Salvatore Bonafede
Existence results for a class of semilinear degenerate elliptic equations


Persistent URL: [http://dml.cz/dmlcz/134032](http://dml.cz/dmlcz/134032)

**Terms of use:**

© Institute of Mathematics AS CR, 2003

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.
EXISTENCE RESULTS FOR A CLASS OF SEMILINEAR DEGENERATE ELLIPTIC EQUATIONS

Salvatore Bonafele, Palermo

(Received February 12, 2002)

Abstract. We prove existence results for the Dirichlet problem associated with an elliptic semilinear second-order equation of divergence form. Degeneracy in the ellipticity condition is allowed.

Keywords: weak subsolution, degenerate equation, critical point, fixed-point theorems

MSC 2000: 35A05, 35J70, 47H10

1. Introduction

We consider the semilinear boundary value problem

\[
\begin{aligned}
\left\{- \sum_{i,j=1}^{m} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial u}{\partial x_j} \right) = f(u) \right. & \text{ in } \Omega \\
\left. u = 0 \right. & \text{ on } \partial \Omega
\end{aligned}
\]

where \(\Omega\) is a bounded open subset of \(\mathbb{R}^m\), \(f\) is a real valued function defined on \(\mathbb{R}\), and the coefficients \(a_{i,j}(x)\) satisfy the ellipticity condition

\[
\sum_{i,j=1}^{m} a_{ij}(x) p_i p_j \geq \alpha \sum_{i=1}^{m} \nu_i(x) p_i^2
\]

for a.e. \(x \in \Omega\) and for any \(p \in \mathbb{R}^m\)

with \(\nu_i(x)\) satisfying sufficiently general hypotheses.

We obtain some results of existence, uniqueness and boundedness for weak solutions of problem (1.0) with minimal hypotheses on \(f\). Similar results, when \(f\) has a natural polynomial growth, have been obtained in [3], [5], [7] and in [8] by pseudomonotone operators’ theory, while our proof uses fixed-point theorems. The paper
is structured as follows. In Sections 2 and 3 we state hypotheses and results. In Section 4 we establish some useful lemmas and, finally, in Section 5 we prove our main theorems.

2. Functional spaces

Let $\mathbb{R}^m$ be the Euclidean $m$-space with a generic point $x = (x_1, x_2, \ldots, x_m)$, $\Omega$ a bounded open subset of $\mathbb{R}^m$. The notation $\text{meas}_x$ will indicate the $m$-dimensional Lebesgue measure.

If $u(x)$ is a measurable function defined in $\Omega$, we will denote by $|u|_p$ ($1 \leq p \leq \infty$) the usual norm in the space $L^p(\Omega)$.

**Hypothesis 2.1.** Let $\nu_i(x)$ ($i = 1, 2, \ldots, m$) be a positive and measurable function defined in $\Omega$ such that

$$\nu_i(x) \in L^1(\Omega), \quad \nu_i^{-1}(x) \in L^{g_i}(\Omega)$$

where $\sum_i \frac{1}{g_i} < 2$ ($g_i > 1$) if $m \geq 3$ ($m = 2$).

The symbol $H^1(\nu, \Omega)$ stands for the completion of $C^1(\overline{\Omega})$ with respect to the norm

$$\|u\|_1 = \left( \int_\Omega \left( |u|^2 + \sum_{i=1}^m \nu_i(x) \left| \frac{\partial u}{\partial x_i} \right|^2 \right) dx \right)^{\frac{1}{2}}$$

$H^1_0(\nu, \Omega)$ denotes the closure of $C^\infty(\Omega)$ in $H^1(\nu, \Omega)$.

Finally, $H^{-1}(\nu^{-1}, \Omega)$ denotes the dual space of $H^1_0(\nu, \Omega)$ (see also [5], [6] and [10] for details concerning the weighted Sobolev spaces).

