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RADIUS-INVARIANT GRAPHS

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Abstract. The eccentricity $e(v)$ of a vertex $v$ is defined as the distance to a farthest vertex from $v$. The radius of a graph $G$ is defined as $r(G) = \min_{u \in V(G)} \{e(u)\}$. A graph $G$ is radius-edge-invariant if $r(G - e) = r(G)$ for every $e \in E(G)$, radius-vertex-invariant if $r(G - v) = r(G)$ for every $v \in V(G)$ and radius-adding-invariant if $r(G + e) = r(G)$ for every $e \in E(G)$. Such classes of graphs are studied in this paper.

Keywords: radius of graph, radius-invariant graphs

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1. Introduction

All graphs considered in this paper are undirected, finite connected without loops or multiple edges.

Let $G$ be a graph. Then $V(G)$ denotes the vertex set of $G$; $E(G)$ the edge set of $G$; $d_G(u, v)$ (or simply $d(u, v)$) the distance between two vertices $u, v$ in $G$; $e(u)$ the eccentricity of $u$. The radius $r(G)$ and the diameter $d(G)$ are the minimum and maximum of the vertex eccentricities, respectively. The center $C(G)$ is the set of vertices with minimum eccentricities and $\Delta(G)$ is the maximum degree of $G$. The notions and notations not defined here are used accordingly to the book [1].

It is well known (see [1]) that: $r(G - e) \geq r(G)$ for all edges $e$ of $G$; $r(G - v) \geq r(G)$ for all vertices $v$ of $G$; $r(G + e) \leq r(G)$ for all edges of the complement of $G$. Using these inequalities there have been defined and studied the following graphs (see [1, 6]):

1. minimal graphs if $r(G - e) > r(G)$ for every $e \in E(G)$;
2. critical graphs if $r(G - v) \neq r(G)$ for every $v \in V(G)$;

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(3) maximal graphs if \( r(G + e) < r(G) \) for every \( e \in E(G) \).

The paper [8] defines three changing and three unchanging invariants by \( i \), where \( i \) is any graph invariant such as radius, diameter, etc. Three changing invariants are usually used in the same way as minimal, critical and maximal graphs by a given parameter \( i \), see [4], [7]. Three unchanging invariants are new and we shall call them radius-edge-invariant, radius-vertex-invariant, and radius-adding-invariant.

**Definition.** A graph \( G \) is:

1. radius-edge-invariant (r.e.i.) if \( r(G - e) = r(G) \) for every \( e \in E(G) \);
2. radius-vertex-invariant (r.v.i.) if \( r(G - v) = r(G) \) for every \( v \in V(G) \);
3. radius-adding-invariant (r.a.i.) if \( r(G + e) = r(G) \) for every \( e \in E(G) \).

These classes of graphs were studied up till now in the two papers [2] and [4]. In our paper we first prove some existence results for these graphs, then we show how we can construct radius invariant graphs by some operations on graphs, and finally we give some bounds for the number of vertices as maximum and minimum degree in such graphs.

### 2. Existence results

Now we give some existence results for radius-edge-invariant, radius-vertex-invariant and radius-adding-invariant graphs. The situation is very easy for \( r = 1 \).

A graph \( G \) with \( n \) vertices is r.e.i. if and only if it contains at least three vertices of degree \( n - 1 \), r.v.i. if and only if it contains at least 2 vertices of degree \( n - 1 \). Every graph of radius 1 is r.a.i.

The existence of such graphs is more complicated for radius \( r \geq 2 \). We were unable to find exact characterization of r.e.i. graphs of radius 2. Walikar et. al [2] gives the following observation:

**Proposition 2.1.** A graph \( G \) with \( n \) vertices and diameter 2 is r.e.i. if and only if \( G \) contains at least three vertices of degree \( n - 1 \) or \( G \) is self-centered.

The next observation is obvious.

**Proposition 2.2.** If a graph \( G \) of radius 2 is r.v.i., then

1. the maximum degree \( \Delta(G) \leq n - 3 \),
2. \( |C(G)| \geq 2 \),
3. \( G \) is vertex 2-connected.
Proposition 2.3. Let $G$ be a graph with $n$ vertices which satisfies all properties (1)–(3) of Proposition 2.2 and, moreover let $r(G) = d(G) = 2$. Then $G$ is r.v.i.

**Proof.** We will prove this proposition by contradiction.

If $r(G - v) < r(G)$ then there exists a vertex $u \in V(G - v)$ such that $\text{deg}_{G - v}(u) = |V(G - v)| - 1$. It must be $\text{deg}_G(u) \geq |V(G)| - 2$, a contradiction.

If $r(G - v) > r(G)$ then for all central vertices $c_i$ of the graph $G$ there holds $d(c_i, v) = 1$. In the other case, for a central vertex $c$ such that $d(c, v) = 2$ there hold $r(G - v) \leq e_{G - v}(c) \leq 2 = r(G)$. As the graph $G$ is self-centered, we have $(u, v) \in E(G)$ for all $u \in V(G - v)$. Therefore $\text{deg}_G(v) = n - 1$ which is a contradiction. □

Proposition 2.4. A graph $G$ of radius 2 is r.a.i. if and only if the maximum degree $\Delta(G) \leq n - 3$.

**Proof.** ($\Rightarrow$) Assume that there is a vertex $v$ such that $\text{deg}(v) = n - 2$. Then there is a unique vertex $u$ such that $(u, v) \notin E(G)$. If we add the edge $(u, v)$ to the graph $G$, then the degree of the vertex $v$ increases to $n - 1$, a contradiction.

