src}(K_{s,t}) = \lceil \sqrt{t} \rceil.$$  

We claim that $\text{src}(G) = \lceil \sqrt{t} \rceil$. Assume, to the contrary, that $\text{src}(G) = l < \lceil \sqrt{t} \rceil$. Then $t > l^s$. This implies that there exists a strong rainbow $l$-coloring $c$ of $G$. Since every vertex of $G$ belonging to $W$ has degree $s$ in $G$, the coloring $c$ produces a color code $\text{code}(w)$ for each vertex $w$ of $W$ consisting of an ordered $s$-tuple, each entry of which is an element of $\{1,2,\ldots,l\}$. Since the number of distinct color codes for the vertices of $W$ is at most $l^s$ and $|W| = t > l^s$, there exist two vertices $w'$ and $w''$ in $W$ having the same color code. This, however, implies that the two edges in each $w' - w''$ geodesic in $G$ have the same color, contradicting the assumption that $c$ is a strong rainbow $l$-coloring of $G$. \hfill $\Box$

According to Theorems 2.4 and 2.5, the strong rainbow connection number of a complete multipartite graph can be arbitrarily large. This is not the case for the rainbow connection number of a complete multipartite graph however, as we show next. We begin with complete bipartite graphs.
Theorem 2.6. For integers \( s \) and \( t \) with \( 2 \leq s \leq t \),

\[
rc(K_{s,t}) = \min\{\lceil \sqrt[4]{t} \rceil, 4\}.
\]

Proof. First, observe that for \( 2 \leq s \leq t \), \( \lceil \sqrt[4]{t} \rceil \geq 2 \). Let \( U \) and \( W \) be the partite sets of \( K_{s,t} \), where \( |U| = s \) and \( |W| = t \). Suppose that \( U = \{u_1, u_2, \ldots, u_s\} \).

We consider three cases.

Case 1. \( \lceil \sqrt[4]{t} \rceil = 2 \). Then \( s \leq t \leq 2^s \). Since

\[
2 \leq rc(K_{s,t}) \leq src(K_{s,t}) = \lceil \sqrt[4]{t} \rceil = 2,
\]

it follows that \( rc(K_{s,t}) = 2 \).

Case 2. \( \lceil \sqrt[4]{t} \rceil = 3 \). Then \( 2^s + 1 \leq t \leq 3^s \). Since

\[
2 \leq rc(K_{s,t}) \leq src(K_{s,t}) = \lceil \sqrt[4]{t} \rceil = 3,
\]

it follows that \( rc(K_{s,t}) = 2 \) or \( rc(K_{s,t}) = 3 \). We claim that \( rc(K_{s,t}) = 3 \). Assume, to the contrary, that there exists a rainbow 2-coloring of \( K_{s,t} \). Corresponding to this rainbow 2-coloring of \( K_{s,t} \), there is a color code \( code(w) \) assigned to each vertex \( w \in W \), consisting of an ordered \( s \)-tuple \( (a_1, a_2, \ldots, a_s) \), where \( a_i = c(u_iw) \in \{1, 2\} \) for \( 1 \leq i \leq s \). Since \( t > 2^s \), there exist two distinct vertices \( w' \) and \( w'' \) of \( W \) such that \( code(w') = code(w'') \). Since the edges of every \( w' - w'' \) path of length 2 are colored the same, there is no rainbow \( w' - w'' \) path in \( K_{s,t} \), a contradiction. Thus, as claimed, \( rc(K_{s,t}) = 3 \).

Case 3. \( \lceil \sqrt[4]{t} \rceil \geq 4 \). Then \( t \geq 3^s + 1 \). We claim that \( rc(K_{s,t}) = 4 \). First, we show that \( rc(K_{s,t}) \geq 4 \). Assume, to the contrary, that there exists a rainbow 3-coloring of \( K_{s,t} \). In this case, corresponding to this rainbow 3-coloring of \( K_{s,t} \), there is a color code, \( code(w) \), assigned to each vertex \( w \in W \), consisting of an ordered \( s \)-tuple \( (a_1, a_2, \ldots, a_s) \), where \( a_i = c(u_iw) \in \{1, 2, 3\} \) for \( 1 \leq i \leq s \). Since \( t > 3^s \), there exist two distinct vertices \( w' \) and \( w'' \) of \( W \) such that \( code(w') = code(w'') \). Since every \( w' - w'' \) path in \( K_{s,t} \) has even length, the only possible rainbow \( w' - w'' \) path must have length 2. However, since \( code(w') = code(w'') \), the colors of the edges of every \( w' - w'' \) path of length 2 are the same. Hence there is no rainbow \( w' - w'' \) path in \( K_{s,t} \), a contradiction. Thus, as claimed, \( rc(K_{s,t}) \geq 4 \).