3. Hypotheses, problems and results

**Hypothesis 3.1.** The coefficients $a_{ij}(x)$ ($i, j = 1, 2, \ldots, m$) are functions defined and measurable in $\Omega$ satisfying

$$a_{ij}(x) = a_{ji}(x),$$

$$\frac{a_{ij}(x)}{\sqrt{\nu_i(x)\nu_j(x)}} \in L^\infty(\Omega) \quad (i, j = 1, 2, \ldots, m).$$

**Hypothesis 3.2.** There exists $\alpha > 0$ such that for almost every $x$ in $\Omega$ we have

$$\sum_{i,j=1}^m a_{ij}(x)p_ip_j \geq \alpha \sum_{i=1}^m \nu_i(x)p_i^2 \quad \text{for any } p \in \mathbb{R}^m.$$
Let $a: H^1_0(\nu_i, \Omega) \times H^1_0(\nu_i, \Omega) \to \mathbb{R}$ be such that

$$a(u, v) = \int_{\Omega} \sum_{ij=1}^{m} a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} \, dx,$$

and define

$$\tau = \inf_{u \in H^1_0(\nu_i, \Omega) \setminus \{0\}} \frac{a(u, u)}{|u|_2}.$$

In Section 4 we prove the following

**Lemma 4.4.** Let us assume that (2.1), (3.1), (3.2) hold. Then $\tau > 0$ and there exists $u_0 \in H^1_0(\nu_i, \Omega)$ such that $\tau = a(u_0, u_0)$ and

$$a(u, u_0) = \tau \int_{\Omega} uu_0 \, dx \quad \text{for any } u \in H^1_0(\nu_i, \Omega);$$

moreover, we can choose $u_0 \geq 0$.

**Definition 3.2.** Let $H$ be a Hilbert space, $f, g \in C^1(H, \mathbb{R})$, and let

$$E = \{ u \in H : g(u) = 0, \quad g'(u) \neq 0 \}.$$

A point $u_0 \in H$ is a critical point of $f|_E$ if $\frac{4}{dt} f(h(t))|_{t=0} = 0$ for all $C^1$ paths $h(t): \mathbb{R} \to E$ such that $h(0) = u_0$.

**Remark 3.3.** If there exists $u_0 \in E$ such that $f(u_0) = \min \{ f(u) : u \in E \}$, then $(f|_E)'(u_0) = 0$.

**Theorem 3.4** (see, e.g., [2]). A point $u_0 \in E$ is a critical point of $f|_E$ if and only if there exists $\lambda \in \mathbb{R}$ such that $f'(u_0) = \lambda g'(u_0)$.

Now, if $f \in C(\mathbb{R})$ satisfies the condition

$$u \in H^1_0(\nu_i, \Omega) \Rightarrow f(u) \in H^{-1}(\nu_i, \Omega),$$

we obtain the following well posed problem

**Problem.** Find a function $u(x) \in H^1_0(\nu_i, \Omega)$ such that

$$(3.1) \quad \int_{\Omega} \sum_{ij=1}^{m} a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} \, dx = (f(u), v)$$

for any $v(x) \in H^1_0(\nu_i, \Omega)$.

(1) We denote by $(\cdot, \cdot)$ the duality pairing between $H^1_0(\nu_i, \Omega)$ and $H^{-1}(\nu_i, \Omega)$.
A function $u(x)$ satisfying (3.1) is a weak solution of Problem (1.0).

**Remark 3.5.** When $f$ does not depend on $u$, $f \in H^{-1}(\nu_i, \Omega)$, the hypotheses (2.1), (3.1), (3.2) are sufficient to ensure existence and uniqueness of a weak solution of problem (1.0), moreover we have

$$\|u\|_{1,0} \leq \|f\|_{H^{-1}(\nu_i, \Omega)}.$$

Proof follows from Lemma 4.1 and the Lax-Milgram theorem (see Remark 4.2 for the definition of $\|u\|_{1,0}$).

In Section 5 we prove

**Theorem 5.1** (Existence, uniqueness and boundedness). Let us assume that (2.1), (3.1), (3.2) hold and let $f$ be Lipschitz continuous with a Lipschitz constant $L < \tau$.

