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SIGNED 2-DOMINATION IN CATERPILLARS

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Abstract. A caterpillar is a tree with the property that after deleting all its vertices of degree 1 a simple path is obtained. The signed 2-domination number $\gamma_2^s(G)$ and the signed total 2-domination number $\gamma_{st}^2(G)$ of a graph $G$ are variants of the signed domination number $\gamma_s(G)$ and the signed total domination number $\gamma_{st}(G)$. Their values for caterpillars are studied.

Keywords: caterpillar, signed 2-domination number, signed total 2-domination number

MSC 2000: 05C69, 05C05

This paper concerns caterpillars. A caterpillar [1] is a tree with the property that after deleting all its vertices of degree 1 a simple path is obtained. According to this definition a caterpillar has at least three vertices. But we need not care about graphs with one or two vertices. For such graphs our considerations are trivial.

Let $G$ be a caterpillar. The mentioned simple path will be denoted by $B$ and called the body of the caterpillar $G$. Let the number of vertices of $B$ be $m$. Let $a_1, \ldots, a_m$ be these vertices and let $a_ia_{i+1}$ for $i = 1, \ldots, m - 1$ be the edges of $B$. By $[m]$ we shall denote the set of integers $i$ such that $1 \leq i \leq m$. For each $i \in [m]$ let $s_i$ be the degree of $a_i$ in $G$. The vector $\overrightarrow{s} = (s_1, \ldots, s_m)$ will be called the degree vector of the caterpillar $G$.

Now we shall define variants of the signed domination number and of the signed total domination number [2] of a graph. For a vertex $u \in V(G)$ the symbol $N(u)$ denotes the open neighbourhood of $u$ in $G$, i.e. the set of all vertices which are adjacent to $u$ in $G$. The closed neighbourhood of $u$ is $N[u] = N(u) \cup \{u\}$. Similarly the open 2-neighbourhood $N^2(u)$ is the set of all vertices having the distance 2 from $u$ in $G$. The closed 2-neighbourhood of $u$ is $N^2[u] = N[u] \cup N^2(u)$. If $f$ is a mapping
of \( V(G) \) into some set of numbers and \( S \subseteq V(G) \), then \( f(S) = \sum_{x \in S} f(x) \) and the weight of \( f \) is \( w(f) = f(V(G)) = \sum_{x \in V(G)} f(x) \).

Let \( f : V(G) \to \{-1, 1\} \). If \( f(N^2[u]) \geq 1 \) (or \( f(N^2(u)) \geq 1 \)) for each \( u \in V(G) \), then \( f \) is called a signed 2-dominating (or signed total 2-dominating, respectively) function on \( G \). The minimum of weights \( w(f) \) taken over all signed 2-dominating (or all signed total 2-dominating) functions \( f \) is the signed 2-dominating number \( \gamma_2^S(G) \) (or the signed total 2-dominating number \( \gamma_2^{st}(G) \), respectively) of \( G \).

For each \( i \in [m] \) let \( t_i \in \{1, 2\} \) and \( t_i \equiv s_i + 1 \) (mod 2).

We shall prove a theorem concerning \( \gamma_2^S(G) \).

**Theorem 1.** Let \( G \) be a caterpillar with the degree vector \( \vec{s} = (s_1, \ldots, s_m) \) such that \( n \geq 2 \) and \( s_i \geq 3 \) for all \( i \in [m] \). Then

\[
\gamma_2^S(G) = \sum_{i=1}^{m} t_i - 2m + 2.
\]

**Proof.** Consider a vertex \( a_i \) with \( i \in [m] \). As \( s_i \geq 3 \), there exists at least one vertex \( u \in N(a_i) \) which does not belong to \( B \) and has degree 1. Then \( N^2[u] = N[a_i] \).

Let \( f \) be a signed 2-dominating function on \( G \). Then \( f(N^2[u]) = f(N[a_i]) \geq 1 \). The set \( N[a_i] \) has \( s_i + 1 \) vertices. If \( s_i \) is even, then \( s_i + 1 \) is odd. At least \( \frac{1}{2}(s_i + 2) = \frac{1}{2}s_i + 1 \) vertices of \( N[a_i] \) must have the value 1 in \( f \) and at most \( \frac{1}{2}s_i \) of them may have the value \(-1\). Then \( f(N^2[u]) \geq (\frac{1}{2}s_i + 1) - \frac{1}{2}s_i + 1 = t_i \). If \( s_i \) is odd, then \( s_i + 1 \) is even and at least \( \frac{1}{2}(s_i + 1) + 1 \) vertices of \( N[a_i] \) must have the value 1 in \( f \) and at most \( \frac{1}{2}(s_i + 1) - 1 \) of them may have the value \(-1\). Then \( f(N^2[u]) \geq 2 = t_i \).

