Bohdan Zelinka
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SIGN ED 2-DOMINATION IN CATERPILLARS

BOHDAN ZELINKA, Liberec

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Abstract. A caterpillar is a tree with the property that after deleting all its vertices of degree 1 a simple path is obtained. The signed 2-domination number $\gamma^2_s(G)$ and the signed total 2-domination number $\gamma^2_{st}(G)$ of a graph $G$ are variants of the signed domination number $\gamma_s(G)$ and the signed total domination number $\gamma_{st}(G)$. Their values for caterpillars are studied.

Keywords: caterpillar, signed 2-domination number, signed total 2-domination number

MSC 2000: 05C69, 05C05

This paper concerns caterpillars. A caterpillar [1] is a tree with the property that after deleting all its vertices of degree 1 a simple path is obtained. According to this definition a caterpillar has at least three vertices. But we need not care about graphs with one or two vertices. For such graphs our considerations are trivial.

Let $G$ be a caterpillar. The mentioned simple path will be denoted by $B$ and called the body of the caterpillar $G$. Let the number of vertices of $B$ be $m$. Let $a_1, \ldots, a_m$ be these vertices and let $a_ia_{i+1}$ for $i = 1, \ldots, m - 1$ be the edges of $B$. By $[m]$ we shall denote the set of integers $i$ such that $1 \leq i \leq m$. For each $i \in [m]$ let $s_i$ be the degree of $a_i$ in $G$. The vector $\vec{s} = (s_1, \ldots, s_m)$ will be called the degree vector of the caterpillar $G$.

Now we shall define variants of the signed domination number and of the signed total domination number [2] of a graph. For a vertex $u \in V(G)$ the symbol $N(u)$ denotes the open neighbourhood of $u$ in $G$, i.e. the set of all vertices which are adjacent to $u$ in $G$. The closed neighbourhood of $u$ is $N[u] = N(u) \cup \{u\}$. Similarly the open 2-neighbourhood $N^2(u)$ is the set of all vertices having the distance 2 from $u$ in $G$. The closed 2-neighbourhood of $u$ is $N^2[u] = N[u] \cup N^2(u)$. If $f$ is a mapping
of $V(G)$ into some set of numbers and $S \subseteq V(G)$, then $f(S) = \sum_{x \in S} f(x)$ and the weight of $f$ is $w(f) = f(V(G)) = \sum_{x \in V(G)} f(x)$.

Let $f: V(G) \to \{-1, 1\}$. If $f(N^2[u]) \geq 1$ (or $f(N^2(u)) \geq 1$) for each $u \in V(G)$, then $f$ is called a signed 2-dominating (or signed total 2-dominating, respectively) function on $G$. The minimum of weights $w(f)$ taken over all signed 2-dominating (or all signed total 2-dominating) functions $f$ is the signed 2-dominating number $\gamma^2_s(G)$ (or the signed total 2-dominating number $\gamma^2_{st}(G)$, respectively) of $G$.

For each $i \in [m]$ let $t_i \in \{1, 2\}$ and $t_i \equiv s_i + 1 \pmod 2$.

We shall prove a theorem concerning $\gamma^2_s(G)$.

**Theorem 1.** Let $G$ be a caterpillar with the degree vector $\vec{s} = (s_1, \ldots, s_m)$ such that $n \geq 2$ and $s_i \geq 3$ for all $i \in [m]$. Then

$$\gamma^2_s(G) = \sum_{i=1}^{m} t_i - 2m + 2.$$ 

**Proof.** Consider a vertex $a_i$ with $i \in [m]$. As $s_i \geq 3$, there exists at least one vertex $u \in N(a_i)$ which does not belong to $B$ and has degree 1. Then $N^2[u] = N[a_i]$. Let $f$ be a signed 2-dominating function on $G$. Then $f(N^2[u]) = f(N[a_i]) \geq 1$. The set $N[a_i]$ has $s_i + 1$ vertices. If $s_i$ is even, then $s_i + 1$ is odd. At least $\frac{1}{2}(s_i + 2) = \frac{1}{2}s_i + 1$ vertices of $N[a_i]$ must have the value 1 in $f$ and at most $\frac{1}{2}s_i$ of them may have the value $-1$. Then $f(N^2[u]) \geq \left(\frac{1}{2}s_i + 1\right) - \frac{1}{2}s_i = 1 = t_i$. If $s_i$ is odd, then $s_i + 1$ is even and at least $\frac{1}{2}(s_i + 1) + 1$ vertices of $N[a_i]$ must have the value 1 in $f$ and at most $\frac{1}{2}(s_i + 1) - 1$ of them may have the value $-1$. Then $f(N^2[u]) \geq 2 = t_i$. We may easily construct the function $f$ such that it has the value $-1$ in exactly $\frac{1}{2}s_i$ vertices of degree 1 in $N[a_i]$ with $i$ even and in exactly $\frac{1}{2}(s_i + 1) - 1 = \frac{1}{2}(s_i - 1)$ vertices of degree 1 in $N[a_i]$ with $i$ odd. In all other vertices (including all vertices of the body) the function $f$ has the value 1.