Then there exists a unique weak solution $u(x)$ of problem (1.0); moreover, $u(x) \in L^\infty(\Omega)$ and

$$\|u\|_{\infty} \leq \gamma(L, g, m, \text{meas}_{x}\Omega).$$

**Theorem 5.2.** Let us assume that (2.1), (3.1), (3.2) hold and let $f$ be a bounded continuous function. Then Problem (1.0) has a weak solution $u(x)$. Moreover, $u(x) \in L^\infty(\Omega)$ and (5.0) holds.

4. Preliminary lemmas

**Lemma 4.1.** If the hypothesis (2.1) is satisfied then there exists a constant $C = C(m, g_i, |\nu_i^{-1}|_{g_i})$ such that

$$|u|_{2^*} \leq C \left( \int_\Omega \sum_{i=1}^m \nu_i(x) \left| \frac{\partial u}{\partial x_i} \right|^2 \, dx \right)^{\frac{1}{2}} \quad \text{for all } u \in H^1_0(\nu_i, \Omega),$$

where $2^* = 2m(m - 2 + \sum_{i=1}^m \frac{1}{g_i})^{-1}$.

Moreover, the imbedding of $H^1_0(\nu_i, \Omega)$ into $L^2(\Omega)$ is compact.
Let us fix $m_i = \frac{2g_i}{g_i + 1}$. Then

$$\left| \frac{\partial u}{\partial x_i} \right|_{m_i} \leq |\nu_i^{-1}|^{\frac{1}{m_i}} |\nu_i^\frac{1}{2} \frac{\partial u}{\partial x_i}|_{2}^{\frac{1}{m_i}}.$$  \hfill (4.2)

Since $\sum_{i=1}^{m} \frac{1}{m_i} = \sum_{i=1}^{m} \frac{g_i + 1}{2g_i} = \frac{1}{2} \left( m + \sum_{i=1}^{m} \frac{1}{g_i} \right) > 1$, Sobolev’s imbedding theorem yields (see, for instance, [12])

$$|u|_q \leq C(m, m_i, q) \prod_{i=1}^{m} \left| \frac{\partial u}{\partial x_i} \right|_{m_i}^{\frac{1}{m_i}}.$$  \hfill (4.3)

where $q = m \left( -1 + \sum_{i=1}^{m} \frac{1}{m_i} \right)^{-1}$.

From (4.2) and (4.3) we obtain

$$|u|_{2^*} \leq C \prod_{i=1}^{m} \left( |\nu_i^{-1}|^{\frac{1}{2g_i}} |\nu_i^\frac{1}{2} \frac{\partial u}{\partial x_i}|_{2}^{\frac{1}{m_i}} \right).$$

Now, let $\{u_n\}$ be a sequence of functions of $H^1_0(\nu_i, \Omega)$ with equibounded norms and let $\{\Pi_k\}$ be a sequence of open intervals in $\Omega$ such that

1. $\Pi_k \subset \Pi_{k+1}$ for any $k \in \mathbb{N}$,
2. $\lim_{k \to +\infty} \Pi_k = \Omega$,
3. for any closed, bounded subset $C$ of $\Omega$ there exists $\bar{k}$: $C \subset \Pi_k$, $k \geq \bar{k}$.

Let us denote by $W^{1,1}(\Pi_1)$ the usual Sobolev space on the set $\Pi_1$.