We may easily construct the function \( f \) such that it has the value \(-1\) in exactly \( \frac{1}{2}s_i \) vertices of degree 1 in \( N[a_i] \) with \( i \) even and in exactly \( \frac{1}{2}(s_i + 1) - 1 = \frac{1}{2}(s_i - 1) \) vertices of degree 1 in \( N[a_i] \) with \( i \) odd. In all other vertices (including all vertices of the body) the function \( f \) has the value 1.

We have \( \bigcup_{i=1}^{m} N[a_i] = V(G) \). The vertex \( a_1 \) is contained in exactly two sets \( N[a_i] \), namely in \( N[a_1] \) and \( N[a_2] \). Similarly \( a_m \) is contained in exactly two sets \( N[a_{m-1}] \), \( N[a_m] \). For \( i \in [m] - \{1, m\} \) the vertex \( a_i \) is contained in exactly three sets \( N[a_{i-1}] \), \( N[a_i] \), \( N[a_{i+1}] \). Each vertex outside the body is contained in exactly one of these sets. By the Inclusion-Exclusion Principle we have

\[
w(f) = f(V(G)) = \sum_{i=1}^{m} f(N[a_i]) - 2 \sum_{i=2}^{m-1} f(a_i) - f(a_1) - f(a_m)
\]

\[
= \sum_{i=1}^{m} t_i - 2(m - 2) - 1 - 1 = \sum_{i=1}^{m} t_i - 2m + 2.
\]

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As f is the minimum function satisfying the requirements, we have
\[ \gamma_2^2(G) = w(f) = \sum_{i=1}^{m} t_i - 2m + 2. \]

An analogous theorem concerns \( \gamma_{st}^2(G) \).

**Theorem 2.** Let \( G \) be a caterpillar with the degree vector \( \vec{s} = (s_1, \ldots, s_m) \) such that \( m \geq 2 \) and \( s_i \geq 4 \) for all \( i \in [m] \). Then
\[ \gamma_{st}^2(G) = \sum_{i=1}^{m} t_i + 2. \]

**Proof.** Consider a vertex \( a_i \) with \( i \in [m] \). As \( s_i \geq 5 \), there exists at least one vertex \( u \in N(a_i) \) which does not belong to \( B \) and has degree 1. Then \( N^2(u) = N(a_i) - \{u\} \). Let \( f \) be a signed total 2-dominating function on \( G \). Then \( f(N^2(u)) = f(N(a_i) - \{u\}) \geq 1 \). The set \( N(a_i) - \{u\} \) has \( s_i - 1 \) vertices. If \( s_i \) is even, then \( s_i - 1 \) is odd. At least \( \frac{1}{2}s_i \) vertices of \( N(a_i) - \{u\} \) must have the value 1 in \( f \) and at most \( \frac{1}{2}(s_i - 2) = \frac{1}{2}s_i - 1 \) of them may have the value \(-1\). Then \( f(N^2(u)) \geq \frac{1}{2}s_i - \left( \frac{1}{2}s_i - 1 \right) = 1 = t_i \). If \( s_i \) is odd, then \( s_i - 1 \) is even and at least \( \frac{1}{2}(s_i - 1) + 1 \) vertices of \( N(a_i) - \{u\} \) must have the value 1 in \( f \) and at most \( \frac{1}{2}(s_i - 1) - 1 \) of them may have the value \(-1\). Then \( f(N^2(u)) \geq 2 = t_i \). As \( s_i \geq 5 \) for \( i \in [m] \), in both these cases we must admit the possibility \( f(u) = 1 \). Then in the case of \( s_i \) even we have \( f(N(a_i)) \geq 2 = t_i + 1 \) and in the case of \( s_i \) odd we have \( f(N(a_i)) \geq 3 = t_i + 1 \).