We have $\bigcup_{i=1}^{m} N[a_i] = V(G)$. The vertex $a_1$ is contained in exactly two sets $N[a_i]$, namely in $N[a_1]$ and $N[a_2]$. Similarly $a_m$ is contained in exactly two sets $N[a_{m-1}]$, $N[a_m]$. For $i \in [m] - \{1, m\}$ the vertex $a_i$ is contained in exactly three sets $N[a_{i-1}]$, $N[a_i]$, $N[a_{i+1}]$. Each vertex outside the body is contained in exactly one of these sets. By the Inclusion-Exclusion Principle we have

$$w(f) = f(V(G)) = \sum_{i=1}^{m} f(N[a_i]) - 2 \sum_{i=2}^{m-1} f(a_i) - f(a_1) - f(a_m)$$

$$= \sum_{i=1}^{m} t_i - 2(m - 2) - 1 - 1 = \sum_{i=1}^{m} t_i - 2m + 2.$$
As $f$ is the minimum function satisfying the requirements, we have

$$\gamma_s^2(G) = w(f) = \sum_{i=1}^{m} t_i - 2m + 2.$$  

□

An analogous theorem concerns $\gamma_{st}^2(G)$.

**Theorem 2.** Let $G$ be a caterpillar with the degree vector $\vec{s} = (s_1, \ldots, s_m)$ such that $m \geq 2$ and $s_i \geq 4$ for all $i \in [m]$. Then

$$\gamma_{st}^2(G) = \sum_{i=1}^{m} t_i + 2.$$  

**Proof.** Consider a vertex $a_i$ with $i \in [m]$. As $s_i \geq 5$, there exists at least one vertex $u \in N(a_i)$ which does not belong to $B$ and has degree 1. Then $N^2(u) = N(a_i) - \{u\}$. Let $f$ be a signed total 2-dominating function on $G$. Then $f(N^2(u)) = f(N(a_i) - \{u\}) \geq 1$. The set $N(a_i) - \{u\}$ has $s_i - 1$ vertices. If $s_i$ is even, then $s_i - 1$ is odd. At least $\frac{1}{2}s_i$ vertices of $N(a_i) - \{u\}$ must have the value 1 in $f$ and at most $\frac{1}{2}(s_i - 2) = \frac{1}{2}s_i - 1$ of them may have the value $-1$. Then $f(N^2(u)) \geq \frac{1}{2}s_i - (\frac{1}{2}s_i - 1) = 1 = t_i$. If $s_i$ is odd, then $s_i - 1$ is even and at least $\frac{1}{2}(s_i - 1) + 1$ vertices of $N(a_i) - \{u\}$ must have the value 1 in $f$ and at most $\frac{1}{2}(s_i - 1) - 1$ of them may have the value $-1$. Then $f(N^2(u)) \geq 2 = t_i$. As $s_i \geq 5$ for $i \in [m]$, in both these cases we must admit the possibility $f(u) = 1$. Then in the case of $s_i$ even we have $f(N(a_i)) \geq 2 = t_i + 1$ and in the case of $s_i$ odd we have $f(N(a_i)) \geq 3 = t_i + 1$.

