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EXISTENCE RESULTS FOR A CLASS OF SEMILINEAR DEGENERATE ELLIPTIC EQUATIONS

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Abstract. We prove existence results for the Dirichlet problem associated with an elliptic semilinear second-order equation of divergence form. Degeneracy in the ellipticity condition is allowed.

Keywords: weak subsolution, degenerate equation, critical point, fixed-point theorems

MSC 2000: 35A05, 35J70, 47H10

1. Introduction

We consider the semilinear boundary value problem

\[
\begin{align*}
- \sum_{i,j=1}^{m} \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial u}{\partial x_j}) &= f(u) \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega
\end{align*}
\]

where $\Omega$ is a bounded open subset of $\mathbb{R}^m$, $f$ is a real valued function defined on $\mathbb{R}$, and the coefficients $a_{i,j}(x)$ satisfy the ellipticity condition

\[
\sum_{i,j=1}^{m} a_{ij}(x)p_ip_j \geq \alpha \sum_{i=1}^{m} \nu_i(x)p_i^2 \quad \text{for a.e. } x \in \Omega \text{ and any } p \in \mathbb{R}^m
\]

with $\nu_i(x)$ satisfying sufficiently general hypotheses.

We obtain some results of existence, uniqueness and boundedness for weak solutions of problem (1.0) with minimal hypotheses on $f$. Similar results, when $f$ has a natural polynomial growth, have been obtained in [3], [5], [7] and in [8] by pseudomonotone operators’ theory, while our proof uses fixed-point theorems. The paper
is structured as follows. In Sections 2 and 3 we state hypotheses and results. In Section 4 we establish some useful lemmas and, finally, in Section 5 we prove our main theorems.

2. Functional spaces

Let $\mathbb{R}^m$ be the Euclidean $m$-space with a generic point $x = (x_1, x_2, \ldots, x_m)$, $\Omega$ a bounded open subset of $\mathbb{R}^m$. The notation $\text{meas}_x$ will indicate the $m$-dimensional Lebesgue measure.

If $u(x)$ is a measurable function defined in $\Omega$, we will denote by $|u|_p$ ($1 \leq p \leq \infty$) the usual norm in the space $L^p(\Omega)$.

2.1. Let $\nu_i(x)$ ($i = 1, 2, \ldots, m$) be a positive and measurable function defined in $\Omega$ such that

$$\nu_i(x) \in L^1(\Omega), \quad \nu_i^{-1}(x) \in L^{g_i}(\Omega)$$

where $\sum_i \frac{1}{g_i} < 2$ ($g_i > 1$) if $m \geq 3$ ($m = 2$).

The symbol $H^1(\nu_i, \Omega)$ stands for the completion of $C^1(\overline{\Omega})$ with respect to the norm

$$\|u\|_1 = \left( \int_\Omega \left( |u|^2 + \sum_{i=1}^m \nu_i(x) \left| \frac{\partial u}{\partial x_i} \right|^2 \right) \, dx \right)^{\frac{1}{2}};$$

$H_0^1(\nu_i, \Omega)$ denotes the closure of $C^\infty_0(\Omega)$ in $H^1(\nu_i, \Omega)$.

Finally, $H^{-1}(\nu_i^{-1}, \Omega)$ denotes the dual space of $H_0^1(\nu_i, \Omega)$ (see also [5], [6] and [10] for details concerning the weighted Sobolev spaces).

3. Hypotheses, Problems and Results

Hypothesis 3.1. The coefficients $a_{ij}(x)$ ($i, j = 1, 2, \ldots, m$) are functions defined and measurable in $\Omega$ satisfying

$$a_{ij}(x) = a_{ji}(x),$$

$$\frac{a_{ij}(x)}{\sqrt{\nu_i(x)\nu_j(x)}} \in L^\infty(\Omega) \quad (i, j = 1, 2, \ldots, m).$$

Hypothesis 3.2. There exists $\alpha > 0$ such that for almost every $x$ in $\Omega$ we have

$$\sum_{i,j=1}^m a_{ij}(x)p_ip_j \geq \alpha \sum_{i=1}^m \nu_i(x)p_i^2 \quad \text{for any } p \in \mathbb{R}^m.$$
Let \( a: H^1_0(\nu_i, \Omega) \times H^1_0(\nu_i, \Omega) \rightarrow \mathbb{R} \) be such that

\[
a(u, v) = \int_{\Omega} \sum_{ij=1}^{m} a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} \, dx,
\]

and define

\[
\tau = \inf_{u \in H^1_0(\nu_i, \Omega) \setminus \{0\}} \frac{a(u, u)}{|u|_2}.
\]