($\Leftarrow$) Since by adding an edge to the graph $G$ the degree of two vertices increases by 1, there will be no vertex with degree $n - 1$ in the graph $G + (u, v)$ and therefore $r(G + (u, v)) > 1$ for any $(u, v) \in E(\overline{G})$. □

Theorem 2.5. Let $r \geq 2$ be a natural number and let $G$ be a graph with at least one vertex. There exists r.e.i. and r.v.i. graph $H$ such that $r(H) = r$ and $G$ is an induced subgraph of $H$.

Let $G$ and $G'$ be disjoint graphs and let $u \in V(G')$. We say that a graph $H$ is a substitution of $G$ into $G'$ in place $u$, if the vertex set $V(H) = (V(G') - \{u\} \cup V(G))$ and the edge set $E(H)$ consists of all edges of the graphs $G' - \{u\}$ and $G$ and, moreover, every vertex of $G$ is joined to every vertex from the neighborhood of $u$ in $G'$.

**Proof.** The cycle $C_{2r+1}$, $r \geq 2$, is r.e.i. and r.v.i. graph having radius and diameter equal to $r$. The demand graph $H$ is obtained from $C_{2r+1}$, by substituting the graph $G$ instead of any vertex $u$ (i.e. we join any neighbor vertex of $u$ in $C_{2r+1}$ to any vertex in $G$). □

Theorem 2.6. Let $r, d$ be natural numbers such that $2 \leq r < d \leq 2r$. Let $G$ be a graph with at least two vertices. Then there exists an r.e.i. and r.v.i. graph $H$ such that $r(H) = r$, $d(H) = d$, $C(H) = V(G)$ and $G$ is an induced subgraph of $H$.

**Proof.** Consider a graph $Q$ depicted here except the case when $d = 2r - 1$. It is clear that $e(c_1) = e(c_2) = r$. Moreover, $d(u, v) = \min\{1 + (d(v, c_1) + d(c_i, u) +
1), \(2(d - r) + 2\} where \(2(d - r) + 2 \leq d\) or \(2r \leq d\) if \(d \neq 2r - 1\). For any other vertex \(x\) we have \(d(x, u) \leq \min\{2(d - r) + 1, 2r - 2\} < d\). Let \(y, z\) be arbitrary vertices except \(u, v, c_i\). Consider a shortest cycle \(F, y \in F, z \in F\). The length of this cycle is at most \(2 + 2(d - r) + 2(r - 1) = 2d\) if it contains vertices \(c_1\) and \(u\) (or \(v\)) and less otherwise. Therefore \(d(y, z) \leq d\). There are \(2(d - r) + 1\) rows and \(2(r - 1) + 1\) columns of vertices in graph \(Q\). To obtain vertices \(o, p\) such that \(d(o, p) = d\) it is sufficient to take a vertex \(o\) in row 1 and column 1 and vertex \(p\) in row \(2(d - r) + 1\) and column \(d + 1\).

Let \(d = 2r - 1\). In that case take only \(d - 1 < 2(d - r) + 2\) rows of vertices. It follows directly that \(d(u, v) = d\). We can prove the fact that \(r(Q) = r\) and \(d(Q) = d\) in the same way as above.

As for each vertex \(w \neq c_i\) such that \(d(w, c_i) = k\) there are at least two edge and vertex disjoint \(c_i - w\) paths of length \(k\) we have \(r(Q - e) = r(Q)\) for all \(e \in E(Q)\) and \(r(Q - x) \leq r(Q)\) for all \(x \in V(Q)\). As there are two vertices \(u, v\) such that \(d(c_i, u) = d(c_i, v) = r(Q) = r\), we have \(r(Q - x) = r\) for all \(v \in V(Q)\). Therefore \(Q\) is r.e.i. and r.v.i.

The desired graph \(H\) is obtained from the graph \(Q\) by substituting the graph \(G\) instead of the vertices \(c_1, c_2\) (i.e. each neighbor vertex \(c_1, c_2\) in \(Q\) will be joined with every vertex of \(G\)). No other vertices and edges are added to \(H\).

It is clear that \(H\) is an r.e.i. and r.v.i. graph of radius \(r\), diameter \(d\), \(C(H) = V(G)\) and \(G\) is an induced subgraph of \(H\). \(\square\)

**Theorem 2.7.** Let \(r \geq 2\) be a natural number and \(G\) be a graph with at least one vertex. Then there exists an r.a.i. graph \(H\) such that \(r(H) = r\), \(C(H) = V(G)\), \(G\) is an induced subgraph of \(H\) and \(C(H) = V(G)\).

**Proof.** Consider a path \(P_4\) with central vertex \(v\) for \(r = 2\) and the following tree for \(r > 2\):
When we add any other edge to this tree, at least one of the distances \(d(a, b), d(a, c), d(b, c)\) remains the same. As \(d(G + e) \leq 2r(G + e)\), it is impossible to get a graph \(G + e\) of radius shorter than \(r\). Thus this graph is r.a.i. A graph \(H\) obtained from this tree by substituting the graph \(G\) for its central vertex \(v\) is r.a.i. and \(C(H) = V(G)\). \(\square\)

**Theorem 2.8.** Let \(r, d\) be natural numbers such that \(r \leq d \leq 2r\). Then there exists an r.a.i. graph \(G\) such that \(r(G) = r\) and \(d(G) = d\).

We first give some propositions and then the proof.

**Proposition 2.9.** Let \(G\) be an r.a.i. graph with \(n \geq 3\) vertices, radius \(r\) and diameter \(d\). Then there exists an r.a.i. graph \(G'\) such that \(r(G') = r + 1\) and \(d(G') = d + 2\).