We may easily construct the function $f$ such that it has the value $-1$ in $\frac{1}{2}s_i - 1$ vertices of degree 1 in $N(a_i)$ for $s_i$ even, in $\frac{1}{2}(s_i - 1) - 1 = \frac{1}{2}(s_i - 3)$ vertices of degree 1 in $S(a_i)$ for $s_i$ odd and the value 1 for all other vertices (including all vertices of $B$). Each vertex $a_j$ for $j \in [m] - \{1, m\}$ is contained in two sets $N(a_i)$, namely in $N(a_{j-1})$ and $N(a_{j+1})$. Each other vertex is contained in exactly one set $N(a_i)$. Again by the Inclusion-Exclusion Principle we have

$$w(f) = f(V(G)) = \sum_{i=1}^{m} f(N(a_i)) - \sum_{i=2}^{m-1} f(a_i)$$

$$= \sum_{i=1}^{m} (t_i + 1) - (m - 2) = \sum_{i=1}^{m} t_i + m - (m - 2) = \sum_{i=1}^{m} t_i + 2.$$  

As $f$ is the minimum function satisfying the requirements, we have

$$\gamma_{st}^2(G) = w(f) = \sum_{i=1}^{m} t_i + 2.$$  

□
In Figs. 1 and 2 a caterpillar $G$ with the degree vector $(5, 6, 7)$ is depicted. We have $t_1 = t_3 = 2$, $t_2 = 1$ and therefore $\gamma^2_{st}(G) = 7$ and $\gamma^2_s(G) = 1$. In Fig. 1 the values of the corresponding signed total 2-dominating function are illustrated; in the vertices denoted by $+$ the value is 1 and in the vertices denoted by $-$ it is $-1$. Similarly in Fig. 2 the corresponding signed 2-dominating function is illustrated.

In Theorems 1 and 2 we had the assumption $m \geq 2$. The following proposition concerns the singular case $m = 1$.

**Proposition 1.** Let $G$ be a caterpillar with the body consisting of one vertex, i.e. a star with the central vertex $a_1$ and with $s_1 \geq 2$ vertices of degree 1. Then $\gamma^2_{st}(G)$ is undefined and $\gamma^2_s(G) = t_1$.

**Proof.** The open 2-neighbourhood $N^2(a_1) = \emptyset$ and thus $f(N^2(a_1)) = 0$ for any function $f : V(G) \to \{-1, 1\}$, hence none of such functions might be signed total 2-dominating in $G$. On the other hand, $N^2[a_1] = V(G)$ and $|V(G)| = s_1 + 1$. Analogously as in the proofs of Theorems 1 and 2 we prove that for $s_1$ even we have $\gamma^2_s(G) = 1 = t_1$ and for $s_1$ odd we have $\gamma^2_s(G) = 2 = t_1$. \[\square\]

**Proposition 2.** Let $G$ be a caterpillar with $m \equiv 2 \pmod{5}$, $m \geq 5$, $s_i = 3$ for all $i \in [m]$. Then $\gamma^2_{st}(G) \leq \frac{4}{3}(m + 3) + 2$, while $\sum_{i=1}^{m} t_i + 2 = 2(m + 1)$.

**Proof.** As $s_i = 3$ for each $i \in [m]$, we have $t_i = 2$ for each $i \in [m]$. Each vertex $a_i$ for $i \in [m] - \{1, m\}$ is adjacent to exactly one vertex $v_i$ of degree 1. The vertex $a_1$ is adjacent to two such vertices $v_1, w_1$ and similarly $a_m$ to $v_m, w_m$. Let $f : V(G) \to \{-1, 1\}$ be defined so that $f(v_i) = -1$ for $i \equiv 0 \pmod{3}$ and $f(u) = 1$ for all other vertices $u$. This is a signed total 2-dominating function on $G$ (this can be easily verified by the reader) and $w(f) = \frac{1}{3}(4m + 10)$. Therefore $\gamma^2_{st}(G) \leq \frac{1}{3}(4m + 10)$, while $\sum_{i=1}^{m} t_i + 2 = 2(m + 1)$. For $m \geq 3$ we have $\frac{1}{3}(4m + 10) < 2(m + 1)$.

In Fig. 3 we see such a caterpillar for $m = 8$ with the corresponding function $f$. In this case $\gamma^2_{st}(G) = 14$, $\sum_{i=1}^{m} t_i + 2 = 18$. For the signed 2-domination number
here Theorem 1 holds. In Fig. 4 the same caterpillar is depicted with the function $f$ realizing the signed domination number $\gamma_s^2(G) = \sum_{i=1}^{m} t_i - 2m + 2 = 2$. □

Proposition 3. Let $G$ be a caterpillar with $m \geq 2$ and $s_i = 2$ for each $i \in [m]$. Then $\sum_{i=1}^{m} t_i - 2m + 2 < \gamma_s^2(G)$, but $\sum_{i=1}^{m} t_i + 2 = \gamma_{st}^2(G)$.