We may easily construct the function \( f \) such that it has the value \(-1\) in \( \frac{1}{2}s_i - 1 \) vertices of degree 1 in \( N(a_i) \) for \( s_i \) even, in \( \frac{1}{2}(s_i - 1) - 1 = \frac{1}{2}(s_i - 3) \) vertices of degree 1 in \( S(a_i) \) for \( s_i \) odd and the value 1 for all other vertices (including all vertices of \( B \)). Each vertex \( a_j \) for \( j \in [m] - \{1, m\} \) is contained in two sets \( N(a_i) \), namely in \( N(a_{j-1}) \) and \( N(a_{j+1}) \). Each other vertex is contained in exactly one set \( N(a_i) \). Again by the Inclusion-Exclusion Principle we have
\[ w(f) = f(V(G)) = \sum_{i=1}^{m} f(N(a_i)) - \sum_{i=2}^{m-1} f(a_i) = \sum_{i=1}^{m} (t_i + 1) - (m - 2) = \sum_{i=1}^{m} t_i + m - (m - 2) = \sum_{i=1}^{m} t_i + 2. \]

As \( f \) is the minimum function satisfying the requirements, we have
\[ \gamma_{st}^2(G) = w(f) = \sum_{i=1}^{m} t_i + 2. \]

\[ \square \]
In Figs. 1 and 2 a caterpillar $G$ with the degree vector $(5, 6, 7)$ is depicted. We have $t_1 = t_3 = 2$, $t_2 = 1$ and therefore $\gamma^2_{\text{st}}(G) = 7$ and $\gamma^2_s(G) = 1$. In Fig. 1 the values of the corresponding signed total 2-dominating function are illustrated; in the vertices denoted by $+$ the value is 1 and in the vertices denoted by $-$ it is $-1$. Similarly in Fig. 2 the corresponding signed 2-dominating function is illustrated.

In Theorems 1 and 2 we had the assumption $m \geq 2$. The following proposition concerns the singular case $m = 1$.

**Proposition 1.** Let $G$ be a caterpillar with the body consisting of one vertex, i.e. a star with the central vertex $a_1$ and with $s_1 \geq 2$ vertices of degree 1. Then $\gamma^2_{\text{st}}(G)$ is undefined and $\gamma^2_s(G) = t_1$.

**Proof.** The open 2-neighbourhood $N^2(a_1) = \emptyset$ and thus $f(N^2(a_1)) = 0$ for any function $f: V(G) \to \{-1, 1\}$, hence none of such functions might be signed total 2-dominating in $G$. On the other hand, $N^2[a_1] = V(G)$ and $|V(G)| = s_1 + 1$. Analogously as in the proofs of Theorems 1 and 2 we prove that for $s_1$ even we have $\gamma^2_s(G) = 1 = t_1$ and for $s_1$ odd we have $\gamma^2_s(G) = 2 = t_1$. \hfill $\square$

**Proposition 2.** Let $G$ be a caterpillar with $m \equiv 2 \pmod{5}$, $m \geq 5$, $s_i = 3$ for all $i \in [m]$. Then $\gamma^2_{\text{st}}(G) \leq \frac{4}{3}(m + 3) + 2$, while $\sum_{i=1}^{m} t_i + 2 = 2(m + 1)$.

**Proof.** As $s_i = 3$ for each $i \in [m]$, we have $t_i = 2$ for each $i \in [m]$. Each vertex $a_i$ for $i \in [m] - \{1, m\}$ is adjacent to exactly one vertex $v_i$ of degree 1. The vertex $a_1$ is adjacent to two such vertices $v_1, w_1$ and similarly $a_m$ to $v_m, w_m$. Let $f: V(G) \to \{-1, 1\}$ be defined so that $f(v_i) = -1$ for $i \equiv 0 \pmod{3}$ and $f(u) = 1$ for all other vertices $u$. This is a signed total 2-dominating function on $G$ (this can be easily verified by the reader) and $w(f) = \frac{1}{3}(4m+10)$. Therefore $\gamma^2_{\text{st}}(G) \leq \frac{1}{3}(4m+10)$, while $\sum_{i=1}^{m} t_i + 2 = 2(m + 1)$. For $m \geq 3$ we have $\frac{1}{3}(4m+10) < 2(m + 1)$.

In Fig. 3 we see such a caterpillar for $m = 8$ with the corresponding function $f$. In this case $\gamma^2_{\text{st}}(G) = 14$, $\sum_{i=1}^{m} t_i + 2 = 18$. For the signed 2-domination number
here Theorem 1 holds. In Fig. 4 the same caterpillar is depicted with the function \( f \) realizing the signed domination number \( \gamma_2^s(G) = \sum_{i=1}^{m} t_i - 2m + 2 = 2 \).

\[ \square \]

**Proposition 3.** Let \( G \) be a caterpillar with \( m \geq 2 \) and \( s_i = 2 \) for each \( i \in [m] \). Then \( \sum_{i=1}^{m} t_i - 2m + 2 < \gamma_2^s(G) \), but \( \sum_{i=1}^{m} t_i + 2 = \gamma_2^{st}(G) \).