**Proof.** Let \(v_1, \ldots, v_n\) be all vertices of the graph \(G\). We obtain the required graph \(G'\) by adding \(n\) new vertices \(u_1, \ldots, u_n\) to \(V(G)\) and \(n\) new edges \((v_i, u_i)\) to \(E(G)\). The fact that \(G'\) is r.a.i. is a special case of Theorem 3.9 given later. \(\square\)

**Proposition 2.10.** For each natural number \(r\) there exists an r.a.i. graph \(G\) of radius and diameter equal to \(r\) and \(|V(G)| \geq 3\).

**Proof.** Consider the group \(\mathbb{Z}_{2r+1}\) and define a graph \(G_{\mathbb{Z}_{2r+1}}\) in the following way:

\[
V(G) = \{(i, j); i, j \in \mathbb{Z}_{2r+1}\}, ((i_1, j_1), (i_2, j_2)) \in E(G) \iff |i_1 - i_2| \leq 1 \land |j_1 - j_2| \leq 1.
\]

If \((i_1, j_1)\) and \((i_2, j_2)\) are two vertices of \(G_{\mathbb{Z}_{2r+1}}\), then \(d((i_1, j_1), (i_2, j_2)) = \max\{\min\{|i_1 - i_2|, 2r + 1 - |i_1 - i_2|\}, \min\{|j_1 - j_2|, 2r + 1 - |j_1 - j_2|\}\} \leq r\). As for each vertex \(u = (i, j)\), there are \(8r\) vertices \(u_k = (i_k, j_k), i_k = i + r \mod (2r + 1) \lor j_k = j + r \mod (2r + 1) \lor j_k = j + r + 1 \mod (2r + 1)\) such that \(d(u, u_k) = r\), the graph \(G_{\mathbb{Z}_{2r+1}}\) is self-centered of radius \(r\).

We will prove the fact that \(G_{\mathbb{Z}_{2r+1}}\) is r.a.i. by contradiction. Consider a graph \(H = G_{\mathbb{Z}_{2r+1}} + (x, y), x = (i_x, j_x), y = (i_y, j_y)\). Let there be a vertex \(u\) such that \(e(u)_H < r\). Then the edge \((x, y)\) lies on each shortest \(u - u_k\) path, \(k = 1, \ldots, 8r\). Without the loss of generality let \(d_{G_{\mathbb{Z}_{2r+1}}}(u, x) \leq d_{G_{\mathbb{Z}_{2r+1}}}(u, y)\) and \(i_y \geq i_x\).
Consider a vertex \( u_l = (i_y + r - |i_x - i| - 1 \mod(2r + 1), j + r) \). We have 
\[ d_{G_2}(u, u_l) = r, d_{G_2}(y, u_l) \geq r - |i_x - i| - 1 \]
and 
\[ d_{G_2}(x, u_l) \geq r - |i_y - y| + r - |i_x - i| - 1, 2r + 1 - (r - |i_x - i| - 1) - |i_x - i| \] \( \geq r - 1 - |i - i_y| \). Therefore the \( u - x - y - u_l \) path has length at least \( |i_x - i| + 1 + (i_y + r - |i_x - i| - 1) = r \)
and the \( u - y - x - u_l \) path has length at least \( |i - i_y| + 1 + (r - 1 - |i - i_y|) = r \).
But then \( d_H(u, u_l) = r \), a contradiction.

The graph \( G_2 \) is r.a.i and \( r(G) = d(G) = r. \)

**Proof of Theorem 2.8.** For \( d = 2r \) the desired graph is depicted in the
proof of Theorem 2.7. For \( d = 2r - 1 \) we could consider the following graph:

![Graph Diagram]

Finally, consider the case when \( d \leq 2r - 2 \). Let mark \( G_0 = G_2 \) where \( k =
2r - d \geq 2 \). We will construct a graph \( G_{d-1} \) from the graph \( G_d \) by the concept
introduced in the proof of Proposition 2.9. For \( i = d - r \) we have an r.a.i. graph \( G_{d-r} \)
such that \( r(G_{d-r}) = i \cdot 1 + d(G_0) = d - r + (2r - d) = r \) and \( d(G_{d-r}) =
2(d - r) + 2r - d = d. \)

\[ 3. \] **Radius invariant graphs and operations on graphs**

**3.1 The cartesian product of graphs.** Let graphs \( G \) and \( H \) have vertex sets
\( \{u_1, u_2, \ldots, u_m\} \) and \( \{v_1, v_2, \ldots, v_n\} \), respectively. Then the Cartesian product \( G \times H \)
has vertex set \( w_{ij}, i = 1, 2, \ldots, m \) and \( j = 1, 2, \ldots, n \), and \( (w_{hi}, w_{jk}) \) is an edge of
\( G \times H \) if and only if either \( h = j \) and \( (v_i, v_k) \in E(H) \) or \( i = k \) and \( (u_h, u_j) \in E(G). \)

Directly from the definition it follows that (see [3]):

\[
\begin{align*}
    d_{G \times H}(w_{hi}, w_{jk}) &= d_G(u_h, u_j) + d_H(v_i, v_k) \\
    e_{G \times H}(w_{ij}) &= e_G(u_i) + e_H(v_j) \\
    r(G \times H) &= r(G) + r(H) \\
    d(G \times H) &= d(G) + d(H) \\
    w_{ij} \in C(G \times H) &\iff u_i \in C(G) \land v_j \in C(H) \\
    w_{ij} \in Cep(G \times H) &\iff u_i \in Cep(G) \land v_j \in Cep(H)
\end{align*}
\]
where $C(G)$ denotes the set of central vertices in $G$, and $C_e(G)$ denotes the set of eccentric vertices in $C(G)$.