Proof. The caterpillar thus described is a simple path of length $m + 1$. It has $m + 2$ vertices. The inequality $\gamma_s^2(G) \leq \sum_{i=1}^{m} t_i - 2m + 2$ would imply that there exists a signed 2-dominating function $f$ which has the value $-1$ in $m$ vertices, while the value 1 only in two vertices. This is evidently impossible. On the other hand the open 2-neighbourhood of any vertex consists of at most two vertices and therefore the unique signed total 2-dominating function is the constant function equal to 1 in the whole set $V(G)$. Then

$$\gamma_{st}^2(G) = w(f) = \sum_{i=1}^{m} t_i + 2 = m + 2.$$ □

Now we shall study the signed 2-domination number of a simple path $P_n$ with $n$ vertices (i.e. of length $n - 1$). We shall not use the notation for caterpillars used above, but we shall denoted the vertices by $u_1, \ldots, u_n$ and edges by $u_iu_{i+1}$ for $i = 1, \ldots, n - 1$.

Theorem 3. Let $P_n$ be a path with $n$ vertices. If $n \equiv 0 \pmod{5}$, then $\gamma_s^2(P_n) = \frac{1}{5}n$. In general, asymptotically $\gamma_s^2(P_n) \approx \lfloor \frac{1}{5}n \rfloor$.

Proof. If $n \equiv 0 \pmod{5}$, then the closed neighbourhood $N^2[u_i] = \{ u_{i-2}, u_{i-1}, u_i, u_{i+1}, u_{i+2} \}$ for $i \equiv 3 \pmod{5}$, $3 \leq i \leq n - 2$, form a partition of $V(P_n)$. Let $f$
be a signed 2-dominating function on $P_n$. Then $f$ must have the value 1 in at least three vertices and may have the value $-1$ in at most two vertices of each class of this partition. Then $w(f) \geq \frac{3}{5}n = \frac{1}{5}n$. A function $f$ for which the equality occurs may be defined so that $f(u_i) = -1$ for $i \equiv 0 \pmod{5}$ and $i \equiv 1 \pmod{5}$ and $f(u_1) = 1$ for $i \equiv 2 \pmod{5}$, $i \equiv 3 \pmod{5}$ and $i \equiv 4 \pmod{5}$. Therefore $\gamma_s^2(P_n) = w(f) = \frac{1}{5}n$.

Now let $m \equiv r \pmod{5}$, $r \leq 4$. Let $q = n - r$. We have $q \equiv 0 \pmod{5}$ and thus $\gamma_s^2(P_q) = \frac{1}{5}q$. The path $P_n$ is obtained from $P_q$ by adding a path with $r$ vertices. Let $g$ be a minimum signed 2-dominating function on $P_n$, let $g_0$ be its restriction to $P_q$. We have $w(g_0) = \frac{1}{5}q$. Now the vertices of $P_n$ not in $P_q$ may have values 1 or $-1$ in $g$ and thus $\frac{1}{5}q - r \leq w(g) \leq \frac{1}{5}q + r$. In general, $\frac{1}{5}q - 4 \leq \gamma_s^2(P_n) \leq \frac{1}{5}q + 4$. This implies

$$\frac{9}{5n} - \frac{4}{n} \leq \frac{\gamma_s^2(P_n)}{n} \leq \frac{9}{5n} + \frac{4}{n}.$$ 

Therefore $\lim_{n \to \infty} \frac{\gamma_s^2(P_n)}{n} = \frac{9}{5m}$ and thus $\gamma_s^2(P_n) \approx \frac{9}{5} = \lfloor \frac{n}{5} \rfloor$. □

In Fig. 5 we see a path $P_{15}$ (with $\gamma_s^2(P_{15}) = 3$) in which the corresponding signed 2-dominating function is illustrated.

As has already been mentioned, $\gamma_{st}^2(P_n) = n$ for each positive integer $n$. Without a proof we shall state the values of $\gamma_s^2(P_n)$ for $n \leq 4$. We have $\gamma_s^2(P_1) = 1$, $\gamma_s^2(P_2) = 2$, $\gamma_s^2(P_3) = 1$, $\gamma_s^2(P_4) = 2$.

References


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