It follows that the norms of $\{u_n\}$ in $W^{1,1}(\Pi_1)$ are equibounded; in fact, applying the Hölder inequality we obtain the following estimate:

$$\|u_n\|_{W^{1,1}(\Pi_1)} = \int_{\Pi_1} |u_n| \, dx + \int_{\Pi_1} \sum_{i=1}^{m} \left| \frac{\partial u_n}{\partial x_i} \right| \, dx \leq \left( \int_{\Pi_1} |u_n|^2 \, dx \right)^{\frac{1}{2}} \left( \text{meas} \, \Pi_1 \right)^{\frac{1}{2}} + \sum_{i=1}^{m} \left( \int_{\Pi_1} \frac{1}{\nu_i(x)} \, dx \right)^{\frac{1}{2}} \|u_n\|_1 \leq \text{const} \|u_n\|_1.$$

Due to the compact imbedding of $W^{1,1}(\Pi_1)$ into $L^1(\Pi_1)$ (see e.g. [1]) there is a subsequence $\{u_{1,n}\}$ from $\{u_n\}$ that converges a.e. in $\Pi_1$.

The same procedure can be done on each $\Pi_j$ for $j = 2, 3, \ldots$. Hence we get a system of sequences $\{u_{j,n}\}$, $n, j = 1, 2, \ldots$ (where $\{u_{j,n}\}$ is a subsequence of $\{u_{j-1,n}\}$) such that $\{u_{j,n}\}$ is convergent a.e. in $\Pi_j$ for $j = 1, 2, \ldots$.

By the diagonals method we obtain that $\{u_{n,n}\}$ converges a.e. in $\Omega$ and, by virtue (4.1), in $L^2(\Omega)$.  

191
Remark 4.2. If the hypothesis (2.1) holds, then \( \left( \int_{\Omega} \sum_{i=1}^{m} \nu_i(x) | \frac{\partial u}{\partial x_i} |^2 \, dx \right)^{1/2} \) constitutes an equivalent norm in \( H^1_0(\nu_i, \Omega) \). We will denote this norm by \( \|u\|_{1,0} \).

Lemma 4.3. Let \( u(x) \in H^1_0(\nu_i, \Omega) \) and \( k \geq 0 \), then the function \( \min(u, k) \) belongs to \( H^1_0(\nu_i, \Omega) \).

Proof. Define \( v = \min(u, k) \) for \( u \in H^1_0(\nu_i, \Omega) \) and let \( \{\varphi_n\} \) be a sequence of functions of \( C^\infty_0(\Omega) \) such that
\[
\lim_{n \to +\infty} \|\varphi_n - u\|_1 = 0.
\]
Let \( \psi_n = \min(\varphi_n, k) \) for any \( n \in \mathbb{N} \).

By regularization, we can prove that \( \psi_n \) belongs to \( H^1_0(\nu_i, \Omega) \); moreover, because the norms of \( \{\psi_n\} \) are equibounded in \( H^1_0(\nu_i, \Omega) \), there exists a subsequence that weakly converges in \( H^1_0(\nu_i, \Omega) \). On the other hand,
\[
|v(x) - \psi_n(x)| \leq |u(x) - \varphi_n(x)| \quad \text{a.e. in } \Omega,
\]
so \( \{\psi_n\} \) converges to \( v \) in \( L^2(\Omega) \).

The conclusion now follows easily.

Proof of Lemma 4.4. We observe that
\[
(4.4) \quad \tau = \inf \left\{ a(u, u): u \in H^1_0(\nu_i, \Omega), \int_{\Omega} u^2 \, dx = 1 \right\},
\]
and we define \( f, g: H^1_0(\nu_i, \Omega) \to \mathbb{R} \) as
\[
f(u) = a(u, u), \quad g(u) = \int_{\Omega} u^2 \, dx - 1.
\]
Let
\[
E = \{u \in H^1_0(\nu_i, \Omega): g(u) = 0\}.
\]
Then
\[
\tau = \inf_{u \in E} f(u).
\]
Let \( \{u_n\} \) be a sequence such that \( a(u_n, u_n) \to \tau \); from (3.2) and Remark 4.2 we have that \( \{u_n\} \) is bounded in \( H^1_0(\nu_i, \Omega) \), so there exist \( \{u_{n_k}\}, u_0 \in H^1_0(\nu_i, \Omega) \) such that \( u_{n_k} \rightharpoonup u_0 \) weakly in \( H^1_0(\nu_i, \Omega) \). By the compact imbedding of \( H^1_0(\nu_i, \Omega) \) into \( L^2(\Omega) \) (Lemma 4.1), \( u_{n_k} \to u_0 \) strongly in \( L^2(\Omega) \), which gives \( \int_{\Omega} u_0^2 \, dx = 1 \). Therefore \( u_0 \in E \).

