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ORTHOGONALLY ADDITIVE FUNCTIONALS ON BV

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Abstract. In this paper we give a representation theorem for the orthogonally additive functionals on the space BV in terms of a non-linear integral of the Henstock-Kurzweil-Stieltjes type.

Keywords: functional, orthogonally additive functional, two-norm space, function of bounded variation, Henstock integral, Stieltjes integral

MSC 2000: 46A70, 26A45, 26A39, 26A42

1. INTRODUCTION

Orthogonally additive functionals on function spaces have been studied by Orlicz and other authors; see references in [4]. In particular, Chew [4] proved a representation theorem for orthogonally additive functionals on the Denjoy space, the space of all Henstock-Kurzweil integrable functions on an interval $[a, b]$, in terms of a nonlinear Henstock-Kurzweil integral. Also, a representation theorem for boundedly continuous linear functionals defined on BV , the space of all functions of bounded variation, has been proved by Hildebrandt [3] using the left Cauchy integral. In this paper we prove a representation theorem for orthogonally additive functionals defined on BV , making use of the nonlinear integral and hence extending the result of Hildebrandt.

Let BV denote the space of functions of bounded variation on $[a, b]$, that is, $f \in BV$ if the total variation $V(f)$ of f on $[a, b]$ is finite. A functional F defined on BV is *orthogonally additive* if $F(f + g) = F(f) + F(g)$ for all $f, g \in BV$ such that $f(x)g(x) = 0$ except for finitely many x in $[a, b]$. A functional F is said to be *boundedly continuous* on BV if $F(f_n) \rightarrow F(f)$ as $n \rightarrow \infty$ whenever for every $x \in [a, b]$, $f_n(x) \rightarrow f(x)$ as $n \rightarrow \infty$ and there exists $M > 0$ such that $V(f_n) \leq M$ for every n . In this paper we shall prove that if F is an orthogonally additive and boundedly

continuous functional on BV , then F can be represented by a non-linear integral of the Henstock-Kurzweil-Stieltjes type. The full detail is given in Theorem 3.

2. A NON-LINEAR INTEGRAL

We introduce a non-linear integral of the Henstock-Kurzweil-Stieltjes type [4, p. 81]. Let $h = h(s, I)$ be a *point-interval function* defined for s being a real number and $I = [u, v] \subset [a, b]$. A real-valued function f is said to be *h -integrable* to A on a compact interval $[a, b]$ if for every $\varepsilon > 0$ there is a function $\delta(\xi) > 0$ for $\xi \in [a, b]$ such that for any division D of $[a, b]$ given by $a = x_0 < x_1 < \dots < x_n = b$, with $\xi_1, \xi_2, \dots, \xi_n$ satisfying $\xi_i \in [x_{i-1}, x_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ for $i = 1, 2, \dots, n$, we have

$$\left| \sum_{i=1}^n h(f(\xi_i), [x_{i-1}, x_i]) - A \right| < \varepsilon.$$

For brevity, we write $D = \{(\xi, [u, v])\}$ where $(\xi, [u, v])$ denotes a typical point-interval pair $(\xi_i, [x_{i-1}, x_i])$ in D , and also we write the Riemann sum in the form $(D) \sum h(f(\xi), [u, v])$. Here D is said to be *δ -fine* if the above condition holds. In short, f is *h -integrable* on $[a, b]$ if for every $\varepsilon > 0$, there is a positive function δ such that for any δ -fine division $D = \{(\xi, [u, v])\}$ of $[a, b]$ we have

$$\left| (D) \sum h(f(\xi), [u, v]) - A \right| < \varepsilon.$$

For simplicity, we write the h -integral $\int_a^b h(f(x), dx) = A$. For example, when δ is a constant function and $h(f(x), [u, v]) = f(x)[g(v) - g(u)]$, the h -integral reduces to the well-known *Riemann-Stieltjes integral*.

We give a list of conditions on $h(s, I)$ which guarantee that the h -integral becomes meaningful.