Now we will prove some facts about stability of Cartesian products.

**Theorem 3.1.** Let $G$ be an r.e.i. graph. Then for any connected graph $H$, $G \times H$ is r.e.i.

**Proof.** Consider a graph $G \times H - e$, where $e$ is an arbitrary edge, which could be removed in a following ways:

**Case 1:** $e = (w_{ij}, w_{jg})$.

It is clear, that for any vertices $w_{hk}, w_{hl}$ we have $d_{G \times H - e}(w_{hk}, w_{hl}) = d_H(v_k, v_l)$. Because of this, as $G$ is r.e.i., we have that for any vertex $w_{st} \in G \times H$ and central vertex $w \in G \times H - e$ it is $d_{G \times H - e}(w, w_{rs}) \leq r(G - (u_i, u_j)) + r(H) = r(G) + r(H) = r(G \times H)$. Because of this $e_{G \times H - e}(w) = r$ and, therefore $r(G \times H - e) = r(G + H)$.

**Case 2:** $e = (w_{hi}, w_{hj})$ and $G$ has at least two central vertices.

As $G$ has at least two central vertices, there exists a central vertex $w_{xy}$ of the graph $G \times H$ such that $x \neq h$. It is clear that $v_y$ is a central vertex of $H$. Because of this, for any vertex $w_{rs}$ we have $d_{G \times H - e}(w_{xy}, w_{rs}) \leq r(G) + r(H) = r(G \times H)$.

**Case 3:** $e = (w_{hi}, w_{hj})$ and $G$ has only one central vertex.

It is clear that $r(G) > 1$. Let $w_{xy}$ be a central vertex of the graph $G \times H$. For any other vertex $w_{rs}$ such that $r \neq x$, $s \neq y$ there are at least two edge disjoint paths of length at most $r(G \times H)$. Then $d_{G \times H - e}(w_{xy}, w_{rs}) \leq r(G \times H)$. If $r \neq x$, $s = y$ then, as $G$ is r.e.i. we have that $d_{G \times H - e}(w_{xy}, w_{rs}) \leq r(G) < r(G \times H)$. If $r = x, s \neq y$ then if $v_z$ is some neighbour vertex of $v_y$ in the graph $G$, thus $d_{G \times H - e}(w_{xy}, w_{rs}) \leq \min\{d_H(u_y, u_s), 2 + d_H(u_y, u_s)\} \leq 2 + r(H) \leq r(G \times H)$.

This completes proof. \qed

**Theorem 3.2.** Let $G$ and $H$ be an r.v.i. graphs. Then $G \times H$ is r.v.i.

**Proof.** It is clear that for each vertex $u_a \in G \ (v_k \in H)$ there are at least two vertices $u_b, u_c (v_l, v_m)$ such that $d(u_a, u_b) \geq r(G)$, $d(u_a, u_c) \geq r(G)$ and $d(v_k, v_l) \geq r(H)$. Because of this for each vertex $w_{ak}$ of the graph $G \times H$ there are at least two vertices at distance at least $r(G) + r(H)$ and for any other $w_{xy} \in G \times H$ there is $e_{G \times H - w_{xy}}(w_{ak}) \geq r(G) + r(H) = r(G \times H)$.

Let $w_{hi} \in V(G \times H)$. Then $u_h \in V(G)$ and $v_i \in V(H)$. Since $G$ and $H$ are r.v.i., there exist vertices $u_r \in C(G - u_h)$ of eccentricity $r(G)$ and $v_s \in C(H - v_i)$ of eccentricity $r(H)$.

Then the vertex $w_{rs}$ of graph $G \times H - w_{hi}$ will have eccentricity equal to $r(G) + r(H)$. So the graph $G \times H$ is r.v.i. The theorem holds. \qed

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Theorem 3.3. Let $G \times H$ be an r.a.i. graph. Then $G$ is r.a.i. and $H$ is r.a.i.

Proof. We will prove the theorem by contradiction. Assume that for example $G$ is not r.a.i. Consider a central vertex $u_s$ of the graph $G + (u_l, u_k)$ of radius less than or equal to $r(G) - 1$. Let $v_t$ be a central vertex of the graph $H$. Now let $G \times H$ be the cartesian product of the graphs $G$ and $H$ and let $F = G \times H + (w_{lt}, w_{kt})$. Now check the eccentricity of the vertex $w_{st}$. Let $w_{mn}$ be an arbitrary vertex of $F$. Then $d_F(w_{st}, w_{mn}) \leq d_{G+(u_l,u_k)}(u_s, u_m) + d_H(v_t, v_n) \leq r(G) - 1 + r(H)$. Then $r(G \times H + (w_{lt}, w_{kt})) < r(G \times H)$.

The converse is false. For example $G = C_5 \times P_4$ is not r.a.i.

We have $r(G) = 4$, but $r(G + uv) = e_{G+(uv)}(v) = 3$.

3.2 The join of graphs. Recall that the join of graphs $G$ and $H$ is denoted $G + H$ and consists of $G \cup H$ and all edges of the form $(u_i, v_j)$ where are $u_i \in G$, $v_j \in H$. It is clear that $r(G + H) = 2$, or $r(G + H) = 1$ if $\Delta(G) = |V(G)| - 1$ or $\Delta(H) = |V(H)| - 1$. Also $\deg_{G\cup H}(v) = \deg_G(v) + |V(H)|$ for all $v \in V(G)$ and $\deg_{G\cup H}(u) = \deg_H(u) + |V(G)|$ for all $u \in V(H)$. If $G + H$ has radius 2, it is self-centered. Now we show how we can construct r.v.i., r.e.i. and r.a.i. graphs by the operation of join.

