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Pseudo $BL$-algebras and $DR\ell$-monoids


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Abstract. It is shown that pseudo $BL$-algebras are categorically equivalent to certain bounded $DR\ell$-monoids. Using this result, we obtain some properties of pseudo $BL$-algebras, in particular, we can characterize congruence kernels by means of normal filters. Further, we deal with representable pseudo $BL$-algebras and, in conclusion, we prove that they form a variety.

Keywords: pseudo $BL$-algebra, $DR\ell$-monoid, filter, polar, representable pseudo $BL$-algebra

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1. Connections between pseudo $BL$-algebras and $DR\ell$-monoids

Recently, pseudo $BL$-algebras were introduced by A. Di Nola, G. Georgescu and A. Iorgulescu in [3] as a noncommutative extension of Hájek’s $BL$-algebras (see [6]).

An algebra $\mathfrak{A} = (A, \lor, \land, \circ, \rightarrow, \leadsto, 0, 1)$ of type $\langle 2, 2, 2, 2, 2, 0, 0 \rangle$ is called a pseudo $BL$-algebra iff $(A, \lor, \land, 0, 1)$ is a bounded lattice, $(A, \circ, 1)$ is a monoid and the following conditions are satisfied for all $x, y, z \in A$:

1. $x \circ y \leq z$ iff $x \leq y \rightarrow z$ iff $y \leq x \leadsto z$,
2. $x \land y = (x \rightarrow y) \circ x = x \circ (x \leadsto y)$,
3. $(x \rightarrow y) \lor (y \rightarrow x) = (x \leadsto y) \lor (y \leadsto x) = 1$.

By [3, Corollary 3.29], pseudo $BL$-algebras satisfying the identity

$$(x \leadsto 0) \rightarrow 0 = (x \rightarrow 0) \leadsto 0 = x$$

are the duals of pseudo $MV$-algebras.

In the same way, (noncommutative) $DR\ell$-monoids extend Swamy’s $DR\ell$-semi-groups which were introduced in [12] as a common generalization of abelian $\ell$-groups and Brouwerian algebras.
An algebra $\mathfrak{A} = (A, +, 0, \lor, \land, \rightarrow, \leftarrow)$ of type $\langle 2, 0, 2, 2, 2, 2 \rangle$ is a \textit{dually residuated lattice ordered monoid}, or simply a \textit{DRℓ-monoid}, iff

1. $(A, +, 0, \lor, \land)$ is an $\ell$-monoid, that is, $(A, +, 0)$ is a monoid, $(A, \lor, \land)$ is a lattice and, for any $x, y, s, t \in A$, the following distributive laws are satisfied:

   $$s + (x \lor y) + t = (s + x + t) \lor (s + y + t),$$
   $$s + (x \land y) + t = (s + x + t) \land (s + y + t);$$

2. for any $x, y \in A$, $x \rightarrow y$ is the least $s \in A$ such that $s + y \geq x$, and $x \leftarrow y$ is the least $t \in A$ such that $y + t \geq x$;

3. $\mathfrak{A}$ fulfils the identities

   $$((x \rightarrow y) \lor 0) + y \leq x \lor y, \quad y + ((x \leftarrow y) \lor 0) \leq x \lor y,$$
   $$x \rightarrow x \geq 0, \quad x \leftarrow x \geq 0.$$

Note that the inequalities $x \rightarrow x \geq 0$ and $x \leftarrow x \geq 0$ can be omitted, and the condition (2) is equivalent to the system of identities (see [10])

$$x \rightarrow y \leq (x \lor z) \rightarrow y, \quad x \leftarrow y \leq (x \lor z) \leftarrow y,$$
$$x + y \rightarrow y \leq x, \quad (y + x) \leftarrow y \leq x.$$

In [11], mutual relationships between $BL$-algebras and bounded representable commutative $DRℓ$-monoids are described.