In Section 4 we prove the following

**Lemma 4.4.** Let us assume that (2.1), (3.1), (3.2) hold. Then \( \tau > 0 \) and there exists \( u_0 \in H^1_0(\nu_i, \Omega) \) such that \( \tau = a(u_0, u_0) \) and

\[
a(u, u_0) = \tau \int_{\Omega} uu_0 \, dx \quad \text{for any } u \in H^1_0(\nu_i, \Omega);
\]

moreover, we can choose \( u_0 \geq 0 \).

**Definition 3.2.** Let \( H \) be a Hilbert space, \( f, g \in C^1(H, \mathbb{R}) \), and let

\[
E = \{ u \in H : g(u) = 0, \quad g'(u) \neq 0 \}.
\]

A point \( u_0 \in H \) is a critical point of \( f|_E \) if \( \frac{d}{dt} f(h(t))|_{t=0} = 0 \) for all \( C^1 \) paths \( h(t): ] - \varepsilon, \varepsilon[ \rightarrow E \) such that \( h(0) = u_0 \).

**Remark 3.3.** If there exists \( u_0 \in E \) such that \( f(u_0) = \min \{ f(u) : u \in E \} \), then \( (f|_E)'(u_0) = 0 \).

**Theorem 3.4** (see, e.g., [2]). A point \( u_0 \in E \) is a critical point of \( f|_E \) if and only if there exists \( \lambda \in \mathbb{R} \) such that \( f'(u_0) = \lambda g'(u_0) \).

Now, if \( f \in C(\mathbb{R}) \) satisfies the condition

\[
u \in H^1_0(\nu_i, \Omega) \Rightarrow f(u) \in H^{-1}(\nu_i, \Omega),
\]

we obtain the following well posed problem

**Problem.** Find a function \( u(x) \in H^1_0(\nu_i, \Omega) \) such that

\[
(3.1) \quad \int_{\Omega} \sum_{ij=1}^{m} a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} \, dx = (f(u), v) \quad (1)
\]

for any \( v(x) \in H^1_0(\nu_i, \Omega) \).

(1) We denote by \( (\cdot, \cdot) \) the duality pairing between \( H^1_0(\nu_i, \Omega) \) and \( H^{-1}(\nu_i, \Omega) \).
A function $u(x)$ satisfying (3.1) is a weak solution of Problem (1.0).

**Remark 3.5.** When $f$ does not depend on $u$, $f \in H^{-1}(\nu_i, \Omega)$, the hypotheses (2.1), (3.1), (3.2) are sufficient to ensure existence and uniqueness of a weak solution of problem (1.0), moreover we have

$$\|u\|_{1,0} \leq \|f\|_{H^{-1}(\nu_i, \Omega)}.$$  

Proof follows from Lemma 4.1 and the Lax-Milgram theorem (see Remark 4.2 for the definition of $\|u\|_{1,0}$).

In Section 5 we prove

**Theorem 5.1** (Existence, uniqueness and boundedness). Let us assume that (2.1), (3.1), (3.2) hold and let $f$ be Lipschitz continuous with a Lipschitz constant $L < \tau$.

Then there exists a unique weak solution $u(x)$ of problem (1.0); moreover, $u(x) \in L^\infty(\Omega)$ and

$$\|u\|_\infty \leq \gamma(L, g, m, \text{meas}_x \Omega).$$

**Theorem 5.2.** Let us assume that (2.1), (3.1), (3.2) hold and let $f$ be a bounded continuous function. Then Problem (1.0) has a weak solution $u(x)$. Moreover, $u(x) \in L^\infty(\Omega)$ and (5.0) holds.

4. Preliminary lemmas

**Lemma 4.1.** If the hypothesis (2.1) is satisfied then there exists a constant $C = C(m, g_i, |\nu_i^{-1}|_{g_i})$ such that

$$\|u\|_{2^*} \leq C \left( \int_\Omega \sum_{i=1}^m \nu_i(x) \left| \frac{\partial u}{\partial x_i} \right|^2 \, dx \right)^{\frac{1}{2}}$$

for all $u \in H^1_0(\nu_i, \Omega)$, where $2^* = 2m(m - 2 + \sum_{i=1}^m \frac{1}{g_i})^{-1}$.