Theorem 3.4. Let graphs $G$ and $H$ have both at least two vertices. Then:

(1) $G + H$ is r.e.i. of radius 1 if and only if the graphs $G$ and $H$ contain together at least three vertices such that $\deg_G(u_i) = |V(G)| - 1$, $u_i \in G$ or $\deg_H(v_j) = |V(H)| - 1$, $v_j \in H$,

(2) $G + H$ is r.e.i. of radius 2 if and only if the graphs $G$ and $H$ have no vertex with $\deg_G(u_i) = |V(G)| - 1$, $u_i \in G$ or $\deg_H(v_j) = |V(H)| - 1$, $v_j \in H$.

Proof. (1) The graph $G + H$ is r.e.i. of radius 1 if and only if it contains at least 3 vertices of degree $|V(G)| + |V(H)| - 1$ which is the same as the requirement (1).

(2) The graph $G + H$ has radius 2 if and only if the graphs $G$ and $H$ have no vertex with $\deg_G(u_i) = |V(G)| - 1$, $u_i \in G$ or $\deg_H(v_j) = |V(H)| - 1$, $v_j \in H$. As every graph $G + H$ of radius 2 is self-centered, it is also r.e.i.
**Theorem 3.5.** Let graphs $G$ and $H$ have both at least two vertices. Then:

1. $G + H$ is r.v.i. of radius 1 if and only if the graphs $G$ and $H$ contain together at least two vertices such that $\deg_G(u_i) = |V(G)| - 1$, $u_i \in G$ or $\deg_H(v_j) = |V(H)| - 1$, $v_j \in H$.

2. $G + H$ is r.v.i. of radius 2 if and only if the graphs $G$ and $H$ have no vertex with $\deg_G(u_i) = |V(G)| - 1$, $\deg_G(u_i) = |V(G)| - 2$, $u_i \in G$ or $\deg_H(v_j) = |V(H)| - 1$, $\deg_H(v_j) = |V(H)| - 2$, $v_j \in H$.

**Proof.** (1) The graph $G + H$ of radius 1 is r.v.i. if and only if it contains at least 2 vertices of degree $|V(G)| + |V(H)| - 1$ which is the same as the requirement (1).

(2) The graph $G + H$ has radius 2 if and only if the graphs $G$ and $H$ have no vertex with $\deg_G(u_i) = |V(G)| - 1$, $u_i \in G$ or $\deg_H(v_j) = |V(H)| - 1$, $v_j \in H$. Moreover, if it is r.v.i. then $\Delta(G) \leq |V(G)| + |V(H)| - 3$ by Proposition 2.2 which is the same as the requirement (2). Conversely, as a consequence of Proposition 2.3, the graph $G + H$ is r.v.i.

**Theorem 3.6.** For any graphs $G$ and $H$

1. $G + H$ is r.a.i. of radius 1 if and only if the graphs $G$ and $H$ contain together at least one vertex $u_i(v_j)$ such that $\deg_G(u_i) = |V(G)| - 1$, $u_i \in G$ or $\deg_H(v_j) = |V(H)| - 1$, $v_j \in H$.

2. $G + H$ is r.a.i. of radius 2 if and only if the graphs $G$ and $H$ have no vertex $u_i(v_j)$ with $\deg_G(u_i) = |V(G)| - 1$, $\deg_G(u_i) = |V(G)| - 2$, $u_i \in G$ or $\deg_H(v_j) = |V(H)| - 1$, $\deg_H(v_j) = |V(H)| - 2$, $v_j \in H$.

**Proof.** (1) Every graph of radius 1 is r.a.i. Therefore this case is obvious.

(2) We can use Proposition 2.4. The graph $G + H$ has $\Delta(G + H) \leq n - 3$ if and only if it satisfies the condition (2).

### 3.3 The corona of graphs.

The corona $G \circ H$ of graphs $G$ and $H$ was defined by Harary and Frucht ([5], see also [1]) as the graph obtained by taking one copy of $G$ of order $p_G$ and $p_G$ copies of $H$, and then joining the $i$'th vertex of $G$ to every vertex in the $i$'th copy of $H$.

It is clear that if $p_G > 1$, $r(G) = r_G$, then $r(G \circ H) = r_G + 1$ and $v$ is a central vertex of $G \circ H$ if and only if $v$ is a central vertex of $G$. Removal of any vertex of $G$ will change the radius of $G \circ H$ to infinity, so for any graphs $G$, $H$ the graph $G \circ H$ is not r.v.i. and we will not discuss r.v.i. graphs later in this section.

One can show that if $G \circ H$ is r.e.i., then $G$ must be r.e.i. and $H$ must be a graph with at least two vertices and no isolated vertex.
Theorem 3.7. Let $G$ be a self-centered graph with at least three vertices and let $H$ have no vertex with $\deg(v) = 0$. Then $G \circ H$ is r.e.i.

Proof. As was shown in [4], every self-centered graph with at least three vertices is r.e.i. Consider three different forms in which an edge could be removed.

Case 1: We remove an edge which joins vertices from $G$. As $G$ is r.e.i., radius of $G$ remains unchanged and radius of $G \circ H$ is still $r_G + 1$.

