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THE 3-PATH-STEP OPERATOR ON TREES
AND UNICYCLIC GRAPHS

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Abstract. E. Prisner in his book Graph Dynamics defines the $k$-path-step operator on the class of finite graphs. The $k$-path-step operator (for a positive integer $k$) is the operator $S'_k$ which to every finite graph $G$ assigns the graph $S'_k(G)$ which has the same vertex set as $G$ and in which two vertices are adjacent if and only if there exists a path of length $k$ in $G$ connecting them. In the paper the trees and the unicyclic graphs fixed in the operator $S'_3$ are studied.

Keywords: 3-path-step graph operator, tree, unicyclic graph

MSC 2000: 05C38, 05C05

In [2], page 168, the $k$-path-step graph operator $S'_k$ for a positive integer $k$ is defined. Let $G$ be a finite graph. The graph $S'_k(G)$ has the same vertex set as $G$ and two vertices are adjacent in it if and only if there exists a path of length $k$ in $G$ connecting them.

Further, in [2] the abstract of [1] is quoted; it is said that the paper [1] never appeared. In the abstract it was claimed that the finite connected graphs which are periodic in $S'_3$ are just the complete graphs, the complete bipartite graphs, the circuits of lengths not divisible by 3, the graphs of one more infinite family and four exceptional graphs (they were not specified). But in [2] some further graphs were presented which are fixed in $S'_3$. Among them there is a family of trees in Fig. 1. The symbols $p$, $q$ signify that the number of vertices most to the left in the figure $p \geq 0, q \geq 0$

Fig. 1
is \( p \) and the number of vertices most to the right is \( q \); these numbers \( p, q \) may be arbitrary non-negative integers (including zero). Therefore Fig. 1 represents a whole family of trees. Among the graphs mentioned there is a similar family of unicyclic graphs and two other unicyclic graphs. They are shown in Fig. 2.

![Fig. 2](image)

In this paper we shall prove that the graphs in Fig. 1 and Fig. 2 are all trees and all unicyclic graphs which are fixed in the operator \( S'_3 \), i.e. graphs \( G \) such that \( S'_3(G) \cong G \). We start with trees. All graphs considered are without loops and multiple edges.

**Lemma 1.** Let \( T \) be a tree such that \( S'_3(T) \cong T \). Then \( T \) contains no subtree isomorphic to any one of the trees \( F_1, F_2, F_3 \) in Fig. 3.

![Fig. 3](image)

**Proof.** If \( T \) contains a subtree isomorphic to \( F_1 \), to \( F_2 \) or to \( F_3 \), then the image \( S'_3(T) \) contains a subgraph isomorphic to \( S'_3(F_1) \), to \( S'_3(F_2) \), or to \( S'_3(F_3) \). These graphs are in Fig. 4. Evidently each of them contains a circuit: therefore \( S'_3(T) \) is then not a tree. \( \square \)

![Fig. 4](image)
The absence of a subtree isomorphic to $F_1$ implies that a tree fixed in $S'_3$ must be a caterpillar. Indeed, caterpillars are characterized among trees by this property. A caterpillar is defined as a tree with the property that by deleting all pendant edges and vertices from it a path is obtained. (A pendant vertex of a tree is its vertex of degree 1, a pendant edge is an edge incident with a pendant vertex.)

Thus, let us have a tree $T$ fixed in $S'_3$. We describe it as a caterpillar. Let the diameter of $T$ be $d$. Let $D$ be a diametral path of $T$. Denote its vertices by $u_0, u_1, \ldots, u_d$ so that the edges of $D$ are $u_i u_{i+1}$ for $i = 0, 1, \ldots, d - 1$. The number of vertices adjacent to $u_1$ (or $u_{d-1}$) and not belonging to $D$ will be denoted by $p - 1$ (or $q - 1$, respectively). We admit that these numbers may be zero. Further, by $k$ we denote the number of vertices $S$ which are adjacent to the vertices $u_i$ for $2 \leq i \leq d - 2$.

**Lemma 2.** Let $k$ be the above defined number. Then $k = 2$.

**Proof.** The number of edges of $T$ is $d + p + q + k - 2$. As $S'_3(T) \cong T$, so must be the number of edges of $S'_3(T)$, i.e. the number of pairs of vertices whose distance in $T$ is 3. On $D$ there are $d - 2$ such pairs, namely the pairs $\{u_i, u_{i+3}\}$ for $i = 0, \ldots, d - 3$. If a vertex is adjacent to $u_1$ or to $u_{d-1}$ and does not belong to $D$, then there exists exactly one vertex at the distance 3 from it, namely $u_3$ or $u_{d-3}$. If a vertex is adjacent to $u_i$ for $2 \leq i \leq d - 2$ then there are exactly two vertices at the distance 3 from it, namely $u_{i-2}$ and $u_{i+2}$ (we must suppose the absence of $F_3$). Hence the number of edges of $S'_3(T)$ is $d + p + q + 2k - 4$. It is equal to the number of edges of $T$ if and only if $k = 2$. \hfill \Box

Thus we may suppose that there exist integers $r, s$ such that $2 \leq r < r + 2 \leq s \leq d - 2$ and there exists a vertex $v_r$ adjacent to $u_r$ and a vertex $v_s$ adjacent to $u_5$ which do not belong to $D$. Note that the case $r = 2$ is possible only if $p = 1$ and the case $s = d - 2$ is possible only if $q = 1$; otherwise a subtree isomorphic to $F_3$ would occur.