(N1) $h(0, I) = 0$ for all intervals $I \subset [a, b]$.

(N2) $h(s, I)$, as a function of s , is continuous on the real line for all intervals $I \subset [a, b]$.

(N3) $h(s, I)$, as a function of I , is additive, i.e., $h(s, I_1 \cup I_2) = h(s, I_1) + h(s, I_2)$ whenever I_1, I_2 are nonoverlapping and adjacent, for all $I_1, I_2 \subset [a, b]$. That is, $I_1 \cup I_2$ is again an interval.

(N4) For every $M > 0$ and for every $\varepsilon > 0$ there exists $\eta > 0$ such that

$$\left| \sum_{i=1}^p h(s_i, I_i) - \sum_{i=1}^p h(t_i, I_i) \right| < \varepsilon$$

whenever $|s_i - t_i| < \eta$, $|s_i| \leq M$, $|t_i| \leq M$ for every i and I_1, I_2, \dots, I_p are pairwise nonoverlapping.

Here we note that N2 follows from N4. For easy reference, we keep to the same labelling of conditions as in P. Y. Lee [4, p. 82] for N1 to N4. In our case, N5 in P. Y. Lee [4, p. 82] is not used. For reference, we state N5 here.

(N5) For every $M > 0$ and for every $\varepsilon > 0$ there exists $\eta > 0$ such that $|\sum_{i=1}^p h(s_i, I_i)| < \varepsilon$ whenever $\sum_{i=1}^p |I_i| < \eta$, $|s_i| \leq M$, for every i and I_1, I_2, \dots, I_p are nonoverlapping.

Furthermore, we state one more condition, namely N6, as required for our case.

(N6) For every $M > 0$ and $|s| \leq M$, the limit $\lim_{u \uparrow c} h(s, [u, c])$ exists for $c \in (a, b)$ and so does $\lim_{v \downarrow c} h(s, [c, v])$ for $c \in [a, b)$.

We remark that N5 is an essential condition in [4]. Here it is N6 that we need. Note that N5 implies N6 but not conversely. In what follows and throughout the paper, we assume that $h(s, I)$ is fixed and satisfies N1–N4 and N6. For other papers on the nonlinear integral, see references in [4], [5].

A function g^* defined on $[a, b]$ is said to be a *normalized function* of g if $g^*(x) = \frac{1}{2}[g(x+) + g(x-)]$ for every $x \in (a, b)$ and $g^*(a) = g(a+)$, $g^*(b) = g(b-)$.

Lemma 1. *If $h(s, I)$ satisfies N1–N4 and N6, then $\int_a^b h(\varphi(x), dx)$ exists for any step function φ .*

Proof. It is sufficient to prove the lemma for $\varphi = \chi_{[c,d]}$, for some $[c, d] \subseteq [a, b]$. In view of N6, we can prove that

$$\int_a^b h(\varphi(x), dx) = \lim_{u \rightarrow c-} h(s, [u, c]) + h(s, [c, d]) + \lim_{v \rightarrow d+} h(s, [d, v]).$$

□

Theorem 1. *Let $\{f_n\}$ be a sequence of h -integrable functions uniformly bounded on $[a, b]$. If $\{f_n\}$ is uniformly convergent to f on $[a, b]$, then f is h -integrable on $[a, b]$ and $\lim_{n \rightarrow \infty} \int_a^b h(f_n(x), dx) = \int_a^b h(f(x), dx)$.*

The proof is standard and therefore omitted [5].

A function f defined on $[a, b]$ is said to be a *regulated function* [1] if f is the limit of a uniformly convergent sequence of step functions on $[a, b]$.