Case 2: We remove an edge which joins vertices from $H$. As the shortest path from central vertex of $G \circ H$ (which is a vertex of $G$) does not contain an edge from any copy of $H$ the radius remains unchanged.

Case 3: We remove an edge which joins the vertex $v$ from $G$ with any vertex $u$ in some copy of $H$. Then $r(G \circ H) \geq 2$. As $G$ is self-centered, $v$ is a central vertex of $G \circ H$. Distances to all vertices of the copy of $H$ except $u$ are 1. As $H$ has no isolated vertices, $d(u, v) = 2$. The distance from the vertex $v$ to any other vertex of $G \circ H$ remains unchanged, so $G \circ H$ is r.e.i. □

Theorem 3.8. The corona $G \circ H$ of graphs $G, H$ is r.e.i. graph if and only if

1. $G$ is r.e.i. graph,
2. for each vertex $v \in G$ there exists a central vertex $c \in G$ such that $d(u, c) \leq r - 1$,
3. $H$ has no vertex $v \in V(G)$ such that $\deg(v) = 0$.

Proof. $(\Leftarrow)$ We will prove this similarly as in the previous theorem. The cases (1), (2) can be proved by the same way as above. Now consider the third case.

Case 3: We remove an edge which joins a vertex $v$ from $G$ with any vertex $u$ in some copy of $H$. Then $r(G \circ H) \geq 2$. For the vertex $v$ there exists a vertex $c \in G$ such that $d(u, c) \leq r - 1$. As $u$ is not isolated in $H$, we have $d(u, v) = 2$. Because of this, the distances from the vertex $c$ are still not longer than $r + 1$. Then $G \circ H$ is r.e.i.

$(\Rightarrow)$ It is clear that $G$ is r.e.i. and $H$ has no isolated vertices. Let there exist a vertex $v$ such that the nearest central vertex lies at distance at least $r$. Assume that we remove some edge $(v, u)$ when $u$ is in the copy of $H$ which belongs to $v$. Then $d_{G \circ H - (vu)}(v, u) = 2$, $d_{G \circ H - (vu)}(u, c) > r + 2$ for all vertices $c$ which were central in $G \circ H$. It follows that $e(w) > r$ for all vertices $w \in G \circ H - (vu)$. Thus $G \circ H$ is not r.e.i., which is a contradiction. □

Theorem 3.9. For any graphs $G, H, |V(G)| \geq 3$ the corona $G \circ H$ is r.a.i. if and only if $G$ is r.a.i.
Proof. ($\Rightarrow$) Assume that $G, r(G) = r_G$ is not r.a.i. Then $r(G + (u, v)) < r_G$. As $r((G + (u, v)) \circ H) = r(G + (u, v)) + 1 < r(G) + 1 = r(G \circ H)$ we have that $G \circ H$ is not r.a.i., a contradiction.

($\Leftarrow$) Assume that $G \circ H$ is not r.a.i. As $G$ is r.a.i., adding an edge which joins vertices of $G$ does not change $r(G \circ H)$. The same is true for any edge between vertices of the unique copy of $H$.

Let $u, v$ be two vertices of $G$ and let an edge be added between two copies $H_u, H_v$ of the graph $H$ where each vertex of $H_u$ is connected to $u$ and each vertex of $H_v$ is connected to $v$.

Let an edge $(h_u, h_v), h_v \in H_v, h_u \in H_u$ be added and assume that the radius has changed. It is enough to consider the case when the new central vertex $c$ lies in $G$. If a vertex $h_d \in H_d$ is a central vertex of the graph $G \circ H + (h_u, h_v)$, then $d$ must be a central vertex of the graph $G \circ H + (u, v)$ and $e_{G \circ H + (u, v)}(d) \leq e_{G \circ H + (h_u, h_v)}(h_d)$ which leads to the previous case.

So let $c \in G$ be a new central vertex and let $r(G \circ H + (h_u, h_v)) < r(G \circ H)$. Then every $c - x$ path which has length at least $r$ in $G \circ H$ was shortened. But then it contains $u - h_u - h_v$ or $v - h_v - h_u$ path. If we add the edge $(v, u)$ to the graph $G \circ H$, then these paths could be shortened by same distance replacing $u - h_u - h_v$ by $u - v - h_v$ or $v - h_v - h_u$ by $v - u - h_u$. But this case was discussed above.

Because of this, if the radius was changed, then an edge between some vertex $v \in G$ and some vertex $w_u \in H_u$ of the copy of the graph $H$ must have been added. By the same arguments as above we should consider only the case when the new central vertex $c$ of the graph $G \circ H + (v, w_u)$ is in $G$. $G$ is r.a.i., so there are at least two vertices $y, z$ at distance $r$ or longer from $c$. If not, then adding the edge $(c, y)$ (if $y$ is the only such vertex) will decrease the radius of $G$.

It follows that distances from the coronas $H_y, H_z$ to $c$ are at least $r + 1$. Then the shortest $c - h_y, c - h_z$ paths in the graph $G \circ H + (u, w_u)$ for some $h_y \in H_y, h_z \in Z$ must contain the edge $(v, w_u)$. Moreover, $y \neq u$ or $z \neq u$. Without the loss of generality let us consider the first case. Then

$$r \geq d(h_y, c) > d(y, c) > d(u, c).$$

All shortest paths satisfying this inequality contain the $v - w_u - u$ path. If we substitute this path by an edge $(v, u)$ then $d(c, b) \leq r$ for all $b \in G \circ H + (v, u)$. But this case was discussed above. The proof is now complete. \qed
4. Some bounds

Now we will prove some estimations for the number of vertices and degree of a vertex for radius invariant graphs.