To verify that \( rc(K_{s,t}) \leq 4 \), we show that there exists a rainbow 4-coloring of \( K_{s,t} \). Let \( A = \{1, 2, 3\} \), \( W = \{w_1, w_2, \ldots, w_t\} \), \( W' = \{w_1, w_2, \ldots, w_{3^s}\} \), and \( W'' = W - W' \). Assign to the vertices in \( W' \) the \( 3^s \) distinct elements of \( A^s \) and assign to the vertices in \( W'' \) the identical code whose first coordinate is 4 and all whose remaining coordinates are 3. Corresponding to this assignment of codes is a coloring.
of the edges of $K_{s,t}$, where $c(w_iu_j) = k$ if the $j$th coordinate of $\text{code}(w_i)$ is $k$. We claim that this coloring is, in fact, a rainbow 4-coloring of $K_{s,t}$. Let $x$ and $y$ be two nonadjacent vertices of $K_{s,t}$. Suppose first that $x, y \in W$. We consider three cases.

Case i. $x, y \in W'$. Since $\text{code}(x) \neq \text{code}(y)$, there exists $i$ with $1 \leq i \leq s$ such that $\text{code}(x)$ and $\text{code}(y)$ have different $i$th coordinates. Then the path $x, u_i, y$ is a rainbow $x - y$ path of length 2 in $K_{s,t}$.

Case ii. $x \in W'$ and $y \in W''$. Suppose that the first coordinate of $\text{code}(x)$ is $a$, where $1 \leq a \leq 3$. Then $x, u_1, y$ is a rainbow $x - y$ path of length 2 in $K_{s,t}$ whose edges are colored $a$ and 4.

Case iii. $x, y \in W''$. Let $z \in W'$ such that the first coordinate of $\text{code}(z)$ is 1 and the second coordinate of $\text{code}(z)$ is 2. Then $x, u_1, z, u_2, y$ is a rainbow $x - y$ path of length 4 in $K_{s,t}$ whose edges are colored 4, 1, 2, 3, respectively.

Finally, suppose that $x, y \in U$. Then $x = u_i$ and $y = u_j$, where $1 \leq i < j \leq s$. Then there exists a vertex $w \in W'$ whose $i$th and $j$th coordinates are distinct. Then $x, w, y$ is a rainbow $x - y$ path in $K_{s,t}$.

Thus this coloring is a rainbow 4-coloring of $K_{s,t}$ and so $\text{rc}(K_{s,t}) = 4$ in this case. □

Next, we determine rainbow connection numbers of all complete multipartite graphs.

**Theorem 2.7.** Let $G = K_{n_1, n_2, \ldots, n_k}$ be a complete $k$-partite graph, where $k \geq 3$ and $n_1 \leq n_2 \leq \ldots \leq n_k$ such that $s = \sum_{i=1}^{k-1} n_i$ and $t = n_k$. Then

$$\text{rc}(G) = \begin{cases} 1 & \text{if } n_k = 1, \\ 2 & \text{if } n_k \geq 2 \text{ and } s > t, \\ \min\{\lceil \sqrt{t} \rceil, 3\} & \text{if } s \leq t. \end{cases}$$

**Proof.** Let $n = s + t = \sum_{i=1}^{k} n_i$. If $n_k = 1$, then $G = K_n$ and by Proposition 1.1, $\text{rc}(G) = 1$. Suppose next that $n_k \geq 2$ and $s > t$. By Theorem 2.5, $\text{src}(G) = 2$ and so $\text{rc}(G) = 2$ by Proposition 1.1.

Next, suppose that $s \leq t$. Since $n_k \geq 2$, it follows that $G \neq K_n$ and so $\text{rc}(G) \geq 2$. By Theorem 2.5, $\text{src}(G) = \lceil \sqrt{s} \rceil$ and so $\text{rc}(G) \leq \lceil \sqrt{s} \rceil$. To show that $\text{rc}(G) \leq 3$ as well, we provide a rainbow 3-coloring of $G$. Let $V_1, V_2, \ldots, V_k$ be the partite sets of $G$ with

$$V_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,n_i}\}$$
for $1 \leq i \leq k$. Furthermore, let

$$U = V_1 \cup V_2 \cup \ldots \cup V_{k-1} = \{u_1, u_2, \ldots, u_s\}$$

such that $u_i = v_{k-1,i}$ for $1 \leq i \leq n_{k-1}$. Thus $|U| = s$. Define a coloring $c^*$ of the edges of $G$ by

$$c^*(e) = \begin{cases} 
1 & \text{if } e = v_{i,j}v_{i+1,j} \text{ for } 1 \leq i \leq k-2 \text{ and } 1 \leq j \leq n_i \text{ or } \\
2 & \text{if } e = u_lv_{k,l} \text{ for } 1 \leq l \leq s, \\
3 & \text{if } e = v_{1,j}v_{k,l} \text{ for } 1 \leq j \leq n_1 \text{ and } s+1 \leq l \leq t, \\
3 & \text{otherwise.}
\end{cases}$$