Moreover, the imbedding of $H^1_0(\nu_i, \Omega)$ into $L^2(\Omega)$ is compact.
Proof. Let us fix \( m_i = \frac{2g_i}{g_i + 1} \). Then

\[
|\frac{\partial u}{\partial x_i}|_{m_i} \leq |\nu_i^{-1}|^{\frac{1}{g_i}} |\nu_i^{\frac{1}{m_i}} \frac{\partial u}{\partial x_i}|_2.
\]

Since \( \sum_{i=1}^{m} \frac{1}{m_i} = \frac{\sum_{i=1}^{m} \frac{g_i + 1}{2g_i}}{\frac{m}{g_i}} = \frac{1}{2} \left( m + \sum_{i=1}^{m} \frac{1}{g_i} \right) > 1 \), Sobolev’s imbedding theorem yields (see, for instance, [12])

\[
|u|_q \leq C(m, m_i, q) \prod_{i=1}^{m} |\frac{\partial u}{\partial x_i}|_{m_i}^{\frac{1}{m_i}}
\]

where \( q = m \left( -1 + \sum_{i=1}^{m} \frac{1}{m_i} \right)^{-1} \).

From (4.2) and (4.3) we obtain

\[
|u|_{2^*} \leq C \prod_{i=1}^{m} \left( |\nu_i^{-1}|^{\frac{1}{g_i}} |\nu_i^{\frac{1}{m_i}} \frac{\partial u}{\partial x_i}|_2 \right).
\]

Now, let \( \{u_n\} \) be a sequence of functions of \( H^1_0(\nu_i, \Omega) \) with equibounded norms and let \( \{\Pi_k\} \) be a sequence of open intervals in \( \Omega \) such that
1. \( \Pi_k \subset \Pi_{k+1} \) for any \( k \in \mathbb{N} \),
2. \( \lim_{k \to +\infty} \Pi_k = \Omega \),
3. for any closed, bounded subset \( C \) of \( \Omega \) there exists \( \bar{k} \): \( C \subset \Pi_k, k \geq \bar{k} \).

Let us denote by \( W^{1,1}(\Pi_1) \) the usual Sobolev space on the set \( \Pi_1 \).

It follows that the norms of \( \{u_n\} \) in \( W^{1,1}(\Pi_1) \) are equibounded; in fact, applying the Hölder inequality we obtain the following estimate:

\[
\|u_n\|_{W^{1,1}(\Pi_1)} = \int_{\Pi_1} |u_n| \, dx + \int_{\Pi_1} \sum_{i=1}^{m} |\frac{\partial u_n}{\partial x_i}| \, dx \\
\leq \left( \int_{\Pi_1} |u_n|^2 \, dx \right) (\text{meas} \Pi_1)^{\frac{1}{2}} + \sum_{i=1}^{m} \left( \int_{\Pi_1} \frac{1}{\nu_i(x)} \, dx \right)^{\frac{1}{2}} \|u_n\|_1 \\
\leq \text{const} \|u_n\|_1.
\]

Due to the compact imbedding of \( W^{1,1}(\Pi_1) \) into \( L^1(\Pi_1) \) (see e.g. [1]) there is a subsequence \( \{u_{1,n}\} \) from \( \{u_n\} \) that converges a.e. in \( \Pi_1 \).

The same procedure can be done on each \( \Pi_j \) for \( j = 2, 3, \ldots \). Hence we get a system of sequences \( \{u_{j,n}\} \), \( n, j = 1, 2, \ldots \) (where \( \{u_{j,n}\} \) is a subsequence of \( \{u_{j-1,n}\} \)) such that \( \{u_{j,n}\} \) is convergent a.e. in \( \Pi_j \) for \( j = 1, 2, \ldots \).

By the diagonals method we obtain that \( \{u_{n,n}\} \) converges a.e. in \( \Omega \) and, by virtue (4.1), in \( L^2(\Omega) \).
Remark 4.2. If the hypothesis (2.1) holds, then \( (\int_{\Omega} \sum_{i=1}^{m} \nu_i(x) |\frac{\partial u}{\partial x_i}|^2 \, dx)^{1/2} \) constitutes an equivalent norm in \( H^1_0(\nu_i, \Omega) \). We will denote this norm by \( \|u\|_{1,0} \).

**Lemma 4.3.** Let \( u(x) \in H^1_0(\nu_i, \Omega) \) and \( k \geq 0 \), then the function \( \min(u, k) \) belongs to \( H^1_0(\nu_i, \Omega) \).