192
Finally, by virtue of
\[ \tau \leq a(u_0, u_0) \leq \liminf_{k \to +\infty} a(u_{n_k}, u_{n_k}) = \tau \]
we obtain
\[ \tau = a(u_0, u_0) \]
and \( f \) attains its minimum at \( u_0 \in E \). By Remark 3.3 we have
\[ (f|_E)'(u_0) = 0. \]
Accordingly, Theorem 3.4 yields
\[ (f)'(u_0) = \lambda (g)'(u_0) \text{ for some } \lambda \in \mathbb{R} \]
or
\[ a(u, u_0) = \lambda \int_{\Omega} uu_0 \, dx \quad \text{for any } u \in H_0^1(\nu_i, \Omega). \]
Choosing \( u = u_0 \) we have
\[ \tau = a(u_0, u_0) = \lambda \int_{\Omega} u_0^2 \, dx \Rightarrow \tau = \lambda. \]
Obviously \( u_0 \in H_0^1(\nu_i, \Omega) \) is such that
\[ a(u, u_0) = \tau \int_{\Omega} uu_0 \, dx \quad \text{for any } u \in H_0^1(\nu_i, \Omega). \]
Next, Lemma 4.3 implies that if \( u \) satisfies (4.4) then \(|u| \) also satisfies (4.4), therefore we can choose \( u_0 \) to be non-negative.

5. PROOF OF MAIN RESULTS

Define \( G: H^{-1}(\nu_i^{-1}, \Omega) \to H_0^1(\nu_i, \Omega) \) as
\[ G(g) = w \quad \text{where } w \text{ is a weak solution of } \begin{cases} - \sum_{i,j=1}^m \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial w}{\partial x_j}) = g & \text{in } \Omega \\ w = 0 & \text{on } \partial \Omega \end{cases} \]
Remark 3.5 ensures that \( G \) is a linear continuous map. For \( u \in H_0^1(\nu_i, \Omega) \) define \( F(u) = G(f(u)) \). Then a fixed point \( u \) of \( F \) is a solution of problem (1.0).
Proof of Theorem 5.1. We claim that

\[ u \in L^2(\Omega) \Rightarrow f(u) \in L^2(\Omega). \]

Indeed,

\[ |f(u)| \leq |f(u) - f(0)| + |f(0)| \leq L|u| + |f(0)|, \]

thus

\[ \int_{\Omega} |f(u)|^2 \, dx \leq 2L^2 \int_{\Omega} |u|^2 \, dx + 2|f(0)|^2 \text{meas}_x \Omega. \]

We proceed to show that \( F \) is a contractive mapping. We see at once that

\[ |f(u) - f(v)|_2 \leq L|u - v|_2 \quad \text{for any } u, v \in H^1_0(\nu_i, \Omega). \]

By (3.1) and Remark 4.2 we deduce that

\[ \alpha \|u\|_{1,0}^2 \leq a(u, u) = (f(u), u) \leq c|f(u)|_2 \|u\|_{1,0} \]

or

\[ \|u\|_{1,0} \leq \frac{c}{\alpha} |f(u)|_2. \]

Consequently, \( G \) is continuous from \( L^2(\Omega) \to L^2(\Omega) \). Therefore

\[ |F(u) - F(v)|_2 = |G(f(u) - f(v))|_2 \leq \|G\|_* |f(u) - f(v)|_2 \leq L\|G\|_* |u - v|_2. \]  

(5.1)

Since \( \tau|u|^2 \leq a(u, u) = \int_{\Omega} f(u)u \, dx \leq |f(u)|_2 \|u\|_2 \) or

\[ \frac{|G(f(u))|_2}{|f(u)|_2} \leq \frac{1}{\tau}, \]

it results that

\[ \|G\|_* \leq \frac{1}{\tau}. \]

We conclude from (5.1) that

\[ |F(u) - F(v)|_2 \leq \frac{L}{\tau} |u - v|_2 \]

and finally that, since \( L < \tau \), \( F \) has a fixed point in \( H^1_0(\nu_i, \Omega) \).