**Theorem 1.1.** Let $\mathfrak{A} = (A, \lor, \land, \odot, \rightarrow, \leftarrow, 0, 1)$ be a pseudo $BL$-algebra. If we set

$$x + y := x \odot y, \quad x \lor_d y := x \land y, \quad x \land_d y := x \lor y,$$
$$x \rightarrow y := y \rightarrow x, \quad x \leftarrow y := y \leftarrow x, \quad 0_d := 1, \quad 1_d := 0$$

for any $x, y \in A$, then $\mathfrak{A}_d = (A, +, 0_d, \lor_d, \land_d, \rightarrow, \leftarrow)$ is a bounded $DRℓ$-monoid with the greatest element $1_d$. In addition, this $DRℓ$-monoid satisfies the identities

$$((x \rightarrow y) \land_d (y \rightarrow x)) = 0_d,$$

$$((x \leftarrow y) \land_d (y \leftarrow x)) = 0_d.$$

**Proof.** Since $(A, \odot, 1, \lor, \land)$ is an $\ell$-monoid, by [3, Propositions 3.3, 3.9], so is $(A, +, 0_d, \lor_d, \land_d)$. The rest follows directly by the definitions. Note that if a $DRℓ$-monoid $\mathfrak{A}_d$ contains the greatest element $1_d$ then $0_d$ is its least element, by [8, Theorem 1.2.3]. □
In view of Theorem 1.1, it is easily seen that in the definition of a pseudo BL-algebra, the condition (1) can be equivalently replaced by the following identities:

\[(x \rightarrow y) \odot x \leq y, \quad x \odot (x \bowtie y) \leq y,\]
\[x \rightarrow y \geq x \rightarrow (y \land z), \quad x \bowtie y \geq x \bowtie (y \land z),\]
\[y \rightarrow (x \odot y) \geq x, \quad y \bowtie (y \circ x) \geq x.
\]

Consequently, pseudo BL-algebras form a variety of algebras of type \(\langle 2, 2, 2, 2, 2, 0, 0 \rangle\). This variety is arithmetical; in accordance with [8, Theorem 3.1.1], the Pixley term of the variety of pseudo BL-algebras can be taken as follows:

\[p(x, y, z) = ((x \bowtie y) \rightarrow z) \land ((z \bowtie y) \rightarrow x) \land (x \lor z).
\]

Theorem 1.2. Let \(\mathfrak{A} = (A, +, 0, \lor, \land, \rightarrow)\) be a DR\(\ell\)-monoid with the greatest element \(1\). For any \(x, y \in A\) set

\[x \odot y := x + y, \quad x \lor_d y := x \land y, \quad x \land_d y := x \lor y,\]
\[x \rightarrow y := y \rightarrow x, \quad x \bowtie y := y \leftarrow x, \quad 0_d := 1, \quad 1_d := 0.
\]

Then \(\mathfrak{A}_d = (A, \lor_d, \land_d, \odot, \rightarrow, \bowtie, 0_d, 1_d)\) is a pseudo BL-algebra if and only if \(\mathfrak{A}\) satisfies \((\ast)\).

Proof. In any DR\(\ell\)-monoid we have

\[x \lor y = ((y \rightarrow x) \lor 0) + x = x + ((y \leftarrow x) \lor 0).
\]

Since \(\mathfrak{A}\) is bounded, that is, \(0 \leq x \leq 1\) for any \(x \in A\), it follows that

\[x \land_d y = (x \rightarrow y) \odot x = x \odot (x \bowtie y).
\]

The rest is obvious. \(\square\)

Let \(\mathcal{PBL}\) be the category of pseudo BL-algebras, that is, the category whose objects are pseudo BL-algebras and morphisms are homomorphisms of pseudo BL-algebras. Let \(\mathcal{DRL}_1(\ast)\) be the category of bounded DR\(\ell\)-monoids satisfying \((\ast)\). Its morphisms are homomorphisms of DR\(\ell\)-monoids which preserve also 1, thus in the sequel, bounded DR\(\ell\)-monoids are regarded as algebras \((A, +, 0, \lor, \land, \rightarrow, \leftarrow, 1)\) of type \(\langle 2, 0, 2, 2, 2, 2, 0 \rangle\).