Let $x$ and $y$ be two nonadjacent vertices of $G$. Then $x, y \in V_i$ for some $i$ with $1 \leq i \leq k$. Let $x = v_{i,p}$ and $y = v_{i,q}$, where $1 \leq p < q \leq n_i$. If $1 \leq i \leq k-1$, then $x, v_{i+1,p}, y$ is a rainbow $x - y$ path in $G$ whose edges are colored 1 and 3. Thus we may assume that $i = k$. If $1 \leq p < q \leq s$, then $x, u_p, y$ is a rainbow $x - y$ path in $G$ whose edges are colored 1 and 3. If $s+1 \leq p < q \leq t$, then $x, v_{1,1}, v_{2,1}, y$ is a rainbow $x - y$ path in $G$ whose edges are colored 2, 1 and 3, respectively. If $1 \leq p \leq s$ and $s+1 \leq q \leq t$, then $x, v_{1,1}, y$ is a rainbow $x - y$ path whose edges are colored 3 and 2. Thus $rc(G) \leq 3$. Therefore, as claimed, $rc(G) \leq \min\{\lceil \sqrt{t} \rceil, 3\}$.

Assume, to the contrary, that $rc(G) < \min\{\lceil \sqrt{t} \rceil, 3\} \leq 3$. Since $rc(G) \geq 2$, it follows that $rc(G) = 2$. Let $c'$ be a rainbow 2-coloring of $G$. Thus, we can associate a color code $\text{code}(w) = (a_1, a_2, \ldots, a_s)$ to each vertex $w \in W$, where $a_i = c(u_iw) \in \{1, 2\}$ for $1 \leq i \leq s$. Since $\sqrt{t} > 2$, it follows that $t > 2^s$ and so there exist two distinct vertices $w'$ and $w''$ of $W$ such that $\text{code}(w') = \text{code}(w'')$. Hence the two edges of each $w' - w''$ path of length 2 are colored the same and so there is no rainbow $w' - w''$ path in $K_{s,t}$, producing a contradiction. Thus, as claimed, $rc(K_{s,t}) = 3 = \min\{\lceil \sqrt{t} \rceil, 3\}$ in this case. \hfill \Box

### 3. ON RAINBOW CONNECTION NUMBERS WITH PRESCRIBED VALUES

We have seen that $rc(G) \leq src(G)$ for every nontrivial connected graph $G$. By Proposition 1.1, it follows that for every positive integer $a$ and for every tree $T$ of size $a$, $rc(T) = src(T) = a$. Furthermore, for $a \in \{1, 2\}$, $rc(G) = a$ if and only if $src(G) = a$. If $a = 3$ and $b \geq 4$, then by Propositions 2.2 and 2.3, $rc(W_{3b}) = 3$ and $src(W_{3b}) = b$. For $a \geq 4$, we have the following.
**Theorem 3.1.** Let $a$ and $b$ be integers with $a \geq 4$ and $b \geq (5a - 6)/3$. Then there exists a connected graph $G$ such that $rc(G) = a$ and $src(G) = b$.

**Proof.** Let $n = 3b - 3a + 6$ and let $W_n$ be the wheel consisting of an $n$-cycle $C_n: v_1, v_2, \ldots, v_n, v_1$ and another vertex $v$ joined to every vertex of $C_n$. Let $G$ be the graph constructed from $W_n$ and the path $P_{a-1}: u_1, u_2, \ldots, u_{a-1}$ of order $a - 1$ by identifying $v$ and $u_{a-1}$.

First, we show that $rc(G) = a$. Since $b \geq (5a - 6)/3$ and $a \geq 4$, it follows that $b > a$ and so $n = 3b - 3a + 6 \geq 7$. By Proposition 2.2, we then have $rc(W_n) = 3$. Define a coloring $c$ of the graph $G$ by

$$c(e) = \begin{cases} 
  i & \text{if } e = u_iu_{i+1} \text{ for } 1 \leq i \leq a - 2, \\
  a & \text{if } e = v_iv \text{ and } i \text{ is odd,} \\
  a - 1 & \text{if } e = v_iw \text{ and } i \text{ is even,} \\
  1 & \text{otherwise.}
\end{cases}$$

Since $c$ is a rainbow $a$-coloring of the edges of $G$, it follows that $rc(G) \leq a$.