Corollary 1. *If f is a regulated function, then it is h -integrable on $[a, b]$.*

3. CONTINUOUS FUNCTIONALS

We introduce two more continuity concepts of functionals on BV . A sequence $\{f_n\}$ is *bounded* in BV if there is $M > 0$ such that $V(f_n) \leq M$ for all n . Also we write $\|f\| = \sup\{|f(x)| : a \leq x \leq b\}$. A functional F is said to be *uniformly continuous* on BV if $F(f_n) - F(g_n) \rightarrow 0$ as $n \rightarrow \infty$ whenever $\|f_n - g_n\| \rightarrow 0$ as $n \rightarrow \infty$ with $\{f_n\}$ and $\{g_n\}$ bounded in BV . Further, a functional F is said to be *two-norm continuous* on BV if it is uniformly continuous with g_n replaced by f . It is obvious that if F is uniformly continuous on BV , then F is two-norm continuous on BV . It is well-known that functions of bounded variation satisfy *Helly's choice property* [2], [8]. It states that if a sequence $\{g_n\}$ is bounded in BV , then there is a subsequence $\{f_k\}$ of $\{g_n\}$ and a function $f \in BV$ such that for every $x \in [a, b]$, $f_k(x) \rightarrow f(x)$ as $n \rightarrow \infty$.

Lemma 2. *If F is boundedly continuous on BV , then F is uniformly continuous on BV .*

Proof. Suppose F is boundedly continuous on BV . Take two sequences $\{f_n\}$ and $\{g_n\}$ in BV such that $\|f_n - g_n\| \rightarrow 0$ as $n \rightarrow \infty$ and $\{f_n\}, \{g_n\}$ are bounded in BV . Since BV satisfies Helly's choice property, we obtain a subsequence of $\{f_n\}$, denoted by $\{f_{n_i}\}$, such that f_{n_i} is boundedly convergent to f on $[a, b]$. Further, take a subsequence of $\{g_n\}$, denoted by $\{g_{n_i}\}$, such that for every $x \in [a, b]$, $g_{n_i}(x) \rightarrow g(x)$. In view of $\|f_n - g_n\| \rightarrow 0$, we have for every $x \in [a, b]$, $g_{n_i}(x) \rightarrow f(x)$ as $n_i \rightarrow \infty$.

Since F is boundedly continuous, $F(f_{n_i}) - F(f) \rightarrow 0$ and $F(g_{n_i}) - F(f) \rightarrow 0$ as $n_i \rightarrow \infty$. Therefore, $F(f_{n_i}) - F(g_{n_i}) \rightarrow 0$ as $n_i \rightarrow \infty$. Consequently, F is uniformly continuous on BV . □

Theorem 2. *Suppose $h(s, I)$ satisfies N1–N4 and N6. If a functional F defined on BV is given by $F(f) = \int_a^b h(f(x), dx)$ for every $f \in BV$, then F is orthogonally additive and uniformly continuous on BV . Furthermore, F is two-norm continuous on BV .*

Proof. The orthogonal additivity follows from N1. Next, take $f_n, g_n \in BV$ such that $\|f_n - g_n\| \rightarrow 0$ as $n \rightarrow \infty$ and $\{f_n\}, \{g_n\}$ are bounded in BV . Since f_n and g_n are h -integrable on $[a, b]$, for every $\varepsilon > 0$ and every n there exists a function $\delta_n(\varepsilon) > 0$ such that for every δ_n -fine division $D = \{(\xi, [u, v])\}$ of $[a, b]$ we have

$$\left| F(f_n) - (D) \sum h(f_n(\xi), [u, v]) \right| < \frac{\varepsilon}{3} \quad \text{and} \quad \left| F(g_n) - (D) \sum h(g_n(\xi), [u, v]) \right| < \frac{\varepsilon}{3}.$$

Since $\|f_n - g_n\| \rightarrow 0$ as $n \rightarrow \infty$ and N4 holds with $s_i = f_n(\xi_i)$ and $t_i = g_n(\xi_i)$, we can show that $|(D) \sum h(f_n(\xi), [u, v]) - (D) \sum h(g_n(\xi), [u, v])| < \frac{\varepsilon}{3}$ for large n .

Consequently, $F(f_n) - F(g_n) \rightarrow 0$ as $n \rightarrow \infty$. Therefore F is a uniformly continuous functional on BV .