Now, let us fix \( k \geq 0 \), then from (3.1) for \( v = u - \min(u, k) \) we get

\[ \alpha \|v\|_{1,0}^2 \leq L \int_{\Omega} |v| |v| \, dx + \int_{\Omega} |f(0)||v|. \]  

(5.2)
Lemma 4.1 and the definition of \( v \) imply
\[
\int_{\Omega} |u| |v| \, dx \leq \int_{\Omega(u > k)} v^2 \, dx + k \int_{\Omega(u > k)} v \, dx
\]
\[
\leq \|v\|_{2^*,1}^2 \left[ \text{meas}_x \Omega(u > k) \right]^{1-\frac{2}{2^*}} + k \int_{\Omega(u > k)} v \, dx
\]
\[
\leq c \|v\|_{1,0}^2 \left[ \text{meas}_x \Omega(u > k) \right]^{1-\frac{2}{2^*}} + k \int_{\Omega(u > k)} v \, dx.
\]

Therefore from (5.2) we obtain
\[
\|v\|_{1,0}^2 \left( \alpha - Lc^2 \left[ \text{meas}_x \Omega(u > k) \right]^{1-\frac{2}{2^*}} \right) \leq (Lk + |f(0)|) \int_{\Omega(u > k)} v \, dx.
\]

Recalling that
\[
\lim_{k \to +\infty} \text{meas}_x \Omega(u > k) = 0
\]
we can certainly choose \( \tilde{k} \geq 0 \) such that for any \( k \geq \tilde{k} \) we have
\[
Lc^2 \left[ \text{meas}_x \Omega(u > k) \right]^{1-\frac{2}{2^*}} \leq \frac{\alpha}{2}.
\]

We apply this inequality to (5.3) obtaining
\[
\|v\|_{1,0} \leq \frac{2c}{\alpha} \left[ \text{meas}_x \Omega(u > k) \right]^{1-\frac{2}{2^*}} (|f(0)| + Lk) \quad \text{for any } k \geq \tilde{k}.
\]

Let \( h, k \) be real numbers, \( h > k \geq \tilde{k} \). Then one has
\[
\|v\|_{2^*} = \left[ \int_{\Omega(u > k)} |u - k|^{2^*} \, dx \right]^{\frac{1}{2^*}} \geq (h - k) \left[ \text{meas}_x \Omega(u > h) \right]^{\frac{1}{2^*}};
\]

furthermore, (5.4) and Lemma 4.1 yield
\[
\left[ \text{meas}_x \Omega(u > h) \right]^{\frac{1}{2^*}} \leq \frac{2c^2}{\alpha(h - k)} (|f(0)| + Lk) \left[ \text{meas}_x \Omega(u > k) \right]^{1-\frac{2}{2^*}}.
\]

Next, if \( k > 0 \), we get
\[
\text{meas}_x \Omega(u > k) \leq \frac{1}{k^{2^*}} \int_{\Omega(u > k)} u^{2^*} \, dx, \quad \frac{2c^2}{\alpha k} (|f(0)| + 2Lk) 2^{\frac{2^*-1}{2^*}} \left[ \text{meas}_x \Omega(u > k) \right]^{1-\frac{2}{2^*}}
\]
\[
\leq \frac{2c^2}{\alpha k^{2^*-1}} (|f(0)| + 2Lk) 2^{\frac{2^*-1}{2^*}} \left( \int_{\Omega(u > k)} u^{2^*} \, dx \right)^{1-\frac{2}{2^*}}.
\]
Now, the first term of the above inequality goes to zero as \( k \to +\infty \), so we can fix \( k_1 (\geq \tilde{k}) \) such that

\[
(5.6) \quad \frac{2c^2}{\alpha (|f(0)| + 2Lk_1)} \left[ \text{meas}_x \Omega(u > k_1) \right]^{1 - \frac{2}{\alpha}} 2^{\frac{2^* - 1}{2}} \leq k_1.
\]