A $k$-depth spanning tree ($k$-DST) of a graph $G$ is a spanning tree of $G$ of height $k$. It must be true that $k \geq r$, and if $k = r$, such trees must be rooted at a central vertex. A breadth first search algorithm beginning with any vertex $v$ such that $e(v) = k$ will always produce an $k$-DST. Moreover, if $d(u, v) = i$ then the vertex $u$ belongs to the level $i$. Vertices of the spanning tree with degree 1 will be called leaves. We would consider only breadth first search distance spanning trees later in this section.

**Theorem 4.1.** Let $G$ be an r.e.i. graph of radius $r \geq 1$. Then $|V(G)| \geq 2r$.

**Proof.** A graph of radius $r$ has at least $2r$ vertices. □

The cycle $C_{2r}$ is r.e.i., so we can’t find any better estimation.

**Theorem 4.2.** Let $G$ be an r.v.i. graph of radius $r \geq 1$. Then $|V(G)| \geq 2r + 1$.

**Proof.** Consider an $r$-DST of the graph $G$. If it has only one vertex on level $r$, its removal will decrease radius to $r - 1$. Therefore there are at least two vertices on level $r$. Assume that there is a level $i$, $1 \leq i < r$ on which there appears only one vertex $a$. Then the graph $G - a$ is disconnected, which is a contradiction. Finally we have the central vertex $v$ and at least two vertices at every level $1, \ldots, r$. □

It’s clear (for example as a consequence of Theorem 2.5) that there is no upper bound for $|V(G)|$ both for r.e.i. and r.v.i. graphs.

**Theorem 4.3.** Let $G$ be an r.a.i. graph of radius $r \geq 3$. Then $|V(G)| \geq 3r$.

At first we give four lemmas and then the proof.

**Lemma 4.4.** Let $G$ be an r.a.i. graph of radius $r \geq 2$. Then every $r$-DST has at least two vertices on level $r$.

**Proof.** Assume that $r$-DST is rooted at central vertex $v$ and there is only one vertex $u$ on level $r$. By adding an edge $(u, v)$ to $G$ we obtain a graph of radius no greater than $r - 1$. Thus $G$ is not r.a.i., a contradiction. □
Lemma 4.5. Let $G$ be an r.a.i. graph of radius $r \geq 3$ and let $v$ be a central vertex of $G$. Consider an $r$-DST rooted at a central vertex $v$ and the set $\{u_1, \ldots, u_s\}$ of all vertices on level $r$. Now if $w$ is some joint vertex of all $v-u_1, \ldots, v-u_n$ paths then $d(v, w) = 1$ or $v = w$.

Proof. Assume that $d(v, w) \geq 2$. Adding an edge $(v, w)$ to $G$ decreases all distances $d(v, u_i)$ by $(d(v, w) - 1)$. The radius of $G+(v, w)$ is smaller than the radius of $G$, a contradiction. \qed

Lemma 4.6. Let $G$ be an r.a.i. graph of radius $r \geq 3$ and let $v$ be a central vertex of $G$. Consider an $r$-DST rooted at central vertex $v$ and the set $\{u_1, \ldots, u_s\}$ of all vertices on level $r$. Let $w$ be a joint vertex of $v-u_1, \ldots, v-u_s$ paths such that $v \neq w$. Then there exist at least two vertices $y, z$ on level $r-1$ for which $w$ does not lie on $v-y$, $v-z$ paths.

Proof. Assume that there is only one such vertex (for example $y$). Now consider distances from the vertex $w$ in graph $G+(w, y)$. We have $d_G(w, u_i) = r - 1 \geq d_{G+(w, y)}(w, u_i), d_{G+(w, y)}(w, y) = 1$ and the distance from the vertex $w$ to all other vertices in the graph $G+(w, y)$ are no longer than in the graph $G$ where it was less than $r$. Adding the edge $(w, y)$ decreases the radius of $G$, a contradiction. \qed

Lemma 4.7. Let $G$ be an r.a.i. graph of radius $r$, $v$ be a central vertex of graph $G$ and let for vertices $u_1, u_2$ on level $r$ of r-DST the paths $v-u_1, v-u_2$ have a unique joint vertex $v$. Then at least one of the following properties holds:

1. there exists a vertex $y$ on level at least $(r - 1)$ such that each pair of paths $v-u_1, v-u_2, v-y$ has only one joint vertex $v$,
2. there exist vertices $p, q$ on level at least $(r - 1)$ such that the paths $v-p, v-q$ and $v-u_1, v-u_2$ have just two joint vertices.

Proof. Let $a_1, a_2$ be the second and the third vertices on the path $v-u_1$, and let $b_1, b_2$ be the second and the third vertices on $v-u_2$ path. Assume that none of the conditions (1), (2) holds. Without loss of generality assume that there does not exist a vertex $p$ on level $r-1$ such that the paths $v-p, v-u_1$ have just the two vertices $v, a_1$ in common. Consider the graph $G+(b_1, a_2)$ and distances from the vertex $b_1$.

Case 1: Let $k$ be a vertex such that the path $v-k$ has no joint vertex with $v-u_1, v-u_2$ paths except $v$. The property (1) is false, so $d(v, k) \leq r - 2$ which implies $d(b_1, k) \leq r - 1$.

Case 2: Let $l$ be a vertex such that the paths $v-l, v-u_2$ have joint vertices $v$ and $b_1$. Then $d(v, l) \leq r$, so $d(b_1, l) \leq r - 1$.
Case 3: Let $m$ be a vertex such that the paths $v - m$, $v - u_1$ have joint vertices
$v, a_1, a_2$. Then $d(m, v) \leq r$, so $d(m, a_2) \leq r - 2$ and $d(m, b_1) \leq r - 1$.