Theorem 1.3. The categories \(\mathcal{PBL}\) and \(\mathcal{DRL}_1(\ast)\) are equivalent.

Proof. Theorems 1.1 and 1.2 enable us to define a functor \(F: \mathcal{PBL} \rightarrow \mathcal{DRL}_1(\ast)\) as follows: (i) \(F(\mathfrak{A}) = \mathfrak{A}_d\) for any pseudo BL-algebra \(\mathfrak{A}\), and (ii) \(F(h) = h\) for any pseudo BL-homomorphism \(h\). It is easy to see that \(F\) is really a categorical equivalence. \(\square\)
2. Filters

According to [3], a subset $F$ of a pseudo $BL$-algebra $\mathfrak{A}$ with the following properties is said to be a filter of $\mathfrak{A}$:

(F1) $1 \in F$;
(F2) $\forall x, y \in F; x \odot y \in F$;
(F3) $\forall x \in F \forall y \in A; x \leq y \Rightarrow y \in F$.

For any subset $M \subseteq A$, the intersection of all filters containing $M$ is called a filter generated by $M$ and denoted by $[M]$. It is clear that

$$[M] = \{x \in A; x \geq a_1 \odot \ldots \odot a_n \text{ for some } a_1, \ldots, a_n \in M \text{ and } n \geq 1\},$$

and if we write briefly $[a]$ for $\{(a)\}$ then

$$[a] = \{x \in A; x \geq a^n \text{ for some } n \geq 1\}.$$

In Section 1, we have already proved that $DRL$-monoids include the duals of pseudo $BL$-algebras. It is obvious that $F \subseteq A$ is a filter of a pseudo $BL$-algebra $\mathfrak{A}$ iff it is an ideal of the induced bounded $DRL$-monoid $\mathfrak{A}_d$, that is,

(I1) $0_d \in F$;
(I2) $\forall x, y \in F; x + y \in F$;
(I3) $\forall x \in F \forall y \in A; x \geq_d y \Rightarrow y \in F$.

Ideals of noncommutative $DRL$-monoids were studied in [9]. Considering the above facts, we immediately obtain the following results.

**Proposition 2.1.** The set of all filters of any pseudo $BL$-algebra $\mathfrak{A}$, ordered by set inclusion, is an algebraic Brouwerian lattice. For any filters $F, G$ of $\mathfrak{A}$, the relative pseudocomplement of $F$ with respect to $G$ is given by

$$F \ast G = \{a \in A; a \lor x \in G \text{ for all } x \in F\}.$$

Let $\mathfrak{A}$ be a pseudo $BL$-algebra and $X \subseteq A$. The set

$$X^\perp = \{a \in A; a \lor x = 1 \text{ for any } x \in X\}$$

is called the polar of $X$. For any $x \in A$ we write $x^\perp$ instead of $\{x\}^\perp$.

A subset $X$ of $A$ is a polar in $\mathfrak{A}$ iff $X = Y^\perp$ for some $Y \subseteq A$.

**Proposition 2.2** [3, Propositions 4.38, 4.39]. For all subsets $X, Y$ of a pseudo $BL$-algebra $\mathfrak{A}$, (i) $X^\perp$ is a filter of $\mathfrak{A}$, (ii) $X \subseteq X^{\perp\perp}$, (iii) $X \subseteq Y$ implies $Y^\perp \subseteq X^\perp$, (iv) $X^\perp = X^{\perp\perp\perp}$.
Proposition 2.3. For any subset $X$ of a pseudo $BL$-algebra $\mathfrak{A}$, $X$ is a polar in $\mathfrak{A}$ iff $X = X^\perp$. 