Moreover, one has

\[
(5.7) \quad \frac{2c^2}{\alpha (h - k)} (|f(0)| + Lk) \leq \frac{2c^2}{(h - k)} (|f(0)| + 2Lk_1) \quad \text{if} \quad 0 \leq k \leq k_1.
\]

Combining (5.5) and (5.7) we obtain

\[
[\text{meas}_x \Omega(u > h)]^{\frac{1}{\alpha}} \leq \frac{2c^2}{\alpha (h - k)} (|f(0)| + 2Lk_1) \left[ \text{meas}_x \Omega(u > k) \right]^{1 - \frac{1}{\alpha}}
\]

for any \( h, k \in \mathbb{R} \) such that \( k_1 \leq k < h \leq 2k_1 \).

Assuming in \([k_1, +\infty[\) that

\[
\varphi(k) = \begin{cases} 
[\text{meas}_x \Omega(u > k)]^{\frac{1}{\alpha}} & \text{if} \quad k_1 \leq k \leq 2k_1 \\
0 & \text{if} \quad k > 2k_1
\end{cases}
\]

we get

\[
\varphi(h) \leq \frac{2c^2}{\alpha (h - k)} (|f(0)| + 2Lk_1) \left[ \varphi(k) \right]^{2^* - 1}
\]

for any \( h, k \in \mathbb{R} \) such that \( k_1 \leq k < h \leq 2k_1 \), and from Stampacchia’s lemma (see [11], p. 212) we deduce

\[
\varphi(k_1 + d) = 0,
\]

where \( d \) is the first term of (5.6).

We can obtain the same conclusion for \(-u\), so the proof of the theorem is complete.

**Proof of Theorem 5.2.** Set \( F \) as in Theorem 5.1. Since the imbedding of \( H_0^1(\nu_i, \Omega) \) into \( L^2(\Omega) \) is compact, we have that \( F \) is also compact from \( L^2(\Omega) \) into \( L^2(\Omega) \); therefore, by Schaeffer’s fixed point theorem, it will be sufficient to prove that the set of all solutions of the equation

\[
(5.8) \quad u = \mu F(u) \quad \text{for} \quad 0 < \mu < 1
\]

is unbounded.
Indeed, if $u$ satisfies (5.8), then $u$ is solution of
\[
\begin{cases}
- \sum_{i,j=1}^{m} \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial u}{\partial x_j}) = \mu f(u) \text{ in } \Omega \\
u = 0 \text{ on } \partial \Omega,
\end{cases}
\]
therefore
\[
\tau |u|^2_2 \leq a(u, u) = \mu \int_{\Omega} f(u) u \, dx \leq M (\text{meas}_x \Omega)^{\frac{1}{2}} |u|_2
\]
or
\[
|u|_2 \leq \frac{M (\text{meas}_x \Omega)^{\frac{1}{2}}}{\tau}.
\]
Now, if we fix in (3.1) $v = u - \min(u, k), k \geq 0$ we get
\[
\alpha \|u\|_{1,0}^2 \leq M \int_{\Omega} v \, dx \leq M |v|_2 \cdot [\text{meas}_x \Omega(u > k)]^{\frac{2^{*}-1}{2^{*}}}. 
\]
This inequality, as in the previous theorem, implies
\[
\|u\|_{\infty} < +\infty.
\]

Acknowledgements. Research was supported by the grant MURST 60% of Italy.

References


Author’s address: Salvatore Bonafede, Dipartimento di Economia dei Sistemi Agro-Forestali, University of Palermo, Viale delle Scienze, 90128 Palermo, Italy, e-mail: bonafede@dmi.unict.it.