Finally, the two-norm continuity follows from the definition. □

We remark that, as shown in [4], if $h(s, I)$ satisfies N1–N5, then the functional F defined by the h -integral is boundedly continuous. Since we have not had N5 in Theorem 2, bounded continuity does not follow.

4. REPRESENTATION THEOREM ON BV

In this section we give a series of lemmas leading to the main theorem which is the representation theorem for boundedly continuous orthogonally additive functionals on BV .

Lemma 3. *If $f \in BV$, then there exists a sequence of step functions f_n such that $\|f_n - f\| \rightarrow 0$ as $n \rightarrow \infty$ and $\{f_n\}$ is bounded in BV .*

The proof is standard and therefore omitted.

Lemma 4. *If F is an orthogonally additive and boundedly continuous functional on BV , then $h(s, I)$ satisfies N1–N4 and N6, where $h(s, I) = F(s\chi_I^*)$ and χ_I^* is the normalized function of the characteristic function χ_I of $I = [u, v]$.*

Proof. The proofs of N1 and N3 follow from the orthogonal additivity and the proof of N2 from the fact that F is a boundedly continuous functional on BV .

We now prove N4. Suppose it is false. Then we shall deduce that F is not uniformly continuous on BV . In view of Lemma 2, this contradicts the fact that F is boundedly continuous on BV .

If N4 does not hold, then there exist $M > 0$ and $\varepsilon > 0$ for every $\eta > 0$ such that there exist x_i, y_i and $I_i, 1 \leq i \leq k$, pairwise nonoverlapping, satisfying $|x_i| \leq M, |y_i| \leq M, |x_i - y_i| < \eta$ for every i and $|\sum_{i=1}^k h(x_i, I_i) - \sum_{i=1}^k h(y_i, I_i)| \geq \varepsilon$. Take $\eta = \frac{1}{n}$. Then there exist $x_{n,i}, y_{n,i}$ and $I_{n,i}, 1 \leq i \leq k_n$, such that $|x_{n,i} - y_{n,i}| < \frac{1}{n}$ for every $i, |x_{n,i}| \leq M, |y_{n,i}| \leq M$ and

$$\left| \sum_{i=1}^{k_n} h(x_{n,i}, I_{n,i}) - \sum_{i=1}^{k_n} h(y_{n,i}, I_{n,i}) \right| \geq \varepsilon.$$

Put $f_n = \sum_{i=1}^{k_n} x_{n,i} \chi_{I_{n,i}}^*$ and $g_n = \sum_{i=1}^{k_n} y_{n,i} \chi_{I_{n,i}}^*$. Then we have $\|f_n\| \leq M, \|g_n\| \leq M$ for every n and $\|f_n - g_n\| \leq \frac{1}{n} \rightarrow 0$ as $n \rightarrow \infty$, but $|F(f_n) - F(g_n)| \geq \varepsilon$. That is, F is not uniformly continuous on BV .

We now prove N6. Take $u_n \uparrow c$ as $n \rightarrow \infty$. Then $s\chi_{[u_n, c]}^*$ is pointwise convergent to $s\chi_{[c, c]}^*$ and $\|s\chi_{[u_n, c]}^*\| \leq |s|$ for every n . Put $f_n = s\chi_{[u_n, c]}^*$ and $f = s\chi_{[c, c]}^*$. Obviously, f_n is boundedly convergent to f in BV . Since F is a boundedly continuous functional on BV , we have $|F(f_n) - F(f)| \rightarrow 0$ as $n \rightarrow \infty$. That is, $\lim_{u_n \uparrow c} h(s, [u_n, c])$ exists and similarly, we can prove that $\lim_{v_n \downarrow c} h(s, [c, v_n])$ exists. \square

Note that we require χ_I^* in the definition of $h(s, I)$ in order to prove N3. Replacing χ_I^* by χ_I would not be sufficient.