It remains to show that $rc(G) \geq a$. Assume, to the contrary, that $rc(G) \leq a - 1$. Let $c'$ be a rainbow $(a - 1)$-coloring of $G$. Since the path $u_1, u_2, \ldots, u_{a-1}$ is the only $u_1 - u_{a-1}$ path in $G$, the edges of this path must be colored differently by $c'$. We may assume, without loss of generality, that $c'(u_iu_{i+1}) = i$ for $1 \leq i \leq a - 2$. For each $j$ with $1 \leq j \leq 3b - 3a + 6$, there is a unique $u_1 - v_j$ path of length $a - 1$ in $G$ and so $c'(vv_j) = a - 1$ for $1 \leq j \leq 3b - 3a + 6$. Consider the vertices $v_1$ and $v_{a+1}$. Since $b \geq (5a - 6)/3$, any $v_1 - v_{a+1}$ path of length $a - 1$ or less must contain $v$ and thus two edges colored $a - 1$, contradicting our assumption that $c'$ is a rainbow $(a - 1)$-coloring of $G$. This implies that $rc(G) \geq a$ and so $rc(G) = a$.

Next, we show that $src(G) = b$. Since $n = 3b - 3a + 6 = 3(b - a + 2) \geq 7$, it follows by Proposition 2.3 that $src(W_n) = b - a + 2$. Let $c_1$ be a strong rainbow $(b - a + 2)$-coloring of $W_n$. Define a coloring $c$ of the graph $G$ by

$$c(e) = \begin{cases} 
  c_1(e) & \text{if } e \in E(W_n), \\
  b - a + 2 + i & \text{if } e = u_iu_{i+1} \text{ for } 1 \leq i \leq a - 2.
\end{cases}$$

Since $c$ is a strong rainbow $b$-coloring of $G$, it follows that $src(G) \leq b$.

It remains to show that $src(G) \geq b$. Assume, to the contrary, that $src(G) \leq b - 1$. Let $c^*$ be a strong rainbow $(b - 1)$-coloring of $G$. We may assume, without loss of generality, that $c^*(u_iu_{i+1}) = i$ for $1 \leq i \leq a - 2$. For each $j$ with $1 \leq j \leq 3b - 3a + 6$, there is a unique $u_1 - v_j$ geodesic in $G$, implying $c^*(vv_j) \in C = \{a - 1, a, \ldots, b - 1\}$. Let $S = \{vv_j: 1 \leq j \leq 3b - 3a + 6\}$. Then $|S| = 3b - 3a + 6$ and $|C| = b - a + 1$. Since at most three edges in $S$ can be colored the same, the $b - a + 1$ colors in $C$ can
color at most $3(b - a + 1) = 3b - 3a + 3$ edges, producing a contradiction. Therefore, $\text{src}(G) \geq b$ and so $\text{src}(G) = b$. \hfill \Box$

Combining Propositions 1.1, 2.2, 2.3 and Theorem 3.1, we have the following.

**Corollary 3.2.** Let $a$ and $b$ be positive integers. If $a = b$ or $3 \leq a < b$ and $b \geq (5a - 6)/3$, then there exists a connected graph $G$ such that $\text{rc}(G) = a$ and $\text{src}(G) = b$.

We conclude with two conjectures and a result.

**Conjecture 3.3.** Let $a$ and $b$ be positive integers. Then there exists a connected graph $G$ such that $\text{rc}(G) = a$ and $\text{src}(G) = b$ if and only if $a = b \in \{1, 2\}$ or $3 \leq a \leq b$.

It is easy to see that if $H$ is a connected spanning subgraph of a nontrivial (connected) graph $G$, then $\text{rc}(G) \leq \text{rc}(H)$. We have already noted that if, in addition, $\text{diam}(H) = 2$, then $\text{src}(G) \leq \text{src}(H)$. However, the question arises as to whether this is true when $\text{diam}(H) \geq 3$.

**Conjecture 3.4.** If $H$ is a connected spanning subgraph of a nontrivial (connected) graph $G$, then $\text{src}(G) \leq \text{src}(H)$.

If Conjecture 3.4 is true, then for every nontrivial connected graph $G$ of order $n$,

$$\text{diam}(G) \leq \text{rc}(G) \leq \text{src}(G) \leq n - 1.$$

The following can be proved immediately.

**Proposition 3.5.** For each triple $d, k, n$ of integers with $2 \leq d \leq k \leq n - 1$, there exists a connected graph $G$ of order $n$ with $\text{diam}(G) = d$ such that $\text{rc}(G) = \text{src}(G) = k$.

**References**


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