**Proof.** Define \( v = \min(u, k) \) for \( u \in H^1_0(\nu_i, \Omega) \) and let \( \{\varphi_n\} \) be a sequence of functions of \( C_0^\infty(\Omega) \) such that

\[
\lim_{n \to +\infty} \|\varphi_n - u\|_1 = 0.
\]

Let \( \psi_n = \min(\varphi_n, k) \) for any \( n \in \mathbb{N} \).

By regularization, we can prove that \( \psi_n \) belongs to \( H^1_0(\nu_i, \Omega) \); moreover, because the norms of \( \{\psi_n\} \) are equibounded in \( H^1_0(\nu_i, \Omega) \), there exists a subsequence that weakly converges in \( H^1_0(\nu_i, \Omega) \). On the other hand,

\[
|v(x) - \psi_n(x)| \leq |u(x) - \varphi_n(x)| \quad \text{a.e. in } \Omega,
\]

so \( \{\psi_n\} \) converges to \( v \) in \( L^2(\Omega) \).

The conclusion now follows easily.

**Proof of Lemma 4.4.** We observe that

(4.4) \[
\tau = \inf \left\{ a(u, u) : u \in H^1_0(\nu_i, \Omega), \quad \int_{\Omega} u^2 \, dx = 1 \right\},
\]

and we define \( f, g : H^1_0(\nu_i, \Omega) \to \mathbb{R} \) as

\[
f(u) = a(u, u), \quad g(u) = \int_{\Omega} u^2 \, dx - 1.
\]

Let

\[
E = \{u \in H^1_0(\nu_i, \Omega) : g(u) = 0\}.
\]

Then

\[
\tau = \inf_{u \in E} f(u).
\]

Let \( \{u_n\} \) be a sequence such that \( a(u_n, u_n) \to \tau \); from (3.2) and Remark 4.2 we have that \( \{u_n\} \) is bounded in \( H^1_0(\nu_i, \Omega) \), so there exist \( \{u_{n_k}\}, u_0 \in H^1_0(\nu_i, \Omega) \) such that \( u_{n_k} \rightharpoonup u_0 \) weakly in \( H^1_0(\nu_i, \Omega) \). By the compact imbedding of \( H^1_0(\nu_i, \Omega) \) into \( L^2(\Omega) \) (Lemma 4.1), \( u_{n_k} \to u_0 \) strongly in \( L^2(\Omega) \), which gives \( \int_{\Omega} u_0^2 \, dx = 1 \). Therefore \( u_0 \in E \).
Finally, by virtue of
\[ \tau \leq a(u_0, u_0) \leq \liminf_{k \to +\infty} a(u_{n_k}, u_{n_k}) = \tau \]
we obtain
\[ \tau = a(u_0, u_0) \]
and \( f \) attains its minimum at \( u_0 \in E \). By Remark 3.3 we have
\[ (f|_E)'(u_0) = 0. \]
Accordingly, Theorem 3.4 yields
\[ (f)'(u_0) = \lambda(g)'(u_0) \text{ for some } \lambda \in \mathbb{R} \]
or
\[ a(u, u_0) = \lambda \int_{\Omega} uu_0 \, dx \text{ for any } u \in H^1_0(\nu_i, \Omega). \]
Choosing \( u = u_0 \) we have
\[ \tau = a(u_0, u_0) = \lambda \int_{\Omega} u_0^2 \, dx \Rightarrow \tau = \lambda. \]
Obviously \( u_0 \in H^1_0(\nu_i, \Omega) \) is such that
\[ a(u, u_0) = \tau \int_{\Omega} uu_0 \, dx \text{ for any } u \in H^1_0(\nu_i, \Omega). \]
Next, Lemma 4.3 implies that if \( u \) satisfies (4.4) then \(|u|\) also satisfies (4.4), therefore we can choose \( u_0 \) to be non-negative.