Lemma 5. *Suppose F is an orthogonally additive and boundedly continuous functional on BV and $h(s, I)$ satisfies N1–N4 and N6 where $h(s, I) = F(s\chi_I^*)$. Then the h -integral exists and $F(f) = \int_a^b h(f(x), dx)$ for every normalized $f \in BV$.*

Proof. In view of Lemma 3, it is sufficient to prove the assertion for every normalized step function φ . Let φ be the step function which we have defined in Lemma 1. Then by Lemma 1, $\int_a^b h(\varphi(x), dx)$ exists and $\int_a^b h(\varphi(x), dx) = A$. Then there exist $\delta_n(\xi) > 0$ for $\xi \in [a, b]$ and a δ_n -fine division $D_n = \{(\xi, [u_n, v_n])\}$ of $[a, b]$ such that

$$\left| (D_n) \sum h(\varphi(\xi), [u_n, v_n]) - A \right| < \frac{1}{n}.$$

Denote $\varphi_n = (D_n) \sum \varphi(\xi) \chi_{[u_n, v_n]}^*$. We may choose D_n so that for every $x \in [a, b]$, $\varphi_n(x) \rightarrow \varphi(x)$ as $n \rightarrow \infty$. Then φ_n is boundedly convergent to φ in BV . Since F is a boundedly continuous functional on BV , we have $F(\varphi_n) \rightarrow F(\varphi)$ as $n \rightarrow \infty$. Since $h(s, I) = F(s\chi_I^*)$, we have $(D_n) \sum h(\varphi(\xi), [u_n, v_n]) = F((D_n) \sum \varphi(\xi) \chi_{[u_n, v_n]}^*)$ and therefore $F(\varphi) = \lim_{n \rightarrow \infty} F(\varphi_n) = A = \int_a^b h(\varphi(x), dx)$. \square

We state the main theorem of this paper as follows:

Theorem 3. *If F is an orthogonally additive and boundedly continuous functional on BV , then there exists $h(s, I)$ satisfying N1–N4 and N6 such that*

$$F(f) = \int_a^b h(f^*(x), dx) + \sum_{i=1}^{\infty} F([f(t_i) - f^*(t_i)]\delta_{t_i})$$

for every $f \in BV$, where $\delta_t(x) = 1$ when $x = t$ and 0 otherwise, $t_i, i = 1, 2, \dots$, are the discontinuity points of f , and f^* is the normalized function of f .

Proof. Let f^* be the normalized function of f . Then $F(f) = F(f^*) + F(f - f^*)$. It follows from Lemma 5 that $F(f^*) = \int_a^b h(f^*(x), dx)$ and it remains to prove $F(f - f^*) = \sum_{i=1}^{\infty} F([f(t_i) - f^*(t_i)]\delta_{t_i})$.

Let $t_i, i = 1, 2, \dots$, be the discontinuity points of f . Then we have $(f - f^*)(x) = \sum_{i=1}^{\infty} [f(t_i) - f^*(t_i)]\delta_{t_i}(x)$. Since $\sum_{i=1}^{\infty} [f(t_i) - f^*(t_i)]\delta_{t_i}(x)$ converges for every x , we have $\sum_{i=n}^{\infty} [f(t_i) - f^*(t_i)]\delta_{t_i}(x) \rightarrow 0$ as $n \rightarrow \infty$. Also, $\{\sum_{i=n}^{\infty} [f(t_i) - f^*(t_i)]\delta_{t_i}\}$ is bounded in BV . By Lemma 2 and Helly's choice property, F is two-norm continuous on BV and therefore $\lim_{n \rightarrow \infty} F(\sum_{i=n}^{\infty} [f(t_i) - f^*(t_i)]\delta_{t_i}) = 0$. That is, $\sum_{i=n}^{\infty} F([f(t_i) - f^*(t_i)]\delta_{t_i})$ converges and

$$F(f - f^*) = \sum_{i=1}^{\infty} F([f(t_i) - f^*(t_i)]\delta_{t_i}).$$

The fact that $h(s, I)$ satisfies N1–N4 and N6 follows from Lemma 4. □

When F is boundedly continuous and linear in Theorem 3, we have $h(s, I) = sg_1(I)$ where $g_1(I) = F(s\chi_I)$. Here $F(s\chi_I^*) = sF(\chi_I)$. In view of N6 and the fact that f is a regulated function if and only if it has one-sided limits, we obtain that g_1 is a regulated function. Further, write $g_2(t) = F(\delta_t)$ for $t \in [a, b]$. We can prove by contradiction to the bounded continuity of F that g_2 is bounded on $[a, b]$. Hence we obtain a corollary of Theorem 3 as follows:

