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INDUCED-PAIRED DOMATIC NUMBERS OF GRAPHS

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Abstract. A subset $D$ of the vertex set $V(G)$ of a graph $G$ is called dominating in $G$, if each vertex of $G$ either is in $D$, or is adjacent to a vertex of $D$. If moreover the subgraph $\langle D \rangle$ of $G$ induced by $D$ is regular of degree 1, then $D$ is called an induced-paired dominating set in $G$. A partition of $V(G)$, each of whose classes is an induced-paired dominating set in $G$, is called an induced-paired domatic partition of $G$. The maximum number of classes of an induced-paired domatic partition of $G$ is the induced-paired domatic number $d_{ip}(G)$ of $G$. This paper studies its properties.

Keywords: dominating set, induced-paired dominating set, induced-paired domatic number

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A subset $D$ of the vertex set $V(G)$ of a graph $G$ is called dominating in $G$, if each vertex of $G$ either is in $D$, or is adjacent to a vertex of $D$. The minimum number of vertices of a dominating set in $G$ is the domination number $\gamma(G)$ of $G$. The maximum number of classes of a partition of $V(G)$, all of whose classes are dominating sets in $G$, is the domatic number $d(G)$ of $G$. This concept was introduced by F. J. Cockayne and S. T. Hedetniemi in [1].

A variant of $\gamma(G)$ was introduced in [3] by D. J. Studer, T. W. Haynes and L. M. Lawson. If a dominating set $D$ in $G$ has the property that the subgraph $\langle D \rangle$ of $G$ induced by $D$ is regular of degree 1, then $D$ is called an induced-paired dominating set in $G$. The minimum number of vertices of an induced-paired dominating set in $G$ is the induced-paired domination number $\gamma_{ip}(G)$ of $G$.

Analogously as to $\gamma(G)$ the domatic number $d(G)$ was introduced, to $\gamma_{ip}(G)$ we introduce the induced-paired domatic number $d_{ip}(G)$. A partition of $V(G)$ is called induced-paired domatic, if all of its classes are induced-paired dominating sets (shortly IPDS) of $G$. The maximum number of classes of an induced-paired domatic
partition of $G$ is the induced-paired domatic number $d_{ip}(G)$ of $G$. Let us recall yet another numerical invariant of a graph which will be useful for our considerations. A dominating set in $G$ which is simultaneously independent (i.e. consisting of pairwise non-adjacent vertices) is an independent dominating set in $G$. The maximum number of classes of a partition of $V(G)$, all of whose classes are independent dominating sets in $G$, is called the independent domatic number (or shortly idomatic number) $d_i(G)$ of $G$ [4].

The numbers $d_i(G)$ and $d_{ip}(G)$ have the property that they are not well-defined for all graphs. Namely, there are graphs whose vertex sets cannot be partitioned into independent dominating sets or into induced-paired dominating sets.

**Proposition 1.** Let $G$ be a graph in which there is at least one independent domatic partition. Then $G \times K_2$ has at least one induced-paired domatic partition and

$$d_{ip}(G \times K_2) \geq d_i(G).$$

**Proof.** The graph $G \times K_2$ consists of two vertex-disjoint copies of $G$ and of edges joining the corresponding vertices in both the copies. Let $D$ be an independent dominating set in one copy of $G$.

Let $D'$ be the set consisting of all vertices of $D$ and of all vertices of the other copy of $G$ which are adjacent to vertices of $D$ in $G \times K_2$. Then evidently $D'$ is an induced-paired dominating set of $G \times K_2$. If some sets $D$ form an independent domatic partition of the chosen copy of $G$, then the sets $D'$ form an induced-paired domatic partition of $G \times K_2$ with the same number of classes. \hfill \Box

**Corollary 1.** Let $G$ be a connected bipartite graph. Then $G \times K_2$ has at least one induced-paired domatic partition and $d_{ip}(G) \geq 2$.

A complete $k$-partite graph for an integer $k \geq 2$ is a graph whose vertex set is the disjoint union of $k$ independent sets $V_1, \ldots, V_k$ and in which two-vertices are adjacent if and only if they belong to the sets $V_i, V_j$ with $i \neq j$.

**Corollary 2.** Let $G$ be a complete bipartite graph. Then $d_{ip}(G \times K_2) = k$.

**Proof.** Any IPDS in $G \times K_2$ is a set consisting of vertices of $V_i$ for some $i$ in one copy of $G$ and of vertices which are adjacent to them in the other copy. A proper subset of such a set is not dominating. Any two edges of $G$ have either a common end vertex, or an edge which has common end vertices with both of them. \hfill \Box

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**Proposition 2.** Let $n$ be an even positive integer. For the complete graph $K_n$ with $n$ vertices we have $d_{ip}(K_n) = \frac{1}{2}n$.