**Proof.** The caterpillar thus described is a simple path of length \( m + 1 \). It has \( m + 2 \) vertices. The inequality \( \gamma_2^s(G) \leq \sum_{i=1}^{m} t_i - 2m + 2 \) would imply that there exists a signed 2-dominating function \( f \) which has the value \(-1\) in \( m \) vertices, while the value 1 only in two vertices. This is evidently impossible. On the other hand the open 2-neighbourhood of any vertex consists of at most two vertices and therefore the unique signed total 2-dominating function is the constant function equal to 1 in the whole set \( V(G) \). Then

\[ \gamma_2^{st}(G) = w(f) = \sum_{i=1}^{m} t_i + 2 = m + 2. \]

\[ \square \]

Now we shall study the signed 2-domination number of a simple path \( P_n \) with \( n \) vertices (i.e. of length \( n - 1 \)). We shall not use the notation for caterpillars used above, but we shall denoted the vertices by \( u_1, \ldots, u_n \) and edges by \( u_iu_{i+1} \) for \( i = 1, \ldots, n - 1 \).

**Theorem 3.** Let \( P_n \) be a path with \( n \) vertices. If \( n \equiv 0 \pmod{5} \), then \( \gamma_2^s(P_n) = \frac{1}{5} n \). In general, asymptotically \( \gamma_2^s(P_n) \approx \lfloor \frac{1}{5} n \rfloor \).

**Proof.** If \( n \equiv 0 \pmod{5} \), then the closed neighbourhood \( N^2[u_i] = \{u_{i-2}, u_{i-1}, u_i, u_{i+1}, u_{i+2}\} \) for \( i \equiv 3 \pmod{5}, 3 \leq i \leq n - 2 \), form a partition of \( V(P_n) \). Let \( f \)
be a signed 2-dominating function on $P_n$. Then $f$ must have the value 1 in at least three vertices and may have the value $-1$ in at most two vertices of each class of this partition. Then $w(f) \geq \frac{2}{5} n = \frac{1}{5} n$. A function $f$ for which the equality occurs may be defined so that $f(u_i) = -1$ for $i \equiv 0 \pmod{5}$ and $i \equiv 1 \pmod{5}$ and $f(u_1) = 1$ for $i \equiv 2 \pmod{5}$, $i \equiv 3 \pmod{5}$ and $i \equiv 4 \pmod{5}$. Therefore $\gamma_s^2(P_n) = w(f) = \frac{1}{5} n$.

Now let $m \equiv r \pmod{5}$, $r \leq 4$. Let $q = n - r$. We have $q \equiv 0 \pmod{5}$ and thus $\gamma_s^2(P_q) = \frac{1}{5} q$. The path $P_n$ is obtained from $P_q$ by adding a path with $r$ vertices. Let $g$ be a minimum signed 2-dominating function on $P_n$, let $g_0$ be its restriction to $P_q$. We have $w(g_0) = \frac{1}{5} q$. Now the vertices of $P_n$ not in $P_q$ may have values 1 or $-1$ in $g$ and thus $\frac{1}{5} q - r \leq w(g) \leq \frac{1}{5} q + r$. In general, $\frac{1}{5} q - 4 \leq \gamma_s^2(P_n) \leq \frac{1}{5} q + 4$. This implies

$$\frac{9}{5n} - 4 \leq \frac{\gamma_s^2(P_n)}{n} \leq \frac{9}{5n} + 4.$$

Therefore $\lim_{n \to \infty} \frac{\gamma_s^2(P_n)}{n} = \frac{9}{5m}$ and thus $\gamma_s^2(P_n) \approx \frac{9}{5} = \left\lfloor \frac{9}{5} \right\rfloor$. $\Box$

In Fig. 5 we see a path $P_{15}$ (with $\gamma_s^2(P_{15}) = 3$) in which the corresponding signed 2-dominating function is illustrated.

Fig. 5.

As has already been mentioned, $\gamma_s^2(P_n) = n$ for each positive integer $n$.

Without a proof we shall state the values of $\gamma_s^2(P_n)$ for $n \leq 4$. We have $\gamma_s^2(P_1) = 1$, $\gamma_s^2(P_2) = 2$, $\gamma_s^2(P_3) = 1$, $\gamma_s^2(P_4) = 2$.

References