**Lemma 3.** Let $T$ be a caterpillar, let $\text{diam } T \leq 6$. Then $T$ is not fixed in the operator $S'_3$.

**Proof.** If $\text{diam } T \leq 2$, then there is no path of length 3 in $T$. If $3 \leq \text{diam } T \leq 5$, then the above mentioned numbers $r, s$ do not exist. If $\text{diam } T = 6$, then the unique possibility is $r = 2$, $s = 4$ but then $S'_3(T)$ is a path. \hfill \Box

Therefore in the sequel we will suppose $d = \text{diam } T \geq 7$. The image $S'_3(D)$ consists of three connected components which are paths $D_0, D_1, D_2$; for $j \in \{0, 1, 2\}$ we denote by $D_j$ the path having the vertices $u_i$ with $i \equiv j \pmod{3}$. In $S'_3(T)$ there are two paths $R, S$ among these components of length 2 with $v_r$ and $v_s$ as inner vertices. One of the paths $D_0, D_1, D_2$ must have the property that both its terminal
vertices are terminal vertices of the paths $R$ and $S$; denote this property as $\mathcal{V}$. We shall treat the possible cases. If $i, j$ are from $\{0, 1, 2\}$, then $C(i, j)$ denotes the case when $d \equiv i \pmod{3}$ and $D_j$ has the property $\mathcal{V}$.

Case $C(0, 0)$. The path $R$ connects $u_0$ with $u_4$, the path $S$ connects $u_d$ with $u_{d-4}$. Hence $r = 2$, $s = d - 2$. The images of pendant vertices $v_r, v_s$ in an isomorphism of $T$ onto $S'_3(T)$ are again pendant vertices $u_1$ and $u_{d-1}$ and the images of $u_r, u_5$ are $u_4$ and $u_{d-4}$. The images of $u_0$ and $u_d$ are again $u_0$ and $u_d$. The distance $d ↓ 11$; it is easy to try the corresponding trees and to recognize that they are not fixed in $S'_3$.

Case $C(0, 1)$. If $d \geq 12$, then there exists a subtree of $S'_3(T)$ isomorphic to $F_1$. It consists of three paths of length 2 with the common terminal $u_{d-6}$; the first has the edges $u_{d-6}u_{d-9}, u_{d-4}u_{d-6}$, the second $u_d u_{d-3}, u_{d-3} u_{d-3}$, the third $u_{d-2} v_{d-4}, v_{d-4} u_{d-6}$. This is a contradiction. The unique case for $d < 12$ is $d = 9$; it is easy to try the corresponding tree and to recognize that it is not fixed in $S'_3$.

Case $C(0, 2)$ may be transferred to $C(0, 1)$ by changing the notation $u_i$ to $u_{d-i}$ for each $i$.

Case $C(1, 0)$. The path $R$ connects $u_0$ with $u_4$, the path $S$ connects $u_{d-1}$ with $u_{d-5}$. Hence $r = 2$, $s = d - 3$. The images of $v_r, v_s$ are $u_1$ and $u_{d-2}$ and the images of $u_r, u_s$ are $u_4, u_{d-5}$. The images of $u_0$ and $u_d$ are $u_2$ and $u_{d}$. The distance between $u_4$ and $u_d$ is $\frac{1}{3}(d - 1) - 1$, the distance between $u_2$ and $u_{d-5}$ is $\frac{1}{3}(d - 1) - 2$. If $S'_3(T) \cong T$, then one of these distances must be equal to $r$ and the other to $d - s$. This is possible only for $d = 13$.

Case $C(1, 1)$ may be transferred to $C(1, 0)$ by changing the notation $u_i$ to $u_{d-i}$ for each $i$.

Case $C(1, 2)$. If $d \geq 13$, then there exists a subtree of $S'_3(T)$ isomorphic to $F_1$. It consists of three paths of length 2 with a common terminal vertex $u_6$; the first has the edges $u_0 u_3, u_3 u_6$, the second $u_2 u_5, u_9 u_6$, the third $u_2 v_4, v_4 u_6$. This is a contradiction. The unique cases for $d < 13$ are $d = 7$ and $d = 10$; it is easy to try the corresponding trees and to recognize that they are not fixed in $S'_3$.

Case $C(2, 0)$. If $d \geq 14$, then there exists a subtree of $S'_3(T)$ isomorphic to $F_1$. It consists of three paths of length 2 with a common terminal vertex $u_{d-6}$; the first has the edges $u_{d-6} u_{d-9}, u_{d-6} u_{d-6}$, the second $u_d u_{d-3}, u_{d-3} u_{d-6}$, the third $u_{d-2} v_{d-4}, v_{d-4} u_{d-6}$. This is a contradiction. The unique cases for $d < 14$ are $d = 8$ and $d = 11$; it is easy to try the corresponding trees and to recognize that they are not fixed in $S'_3$.