Proof. Let $X = Y^\perp$; then $X^\perp = Y^\perp\perp = Y = X$. \hfill \Box

By Proposition 2.1, the pseudocomplement of a filter $F$ is 

$$F^\ast = \{ a \in A; \ a \lor x = 1 \ \text{for any} \ x \in F \}.$$ 

Moreover, it is clear that $F^\perp = F^\ast$ whenever $F$ is a filter, and conversely, any polar is the pseudocomplement of some filter; in fact, $X = (X^\perp)^\ast$. Thus the polars in any pseudo $BL$-algebra are precisely the pseudocomplements in the lattice of its filters. Therefore, by the Glivenko-Frink Theorem, we directly obtain

Theorem 2.4. The set of all polars in any pseudo $BL$-algebra, ordered by set inclusion, is a complete Boolean algebra.

A filter $F$ of a pseudo $BL$-algebra $\mathfrak{A}$ is said to be normal iff it satisfies the following condition for each $x, y \in A$:

$$x \rightarrow y \in F \iff x \sim y \in F.$$ 

Proposition 2.5. For any filter $F$, the following conditions are equivalent:

(i) $F$ is normal;

(ii) $x \odot F = F \odot x$ for each $x \in A$.

Proposition 2.6. If $F$ and $G$ are normal filters of $\mathfrak{A}$ then

$$F \lor G = \{ x \in A; \ x \geq a \odot b \ \text{for some} \ a \in F, b \in G \}.$$ 

In addition, $F \lor G$ is a normal filter. Consequently, normal filters of any pseudo $BL$-algebra form a complete sublattice of the lattice of all its filters.

Theorem 2.7. In any pseudo $BL$-algebra, there is a one-to-one correspondence between the normal filters and the congruence relations. In fact, $F$ corresponds to $\Theta(F)$ defined by

$$\langle x, y \rangle \in \Theta(F) = \Theta_1(F) \iff (x \rightarrow y) \land (y \rightarrow x) \in F,$$

or equivalently,

$$\langle x, y \rangle \in \Theta(F) = \Theta_2(F) \iff (x \rightsquigarrow y) \land (y \rightsquigarrow x) \in F.$$
As proved in [3], and in general for noncommutative $DR\ell$-monoids in [9], if $F$ is not a normal filter then the binary relations defined in the previous theorem, $\Theta_1(F)$ and $\Theta_2(F)$, are two distinct congruence relations on the distributive lattice $\mathcal{L}(A) = (A, \vee, \wedge, 0, 1)$. In the quotient lattices $\mathcal{L}(A)/\Theta_1(F)$ and $\mathcal{L}(A)/\Theta_2(F)$ we have

\[(2.1)\quad [x]_{\Theta_1(F)} \leq [y]_{\Theta_1(F)} \iff x \rightarrow y \in F\]
and

\[(2.2)\quad [x]_{\Theta_2(F)} \leq [y]_{\Theta_2(F)} \iff x \Rightarrow y \in F,\]

respectively.

Let $\mathfrak{A}$ be a pseudo $BL$-algebra. A filter $F$ of $\mathfrak{A}$ is said to be prime if it is a finitely meet-irreducible element in the lattice of filters of $\mathfrak{A}$.

By [3, Theorem 4.28], for any filter $F$ of a pseudo $BL$-algebra $\mathfrak{A}$ and for each ideal $I$ of the lattice $\mathcal{L}(\mathfrak{A})$, if $F \cap I = \emptyset$ then there exists a prime filter $P$ of $\mathfrak{A}$ with $F \subseteq P$ and $P \cap I = \emptyset$. Consequently, every proper filter is the intersection of all prime filters including it. In particular, the intersection of all prime filters is equal to $\{1\}$.  