Case 4: At last let $o$ be a vertex such that the paths $v - o$, $v - u_1$ have joint
vertices $v$ and $a_1$. The property (2) is false, so $d(v, o) \leq r - 2 \Rightarrow d(a_1, o) \leq r - 3 \Rightarrow
d(a_2, o) \leq r - 2 \Rightarrow d(b_1, o) \leq r - 1$.

Hence in the graph $G + (b_1, a_2)$ we have $e_{G + (b_1, a_2)}(b_1) = r - 1$, so the graph $G$
is not r.a.i., a contradiction. \hfill \Box

Proof of Theorem 4.3. An r.a.i. graph $G$ with $r \geq 3$ must satisfy the
property from Lemma 4.6, or at least one property (1), (2) from Lemma 4.7. In
every case it has at least $3r$ vertices. \hfill \Box

This estimation is the best possible (see for example the tree from proof of Theo-
rem 2.7).

There is no upper bound for the number of vertices of r.a.i. graph of radius $r$. For
example if $p = r \cdot i + j + 1 \geq 3r$, $i, j \in \mathbb{N}$, the tree with central vertex $v$ depicted here
has radius $r$ and is r.a.i.
Theorem 4.8. Let $G$ be an r.e.i. graph with $n$ vertices and radius $r$. Then for each vertex $v \in G$

$$2 \leq \deg(v) \leq n - 2r + 2.$$ 

Proof. As $G$ is r.e.i., it has no bridges and because of this no vertex of degree 1. Thus the first inequality is obvious. The second inequality holds for all graphs (for example see [9]).

Theorem 4.9. Let $G$ be an r.v.i. graph with $n$ vertices and radius $r$. Then for each vertex $v \in G$

$$2 \leq \deg(v) \leq n - 2r + 1.$$ 

Moreover, if $e(v) = k > r$, then

$$\deg(v) \leq n - 2e(v).$$

Proof. As $G$ is r.v.i., it has no cut-vertices and because of this no vertex of degree 1. Then the first inequality is obvious. Let $e(v) = k$. Consider a $k$-DST rooted at the vertex $v$. If $v$ is a central vertex of $G$, then there are at least two vertices at every level 1, 2, ..., $r$. If $v$ is not a central vertex, then there are at least two vertices at every level 1, 2, ..., $k - 1$ and at least one vertex at level $k$.

Theorem 4.10. Let $G$ be an r.a.i. graph with $n$ vertices and radius $r$. Then

$$\Delta(G) \leq n - 3r + 3.$$ 

Proof. We prove this similarly as Theorem (4.3). Consider an arbitrary vertex $v \in V(G)$. If $v \in C(G)$, then we should use Lemmas (4.6), (4.7) and the proof is obvious. Let $v \notin C(G)$. Consider an $k$-DST rooted at the vertex $v$. Then

$$k = e(v) > r.$$ 

Let $x$ be a vertex on level $k$ and $u$ be a vertex such that $d(u, x) = r - 1$. Then there are at least two or more leaves $u_1, u_2, \ldots, u_n$ such that $d(u, u_i) \geq r$. Otherwise if $d(u, v) < r$ and $u_1$ is a leaf such that $d(u, u_1) \geq r$, then $r(G + (u, u_1)) \leq e_{G+(u,u_1)}(u) = r - 1$. If $d(u, v) \geq r$ and there is only one such leaf, it must be $\deg(v) \leq 2$. But as $|V(G)| \geq 3r$ it is immediate that $\deg(v) \leq n - 3r + 3$.

Let $u_1$ be a leaf such that $d(u, u_1) = \max_{u_i \in G} d(u, u_i)$ and $w$ be a vertex such that $d(w, u_1) = r - 2$. It follows that there is at least one leaf $u_j$ with $d(u_j, w) = r - 1$. Otherwise $r(G + (u, w)) \leq r - 1$ as $e_{G+(u,w)}(u) = r - 1$.

Consider two cases:
C a s e 1: $w$ lies on the path $u - u_j$ for all $u_j$. But then $d(w, u_1) = r - 2 < r - 1 \leq d(w, u_j)$ which is not possible as $u_1$ is a vertex at maximal distance from $u$, and then must be a vertex (of $u_1, u_2, \ldots$) at maximum distance from $w$.

C a s e 2: $w$ is not on the path $u - u_2$. Let $z$ be the last joint vertex of the paths $u - u_1, u - u_2$.

A) $z = v$, $w$ is joined to $z$.

In this case all vertices $u_1, u_2, \ldots$ are at level at most $r - 1$. But as $G$ is r.a.i., there are at least 2 vertices at level $r$. Finally, we have $r$ vertices on the path $x - u$, $r$ vertices on the $u - u_2$ path without the vertex $u$, $r - 1$ vertices on the $z - u_1$ path without the vertex $z$ and at least one additional vertex on level $r$. At most 3 of these $3r$ vertices are joined to $v$.

B) $z = v$, $w$ is not joined to $z$.

We have $r$ vertices on the $x - u$ path, $r$ vertices on the $u - u_2$ path without the vertex $u$. As $w$ is at level at least 2 and $d(w, u_1) = r - 2$, we have at least $r$ vertices on the $z - u_1$ path without the vertex $z$. At most 3 of these vertices are joined to $v$.

C) $z \neq v$.

We have already $3r - 1$ vertices of which only 2 are joined to $v$. This completes the proof. \[ \square \]

References


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