Case $C(2, 1)$. The path $R$ connects $u_1$ with $u_5$, the path $S$ connects $u_{d-1}$ with $u_{d-5}$. Hence $r = 3$, $s = d - 3$. The images of $v_r, v_s$ are $u_2$ and $u_{d-2}$ and the images of $u_r, u_5$ are $u_5, u_{d-5}$. The images of $u_0$ and $u_d$ are again $u_0$ and $u_d$. The distance
between $u_5$ and $u_d$ is $\frac{1}{3}(d - 2) - 1$ and the distance between $u_0$ and $u_{d-5}$ is also $\frac{1}{3}(d - 2) - 1$. If $S'_1(T) \cong T$, then $r = d - s - \frac{1}{3}(d - 2) - 1$ and $d = 14$.

Case $C(2, 2)$ may be transferred to $C(2, 0)$ by changing the notation $u_i$ to $u_{d-i}$ for each $i$.

By these considerations we have proved the following lemma.

**Lemma 4.** Let $T$ be a tree such that $S'_1(T) \cong T$. Then $T$ is a caterpillar, $12 \leq \text{diam} T \leq 14$ and in $T$ there exist exactly two vertices of degree 3 with the distance from both the terminal vertices of a diametral path greater than or equal to 2.

From our lemmas and from the considerations which precede Lemma 4 we obtain a theorem.

**Theorem 1.** Let $T$ be a finite tree such that $S'_3(T) \cong T$. Then $T$ belongs to the family of trees depicted in Fig. 1.

The family from Fig. 1 is again depicted in Fig. 5 (diameter 12), Fig. 6 (diameter 13) and Fig. 7 (diameter 14). For $d = 12$ there is only one tree; to $u_1$ and $u_{11}$ no vertices not belonging to $D$ may be adjacent, because then a subtree isomorphic to $F_3$ would occur. For $d = 13$ it is possible for only one of the vertices $u_1, u_{12}$: In Fig. 6 it is $u_{12}$. The second possibility would be a mirror image of the former. For $d = 14$ vertices not belonging to $D$ may be adjacent to both $u_1$ and $u_{13}$.

Now we turn to unicyclic graphs.
**Theorem 2.** Let $G$ be a finite unicyclic graph such that $S'_3(G) \cong G$. Then either $G$ is a circuit of length not divisible by 3, or it is some of the graphs depicted in Fig. 2.

**Proof.** Let $G$ be an acyclic graph, let $\ell(G)$ be the length of its circuit. If $\ell(G) = 3$ and $G$ is not isomorphic to the graph of $\ell(G) = 3$ from Fig. 2, then either it is isomorphic to a subgraph of $H_1$, or contains a subgraph isomorphic to $H_2$, $H_3$ or $H_4$ in Fig. 8. The images of those graphs are in Fig. 9. In the first case $S'_3(G)$ is isomorphic to a subgraph of $S'_3(H_1)$ and it is a forest, in the second case $S'_3(G)$ has a subgraph isomorphic to $S'_3(H_2)$, $S'_3(H_3)$ or $S'_3(H_4)$ and thus it has a circuit of length 4 or of length 6, therefore it is not isomorphic to $G$. If $\ell(G) = 4$ and $G$ is not isomorphic to any graph of the family of graphs with $\ell(G) = 4$ depicted in Fig. 2, then it has a subgraph isomorphic to $H_5$ or $H_6$ in Fig. 10. The graph $S'_3(G)$ has then a subgraph isomorphic to $S'_3(H_5)$ or to $S'_3(H_6)$ in Fig. 11; in both the cases

![Diagram of graphs](image-url)
it contains two circuits of length 4 and cannot be isomorphic to $G$. If $\ell(G) = 6$ and $G$ is not isomorphic to the graph with this $\ell(G)$ in Fig. 2, then it is either a circuit of length 6, or it is isomorphic to $H_1$, $H_8$ or $H_9$ in Fig. 12. In the first case $S'_3(H)$ consists of three connected components being complete graphs with two vertices, in the second case it contains a subgraph isomorphic to $S'_3(H_1)$, $S'_3(H_8)$ or $S'_3(H_9)$ in Fig. 13 and thus it is not isomorphic to $G$. Finally, let $\ell(G) = 5$ or $\ell(G) \geq 7$. If $\ell(G)$ is divisible by 3, then $S'_3(G)$ contains three circuits of length $\frac{1}{3}\ell(G)$ and is not isomorphic to $G$. Thus suppose that $\ell(G)$ is not divisible by 3. If $G$ is not only a circuit, then $G$ contains a vertex not belonging to its circuit, but adjacent to one of its vertices. Such a vertex has distance 3 from the vertices of the circuit. The graph $S'_3(G)$ contains a circuit of length $\ell(G)$ and a vertex adjacent to two vertices of that circuit (in Fig. 14 for $\ell(G) = 5$), and thus it is not isomorphic to $G$.
References


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