**Theorem 2.8.** For any filter $F$ of a pseudo $BL$-algebra $\mathfrak{A}$, the following conditions are equivalent:

(i) $F$ is prime;

(ii) for all filters $G, H$ of $\mathfrak{A}$, $G \cap H \subseteq F$ implies $G \subseteq F$ or $H \subseteq F$;

(iii) for any $x, y \in A$, $x \vee y \in F$ implies $x \in F$ or $y \in F$;

(iv) for any $x, y \in A$, $x \vee y = 1$ implies $x \in F$ or $y \in F$;

(v) for any $x, y \in A$, $x \rightarrow y \in F$ or $y \rightarrow x \in F$;

(vi) for any $x, y \in A$, $x \Rightarrow y \in F$ or $y \Rightarrow x \in F$;

(vii) $\mathcal{L}(\mathfrak{A})/\Theta_1(F)$ is totally ordered;

(viii) $\mathcal{L}(\mathfrak{A})/\Theta_2(F)$ is totally ordered;

(ix) the set of all filters including $F$ is totally ordered under set inclusion.

**Remark.** The equivalence of (iii), (v), (vi), (vii) and (viii) is due to [3, Proposition 4.25].

**Proof.** (i) $\Rightarrow$ (ii): Using the distributivity of the lattice of filters, $G \cap H \subseteq F$ implies $F = F \vee (G \cap H) = (F \vee G) \cap (F \vee H)$, whence $F = F \vee G$ or $F = F \vee H$, that is, $F \supseteq G$ or $F \supseteq H$.

(ii) $\Rightarrow$ (iii): Obviously, $x \vee y \in F$ yields $[x] \cap [y] = [x \vee y] \subseteq F$. Hence, by (ii), $[x] \subseteq F$ or $[y] \subseteq F$ and thus $x \in F$ or $y \in F$.

(iii) $\Rightarrow$ (iv): This is evident since $1 \in F$.  

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(iv) ⇒ (v) and (iv) ⇒ (vi): By the definition of a pseudo BL-algebra,

$$(x \rightarrow y) \lor (y \rightarrow x) = (x \sqsubseteq y) \lor (y \sqsubseteq x) = 1,$$

which implies the assertion by (iv).

(v) ⇒ (vii) and (vi) ⇒ (viii): This is obvious from (2.1) and (2.2), respectively.

(vii) ⇒ (ix): If $F \subseteq G, H$ and neither $G \subseteq H$ nor $H \subseteq G$ then there exist $a, b \in A$ with $a \in G \setminus H$ and $b \in H \setminus G$. For instance, let $a \rightarrow b \in F$. Then $b \supseteq a \land b = (a \rightarrow b) \otimes a \in G$, whence $b \in G$; a contradiction. Similarly (viii) ⇒ (ix).

(ix) ⇒ (i): $F = G \cap H$ entails $F = G$ or $F = H$, because either $G \subseteq H$ or $H \subseteq G$. \[\square\]

3. Representable pseudo BL-algebras

**Proposition 3.1.** If $P$ is a minimal prime filter of a pseudo BL-algebra $\mathfrak{A}$ then $A \setminus P$ is a maximal ideal of the lattice $\mathcal{L}(\mathfrak{A})$.

**Proof.** By Zorn’s Lemma, there is a maximal ideal $I$ of $\mathcal{L}(\mathfrak{A})$ with $A \setminus P \subseteq I$. (Since $P$ is also a prime filter of $\mathcal{L}(\mathfrak{A})$, it follows that $A \setminus P$ is a prime ideal of $\mathcal{L}(\mathfrak{A})$ which is included in some maximal (prime) ideal.) We will show that $I = A \setminus P$. Denote $Q = \bigcup \{a^\perp; a \in I\}$. We claim that $P = Q$.