5. Proof of main results

Define \( G: H^{-1}(\nu_i^{-1}, \Omega) \to H^1_0(\nu_i, \Omega) \) as
\[ G(g) = w \text{ where } w \text{ is a weak solution of } \begin{cases} -\sum_{i,j=1}^{m} \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial w}{\partial x_j}) = g & \text{in } \Omega \\ w = 0 & \text{on } \partial \Omega \end{cases} \]
Remark 3.5 ensures that \( G \) is a linear continuous map. For \( u \in H^1_0(\nu_i, \Omega) \) define \( F(u) = G(f(u)) \). Then a fixed point \( u \) of \( F \) is a solution of problem (1.0).
Proof of Theorem 5.1. We claim that

\[ u \in L^2(\Omega) \Rightarrow f(u) \in L^2(\Omega). \]

Indeed,

\[ |f(u)| \leq |f(u) - f(0)| + |f(0)| \leq L|u| + |f(0)|, \]

thus

\[ \int_{\Omega} |f(u)|^2 \, dx \leq 2L^2 \int_{\Omega} |u|^2 \, dx + 2|f(0)|^2 \text{meas}_x \Omega. \]

We proceed to show that \( F \) is a contractive mapping. We see at once that

\[ |f(u) - f(v)|_2 \leq L|u - v|_2 \quad \text{for any } u, v \in H^1_0(\nu_i, \Omega). \]

By (3.1) and Remark 4.2 we deduce that

\[ \alpha \|u\|_{1,0}^2 \leq a(u, u) = (f(u), u) \leq c|f(u)|_2 \|u\|_{1,0} \]

or

\[ \|u\|_{1,0} \leq \frac{c}{\alpha} |f(u)|_2. \]

Consequently, \( G \) is continuous from \( L^2(\Omega) \to L^2(\Omega) \). Therefore

\[ |F(u) - F(v)|_2 = |G(f(u) - f(v))|_2 \leq \|G\|_* |f(u) - f(v)|_2 \]

\[ \leq L\|G\|_* |u - v|_2. \]

Since \( \tau|u|^2 \leq a(u, u) = \int_{\Omega} f(u)u \, dx \leq |f(u)|_2 |u|_2 \) or

\[ \frac{|G(f(u))|_2}{|f(u)|_2} \leq \frac{1}{\tau}, \]

it results that

\[ \|G\|_* \leq \frac{1}{\tau}. \]

We conclude from (5.1) that

\[ |F(u) - F(v)|_2 \leq L \frac{\tau}{\tau} |u - v|_2 \]

and finally that, since \( L < \tau \), \( F \) has a fixed point in \( H^1_0(\nu_i, \Omega) \).

Now, let us fix \( k \geq 0 \), then from (3.1) for \( v = u - \min(u, k) \) we get

\[ \alpha \|v\|_{1,0}^2 \leq L \int_{\Omega} |u||v| \, dx + \int_{\Omega} |f(0)||v|. \]

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Lemma 4.1 and the definition of $v$ imply
\[ \int_{\Omega} |u||v| \, dx \leq \int_{\Omega(u>k)} v^2 \, dx + k \int_{\Omega(u>k)} v \, dx \leq |v|_2^2 \cdot [\text{meas}_x \Omega(u > k)]^{1-\frac{2}{\alpha}} + k \int_{\Omega(u>k)} v \, dx \leq c^2 \|v\|_{1,0}^2 [\text{meas}_x \Omega(u > k)]^{1-\frac{2}{\alpha}} + k \int_{\Omega(u>k)} v \, dx. \]

Therefore from (5.2) we obtain
\[ (5.3) \quad \|v\|_{1,0}^2 (\alpha - Lc^2[\text{meas}_x \Omega(u > k)]^{1-\frac{1}{\alpha}}) \leq (Lk + |f(0)|) \int_{\Omega(u>k)} v \, dx. \]

Recalling that
\[ \lim_{k \to +\infty} \text{meas}_x \Omega(u > k) = 0 \]
we can certainly choose $\bar{k} \geq 0$ such that for any $k \geq \bar{k}$ we have
\[ Lc^2[\text{meas}_x \Omega(u > k)]^{1-\frac{1}{\alpha}} \leq \frac{\alpha}{2}. \]

We apply this inequality to (5.3) obtaining
\[ (5.4) \quad \|v\|_{1,0} \leq \frac{2c}{\alpha} [\text{meas}_x \Omega(u > k)]^{1-\frac{1}{\alpha}} (|f(0)| + Lk) \text{ for any } k \geq \bar{k}. \]

Let $h, k$ be real numbers, $h > k \geq \bar{k}$. Then one has
\[ |v|_2 = \left[ \int_{\Omega(u>k)} |u-k|^2 \, dx \right]^{\frac{1}{2}} \geq (h-k)[\text{meas}_x \Omega(u > h)]^{\frac{1}{2}}; \]

furthermore, (5.4) and Lemma 4.1 yield
\[ (5.5) \quad [\text{meas}_x \Omega(u > h)]^{\frac{1}{2}} \leq \frac{2c^2}{\alpha(h-k)} (|f(0)| + Lk)[\text{meas}_x \Omega(u > k)]^{1-\frac{1}{\alpha}}. \]