**Proof.** We choose a linear factor in $K_n$. Each of its edges forms a one-element IPDS and this implies the result. □

**Proposition 3.** Let $K_{m,n}$ be a complete bipartite graph. An induced-paired domatic partition of $K_{m,n}$ exists if and only if $m = n$, and then $d_{ip}(K_{n,n}) = n$.

**Proof.** The first part is evident, the second part may be proved in the same way as Proposition 2. □

**Proposition 1.** Let $C_n$ be the circuit of length $n$. An induced paired domatic partition of $C_n$ exists if and only if $n$ is divisible by 4, and then $d_{ip}(C_n) = 2$.

**Proof.** Let the edges going around $C_n$ be $e_1, e_2, \ldots, e_n$. Evidently, if $n$ is not divisible by 4, an induced-paired domatic partition does not exist. If $n$ is divisible by 4, then let $E_1 = \{e_i; \ i \equiv 0 \pmod{4}\}$, $E_1 = \{e_i; \ i \equiv 0 \pmod{4}\}$. Let $D_1$ (or $D_2$) be the set of all end vertices of edges of $E_1$ (or of $E_2$ respectively). Then $\{D_1, D_2\}$ is an induced-paired domatic partition of $C$ and $d_{ip}(C_n) = 2$. □

Now we state two general assertions.

**Proposition 5.** If there exists an induced-paired domatic partition of $G$, then $G$ has an even number of vertices.

Proof is straightforward.

**Proposition 6.** Let there exist $d_{ip}(G)$ for a graph $G$, let $\delta(G)$ be the minimum degree of a vertex of $G$. Then $d_{ip}(G) \leq \delta(G)$.

**Proof.** Each vertex $v$ of $G$ must be adjacent to at least one vertex of each class of an induced-paired domatic partition to which $v$ does not belong. Moreover, it is incident with an edge of the subgraph of $G$ induced by the class to which $v$ belongs. Hence the degree of $v$ is at least $d_{ip}(G)$. □

Now we will study graphs $G$ with $d_{ip}(G) = 2$.

**Theorem 1.** Let $G$ be a graph with $n$ vertices such that $d_{ip}(G) = 2$. Then

$$n \leq |E(G)| \leq \frac{1}{4}n^2 - 1.$$ 

Both the bounds are attained.

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Proof. Let \( \{D_1, D_2\} \) be an induced-paired domatic partition of \( G \). Let \( |D_1| = a, |D_2| = b, a \leq b \). Therefore \( a + b = n, b \geq \frac{1}{2}n \). The subgraph of \( G \) induced by \( D_1 \) (or \( D_2 \)) has \( \frac{1}{2}a \) (or \( \frac{1}{2}b \) respectively) edges. As \( a \leq b \), there exists at least \( b \) edges joining vertices of \( D_1 \) with vertices of \( D_2 \). The number of edges of \( G \) is at least \( b + \frac{1}{2}a + \frac{1}{2}b = b + \frac{1}{2}n \). This expression has its minimum value for \( b = \frac{1}{2}n \). Then \( G \) has \( n \) edges, in the other cases they are at least \( n \).

Now suppose that \( G \) contains all edges which join vertices of \( D_1 \) with vertices of \( D_2 \). We must exclude the possibility \( a = b \); otherwise \( G \) would contain a factor isomorphic to \( K_{a,a} \) and the inequality \( d_{ip}(G) \geq a \) would hold. Again the subgraph of \( G \) induced by \( D_1 \) (or \( D_2 \)) has \( \frac{1}{2}a \) (or \( \frac{1}{2}b \) respectively) edges. The number of other edges is \( ab \). The number of edges of \( G \) is \( ab + \frac{1}{2}a + \frac{1}{2}b = ab + \frac{1}{2}n \). The maximum value of this expression is for \( a = b \); but we have excluded this case. The maximum in the other cases occurs for \( a = \frac{1}{2}n - 1, b = \frac{1}{2}n + 1 \) and it is \( \frac{1}{4}n^2 - 1 \). □

Return to Theorem 1. Evidently a graph \( G \) having the minimum number of edges at \( d_{ip}(G) = 2 \) is a regular graph of degree 2, i.e. a graph all of whose connected components are circuits.

According to Proposition 4 these circuits have lengths divisible by 4. In a graph \( G \) with \( d_{ip}(G) > 2 \) this holds for the subgraph of \( G \) induced by the union of two classes of the induced-paired domatic partition. This implies the following proposition.

**Proposition 7.** Let \( G \) be a graph with the minimum number of edges at a given \( d_{ip}(G) \geq 3 \). Then \( G \) is the union of circuits of lengths divisible by 4. The edges of each circuit may be coloured alternatingly in red and blue in such a way that each red edge is contained in \( d_{ip}(G) - 1 \) circuits, while each blue edge is contained in only one of them. Each vertex is incident with one red edge and \( d_{ip}(G) - 1 \) blue edges.