If $x \in a^\perp$ for some $a \in I$, then $x \lor a = 1$ and $x \notin I$. Indeed, if $x \in I$ then $x \lor a \neq 1$ since $x \lor a = 1$ would mean $I = A$. Thus $x \in A \setminus I \subseteq A \setminus (A \setminus P) = P$, whence $a^\perp \subseteq A \setminus I \subseteq P$ and consequently, $Q \subseteq A \setminus I \subseteq P$.

We shall now prove that $Q$ is a prime filter of $\mathfrak{A}$. (F1): Since any principal polar $a^\perp$ contains 1, so does $Q$. (F2): If $x, y \in Q$, that is, $x \in a^\perp, y \in b^\perp$ for some $a, b \in I$, then $a \lor b \in I$ and

$$(x \otimes y) \lor a \lor b \geq (x \lor a \lor b) \lor (y \lor a \lor b) = 1 \lor 1 = 1.$$

Therefore $x \otimes y \in (a \lor b)^\perp \subseteq Q$. (F3): It is obvious since $a^\perp$ is a filter of $\mathfrak{A}$ for each $a \in I$.

To prove that $Q$ is prime, suppose $x \lor y = 1$ and $x \notin Q$, that is, $x \lor a \neq 1$ for all $a \in I$. If $x \notin I$ then the ideal in the lattice $\mathcal{L}(\mathfrak{A})$ generated by $I \cup \{x\}, (I \cup \{x\})$, is proper, i.e., $A \setminus P \subseteq I \subset (I \cup \{x\}) \neq A$, since $(I \cup \{x\}) = A$ would entail $1 \leq x \lor a$ for some $a \in I$; a contradiction. Hence $x \in I$ and thus $y \in x^\perp \subseteq Q$, proving that $Q$ is prime.

However, $P$ is a minimal prime filter of $\mathfrak{A}$; thus $Q \subseteq A \setminus I \subseteq P$ yields $Q = A \setminus I = P$ as claimed. Therefore $I = A \setminus P$. \[\square\]
Corollary 3.2. If $P$ is a minimal prime filter then

$$P = \bigcup \{a^\perp; a \notin P\}.$$

Proof. By the proof of the previous proposition, $P = \bigcup \{a^\perp; a \in I\}$, where $I = A \setminus P$. □

A pseudo BL-algebra is said to be representable if it is a subdirect product of linearly ordered pseudo BL-algebras.

By Theorems 2.7 and 2.8, subdirect representations by totally ordered pseudo BL-algebras are associated with families of normal prime filters whose intersections are precisely \{1\}. Therefore it is obvious that every BL-algebra is representable (see also [11]). In contrast, for pseudo BL-algebras, this assertion fails.

The following results generalize the similar properties of pseudo MV-algebras, [4, Theorem 2.20], [1, Theorem 5.9], and [2, Theorem 6.11].

Theorem 3.3. For any pseudo BL-algebra $\mathfrak{A}$, the following statements are equivalent.

(i) $\mathfrak{A}$ is representable.

(ii) There exists a family $\{P_i\}_{i \in I}$ of normal prime filters of $\mathfrak{A}$ such that

$$\bigcap_{i \in I} P_i = \{1\}.$$

(iii) Any polar of $\mathfrak{A}$ is a normal filter of $\mathfrak{A}$.

(iv) Any principal polar is a normal filter.

(v) Any minimal prime filter is normal.

Proof. As argued above, the equivalence of (i) and (ii) is clear.

(i) $\Rightarrow$ (iii): Suppose that $\mathfrak{A}$ is a subdirect product of linearly ordered pseudo BL-algebras $\{\mathfrak{A}_i\}_{i \in I}$. Observe that

$$(3.1) \quad x \lor y = 1 \text{ iff } \{i \in I; x_i \neq 1_i\} \cap \{i \in I; y_i \neq 1_i\} = \emptyset$$

for all $x, y \in A$, since $\mathfrak{A}_i$ are totally ordered.