Next, if $k > 0$, we get
\[
\text{meas}_x \Omega(u > k) \leq \frac{1}{k^{2\frac{1}{\alpha}}} \int_{\Omega(u>k)} u^2 \, dx, \quad \frac{2c^2}{\alpha k} (|f(0)| + 2Lk)^{2\frac{1}{\alpha}-1} [\text{meas}_x \Omega(u > k)]^{1-\frac{2}{\alpha}}
\leq \frac{2c^2}{\alpha k^{2\frac{1}{\alpha}-1}} (|f(0)| + 2Lk)^{2\frac{1}{\alpha}-1} \left( \int_{\Omega(u>k)} u^2 \, dx \right)^{1-\frac{2}{\alpha}}. 
\]
Now, the first term of the above inequality goes to zero as \( k \to +\infty \), so we can fix \( k_1 (\geq \tilde{k}) \) such that
\[
2c^2 \alpha (|f(0)| + 2Lk_1) [\text{meas}_x \Omega(u > k_1)]^{1 - \frac{2}{2^*}} 2^{\frac{2^* - 1}{2}} \leq k_1.
\]

Moreover, one has
\[
\frac{2c^2}{\alpha(h - k)} (|f(0)| + Lk) \leq \frac{2c^2}{(h - k)} (|f(0)| + 2Lk_1) \text{ if } 0 \leq k \leq k_1.
\]

Combining (5.5) and (5.7) we obtain
\[
[\text{meas}_x \Omega(u > h)]^{\frac{1}{2^*}} \leq \frac{2c^2}{\alpha(h - k)} (|f(0)| + 2Lk_1) [\text{meas}_x \Omega(u > k)]^{1 - \frac{1}{2^*}}
\]
for any \( h, k \in \mathbb{R} \) such that \( k_1 \leq k < h \leq 2k_1 \).

Assuming in \([k_1, +\infty[\) that
\[
\varphi(k) = \begin{cases} 
[\text{meas}_x \Omega(u > k)]^{\frac{1}{2^*}} \text{ if } k_1 \leq k \leq 2k_1 \\
0 \text{ if } k > 2k_1 
\end{cases}
\]
we get
\[
\varphi(h) \leq \frac{2c^2}{\alpha(h - k)} (|f(0)| + 2Lk_1) [\varphi(k)]^{2^* - 1}
\]
for any \( h, k \in \mathbb{R} \) such that \( k_1 \leq k < h \leq 2k_1 \), and from Stampacchia’s lemma (see [11], p. 212) we deduce
\[
\varphi(k_1 + d) = 0,
\]
where \( d \) is the first term of (5.6).

We can obtain the same conclusion for \(-u\), so the proof of the theorem is complete.

**Proof of Theorem 5.2.** Set \( F \) as in Theorem 5.1. Since the imbedding of \( H_0^1(\nu_i, \Omega) \) into \( L^2(\Omega) \) is compact, we have that \( F \) is also compact from \( L^2(\Omega) \) into \( L^2(\Omega) \); therefore, by Schaefer’s fixed point theorem, it will be sufficient to prove that the set of all solutions of the equation
\[
u = \mu F(u) \quad \text{for } 0 < \mu < 1
\]
is unbounded.

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Indeed, if \( u \) satisfies (5.8), then \( u \) is solution of
\[
\begin{cases}
- \sum_{i,j=1}^{m} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial u}{\partial x_j} \right) = \mu f(u) \text{ in } \Omega \\
u = 0 \text{ on } \partial \Omega,
\end{cases}
\]
therefore
\[\tau |u|^2 \leq a(u,u) = \mu \int_{\Omega} f(u)u \, dx \leq M (\text{meas}_x \Omega)^{\frac{1}{2}} |u|_2\]
or
\[|u|_2 \leq \frac{M (\text{meas}_x \Omega)^{\frac{1}{2}}}{\tau}.
\]
Now, if we fix in (3.1) \( v = u - \min(u, k), k \geq 0 \) we get
\[\alpha \|u\|_{1,0}^2 \leq M \int_{\Omega} v \, dx \leq M |v|_2 \cdot [\text{meas}_x \Omega(u > k)]^{\frac{2^* - 1}{2^*}}.
\]
This inequality, as in the previous theorem, implies
\[\|u\|_{\infty} < +\infty.
\]

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**References**


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