It is evident that red edges are exactly the edges of the subgraph of \( G \) induced by classes of the induced-paired domatic partition, while blue edges are the others.

**Theorem 2.** Let \( G \) be a graph with \( n \) vertices such that \( d_{ip}(G) = \frac{1}{2}n \). Then

\[
\frac{1}{4}n^2 \leq |E(G)| \leq \frac{1}{2}n(n - 1).
\]

Both the bounds are attained.

**Proof.** Let \( D \) be an induced-paired domatic partition of \( G \). Each vertex of \( G \) must be adjacent to vertices of all classes of \( D \) and have degree at least \( \frac{1}{2}n \). Hence \( |E(G)| \geq \frac{1}{4}n^2 \). The equality \( |E(G)| = \frac{1}{4}n^2 \) is attained in the case when \( G \cong K_{n/2} \times K_2 \). The upper bound follows from the fact that \( \frac{1}{2}n \) \((n - 1)\) is the number of edges of \( K_n \). And, as we have seen in Proposition 2, \( d_{ip}(K_n) = \frac{1}{2}n \). □
Obviously \( d_{ip}(G) \leq \frac{1}{2}n \), whenever \( d_{ip}(G) \) exists; this follows from the definition. Also the following proposition is evident.

**Proposition 8.** Let \( G \) be a graph. The equality \( d_{ip}(G) = 1 \) holds if and only if \( G \) is regular of degree 1.

Now we shall treat interconnections among graphs and interconnections among IPDS.

Note that in some cases there is no analogy between \( d_{ip}(G) \) and \( d(G) \). If \( D \subseteq S \subseteq V(G) \), where \( D \) is an IPDS, then \( S \) need not be an IPDS.

**Proposition 9.** Let \( G \) be the disjoint union of two graphs \( G_1, G_2 \). A subset \( S \subseteq V(G) \) is an IPDS in \( G \) if and only if \( S = S_1 \cup S_2 \), where \( S_1 \) is an IPDS in \( G_1 \) and \( S_2 \) is an IPDS in \( G_2 \).

Proof is easy.

**Corollary 3.** Let \( G \) be the disjoint union of two graphs \( G_1, G_2 \). If \( d_{ip}(G_1) = d_{ip}(G_2) \), then also \( d_{ip}(G) = d_{ip}(G_1) = d_{ip}(G_2) \).

If \( G_1, G_2 \) are two vertex-disjoint graphs, then the Zykov sum \( G_1 \oplus G_2 \) of \( G_1 \) and \( G_2 \) is the graph obtained from \( G_1 \) and \( G_2 \) by \( G \) adding all edges which join a vertex of \( G_1 \) with a vertex of \( G_2 \).

**Theorem 4.** Let \( G \) be the Zykov sum \( G_1 \oplus G_2 \). A subset \( S \subseteq V(G) \) is an IPDS of \( G \) if and only if it is an IPDS in \( G_1 \) or an IPDS in \( G_2 \).

Proof is again easy.

**Corollary 4.** Let there exist numbers \( d_{ip}(G_1) \) and \( d_{ip}(G_2) \) for graphs \( G_1, G_2 \). Then \( d_{ip}(G_1 \oplus G_2) = d_{ip}(G_1) + d_{ip}(G_2) \).

It is easy to compare \( d_{ip}(G) \) with the domatic number \( d(G) \) and with the total domatic number \( d_t(G) \) (see e.g. [2]). Every IPDS of \( G \) is also a total dominating set in \( G \) and every total dominating set in \( G \) is a dominating set in \( G \).

**Proposition 10.** Let \( G \) be a graph for which \( d_{ip}(G) \) is defined. Then \( d(G) \leq d_t(G) \leq d_{ip}(G) \).

Now we compare \( d_{ip}(G) \) with the chromatic number \( \chi(G) \) of \( G \).

**Proposition 11.** Let \( G \) be a graph for which \( d_{ip}(G) \) is defined. Then \( \chi(G) \leq 2d_{ip}(G) \).
Proof. Let $s = d_{ip}(G)$ and let $\{D_1, \ldots, D_s\}$ be an induced-paired domatic partition of $G$. Let us have $2s$ colours $c_1^1, c_2^1, \ldots, c_s^1, c_s^2$. We colour vertices of $G$ by these colours. The vertices of each $D_i$ for $i = 1, \ldots, s$ will be coloured by $c_i^1$ and $c_i^2$; obviously adjacent vertices are coloured by different colours. The colouring thus obtained is an admissible colouring of $G$ and this yields the assertion. □

References


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