Let now $P$ be a polar in $\mathfrak{A}$, i.e. $P = P^\perp \perp$. Let $x \in A, a \in P$ and $y \in P^\perp$. Then $x \circ a \leq x$ implies $x \circ a = (x \circ a) \land x = (x \rightarrow (x \circ a)) \circ x$. Further, $\{i \in I; x_i \rightarrow (x_i \circ a_i) \neq 1_i\} \subseteq \{i \in I; a_i \neq 1_i\}$. Indeed, if $a_i = 1_i$ then $x_i \rightarrow (x_i \circ a_i) = x_i \rightarrow (x_i \circ 1_i) = x_i \rightarrow x_i = 1_i$. Hence we obtain

$$\{i \in I; x_i \rightarrow (x_i \circ a_i) \neq 1_i\} \cap \{i \in I; y_i \neq 1_i\} \subseteq \{i \in I; a_i \neq 1_i\} \cap \{i \in I; y_i \neq 1_i\} = \emptyset$$
by (3.1), since $a \in P$ and $y \in P^\perp$. Therefore $(x \to (x \circ a)) \lor y = 1$, and thus $x \to (x \circ a) \in P^{\perp \perp} = P$. Hence $x \circ a = (x \to (x \circ a)) \circ x \in P \circ x$, proving $x \circ P \subseteq P \circ x$.

(iii) $\Rightarrow$ (iv): Obvious.

(iv) $\Rightarrow$ (v): By Corollary 3.2, $P = \bigcup \{a^\perp; a \notin P\}$ for any minimal prime filter $P$. If $x \to y \in P$ then there is $a \notin P$ with $x \to y \in a^\perp$ which is a normal filter, and hence $x \leadsto y \in a^\perp \subseteq P$. Summarizing, $x \to y \in P$ iff $x \leadsto y \in P$.

(v) $\Rightarrow$ (i): Since any prime filter contains a minimal prime filter and the intersection of all prime filters of $\mathfrak{A}$ is obviously $\{1\}$, so does the intersection of minimal prime filters. Thus, by (ii), $\mathfrak{A}$ is representable.

\begin{proof}

Theorem 3.4. A pseudo BL-algebra is representable if and only if it satisfies the identities

\begin{align*}
(3.2) & \quad (y \to x) \lor (z \leadsto ((x \to y) \circ z)) = 1, \\
(3.3) & \quad (y \leadsto x) \lor (z \to (z \circ (x \leadsto y))) = 1.
\end{align*}

Consequently, the class of representable pseudo BL-algebras is a variety.

In any linearly ordered pseudo BL-algebra $\mathfrak{A}$, either $y \to x = 1$ or $x \to y = 1$ (and also $y \leadsto x = 1$ or $x \leadsto y = 1$), and so it is easy to verify that $\mathfrak{A}$ fulfils (3.2) and (3.3). Therefore the part “only if” is obvious.

Conversely, suppose that (3.2) and (3.3) are satisfied by $\mathfrak{A}$. In view of Theorem 3.3 (iv), it suffices to prove that any principal polar $x^\perp$ is a normal filter of $\mathfrak{A}$.

Let $y \in x^\perp$, that is, $y \lor x = 1$. Observe that in this case

$$x = 1 \to x = (y \lor x) \to x = (y \to x) \land (x \to x) = (y \to x) \land 1 = y \to x$$

and similarly $y = x \to y$. Hence, by (3.2),

$$x \lor (z \leadsto (y \circ z)) = (y \to x) \lor (z \leadsto ((x \to y) \circ z)) = 1,$$

thus $z \leadsto (y \circ z) \in x^\perp$. Further, $y \circ z \leq z$ implies $y \circ z = (y \circ z) \land z = z \circ (z \leadsto (y \circ z)) \in z \circ x^\perp$, which shows $x^\perp \circ z \subseteq z \circ x^\perp$. The other inclusion follows similarly by (3.3). \hfill \Box

\end{proof}
References


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