Corollary 2. *If F is a linear and boundedly continuous functional on BV , then there exist a regulated function g_1 and a bounded function g_2 such that*

$$F(f) = \int_a^b f^* dg_1 + \sum_{i=1}^{\infty} [f(t_i) - f^*(t_i)]g_2(t_i)$$

for every $f \in BV$, where $t_i, i = 1, 2, \dots$, are the discontinuity points of f and f^* is the normalized function of f .

This is equivalent to a result by Hildebrandt [3]. In his version, he expressed it in terms of the left or right Cauchy integral.

5. THE SPACE OF REGULATED FUNCTIONS

A corresponding result of Theorem 3 holds true for the space RF of all regulated functions. We shall sketch a proof in this section. We shall define the boundedness of a sequence in RF and the bounded continuity of a functional on RF .

It is known [2, p. 48] that $f \in RF$ if and only if for every $\varepsilon > 0$ the bounded ε -variation $V_\varepsilon(f)$ of f on $[a, b]$ is finite, where

$$V_\varepsilon(f) = \inf\{V(g) : g \in BV \text{ and } |f(x) - g(x)| \leq \varepsilon \text{ for every } x \in [a, b]\},$$

where ε is given and fixed. A sequence $\{f_n\}$ is said to be *bounded in RF* if for every $\varepsilon > 0$ there exists $M_\varepsilon > 0$ such that $\|f_n\| \leq M_\varepsilon$ and $V_\varepsilon(f_n) \leq M_\varepsilon$ for all n . Then a functional F on RF is said to be *boundedly continuous* if $F(f_n) \rightarrow F(f)$ as $n \rightarrow \infty$ whenever for every $x \in [a, b]$, $f_n(x) \rightarrow f(x)$ as $n \rightarrow \infty$ and $\{f_n\}$ is bounded in RF . Furthermore, *Helly's choice theorem* for RF was proved by Dana Franková in [2, Theorem 3.8, p. 51].

Theorem 4. *Suppose $\{g_n\}$ is bounded in RF . Then there is a subsequence $\{f_k\}$ of $\{g_n\}$ and a function $f \in RF$ such that for every $x \in [a, b]$, $f_k(x) \rightarrow f(x)$ as $n \rightarrow \infty$.*

Then following the same argument as above, we obtain results analogous to Theorem 3 and Corollary 2 with BV replaced by RF .

Theorem 5. *If F is an orthogonally additive and boundedly continuous functional on RF , then there exists $h(s, I)$ satisfying N1–N4 and N6 such that*

$$F(f) = \int_a^b h(f^*(x), dx) + \sum_{i=1}^{\infty} F([f(t_i) - f^*(t_i)]\delta_{t_i})$$

for every $f \in RF$, where $\delta_t(x) = 1$ when $x = t$ and 0 otherwise, t_i , $i = 1, 2, \dots$, are the discontinuity points of f , and f^* is the normalized function of f .

Corollary 3. *If F is a linear and boundedly continuous functional on RF , then there exist a function $g_1 \in BV$ and a function g_2 such that the infinite series below converges and*

$$F(f) = \int_a^b f^* dg_1 + \sum_{i=1}^{\infty} [f(t_i) - f^*(t_i)]g_2(t_i)$$

for every $f \in RF$, where t_i , $i = 1, 2, \dots$, are the discontinuity points of f , and f^* is the normalized function of f .

A special case of Corollary 3 has been proved by Tvrdý [7], where every function in RF is assumed to be normalized.

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