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Graphs isomorphic to their path graphs


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Abstract. We prove that for every number $n \geq 1$, the $n$-iterated $P_3$-path graph of $G$ is isomorphic to $G$ if and only if $G$ is a collection of cycles, each of length at least 4. Hence, $G$ is isomorphic to $P_3(G)$ if and only if $G$ is a collection of cycles, each of length at least 4. Moreover, for $k \geq 4$ we reduce the problem of characterizing graphs $G$ such that $P_k(G) \cong G$ to graphs without cycles of length exceeding $k$.

Keywords: line graph, path graph, cycles

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1. Introduction

Let $G$ be a graph, $k \geq 1$, and let $P_k$ be the set of paths of length $k$ in $G$. The vertex set of a path graph $P_k(G)$ is the set $P_k$. Two vertices of $P_k(G)$ are joined by an edge if and only if the edges in the intersection of the corresponding paths form a path of length $k - 1$ in $G$, and their union forms either a cycle or a path of length $k + 1$. It means that the vertices are adjacent if and only if one can be obtained from the other by “shifting” the corresponding paths in $G$.

Path graphs were investigated by Broersma and Hoede in [2] as a natural generalization of line graphs, since $P_1(G)$ is the line graph $L(G)$ of $G$ (for further connections to line graphs see [6]). Traversability of $P_2$-path graphs is studied in [9], and a characterization of $P_2$-path graphs is given in [2] and [7]. Distance properties of path graphs are studied in [1], [3] and [5], and in [4] graphs with connected $P_3$-path graphs are characterized.

When a new function on graphs appears, one of the very first problems is to determine the fixed points of the function, i.e., graphs that are isomorphic to their

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images. It is well known (and trivial to prove) that a connected graph $G$ is isomorphic to its line graph $L(G)$ if and only if $G$ is a cycle.

In [2] it is proved that a connected graph $G$ is isomorphic to its $P_2$-path graph if and only if $G$ is a cycle. We remark that $P_k(G)$ is not necessarily a connected graph if $G$ is connected and $k \geq 2$. However, slight modifications of the proof in [2] give rise to the following theorem:

**Theorem A.** Let $G$ be a graph isomorphic to $P_2(G)$. Then each component of $G$ is a cycle.

However, even stronger theorem follows from [5]. Let $P^i_k(G)$ denote the $i$-iterated $P_k$-path graph of $G$, i.e., $P^i_k(G) = P_k(P^{i-1}_k(G))$ if $i > 0$, and $P^0_k(G) = G$. We have

**Theorem B.** Let $G$ be a graph and $n$ a number, $n \geq 1$, such that $G$ is isomorphic to $P^*_2(G)$. Then each component of $G$ is a cycle.

By now, only a little is known about $P_k$-path graphs for $k \geq 3$. In [8] Li and Zhao proved the following theorem:

**Theorem C.** Let $G$ be a connected graph isomorphic to $P_3(G)$. Then $G$ is a cycle of length greater than or equal to 4.

In this paper we generalize Theorem C to an analogue of Theorem A for $P_3$-path graphs:

**Theorem 1.** Let $G$ be a graph isomorphic to $P_3(G)$. Then each component of $G$ is a cycle of length greater than or equal to 4.

In fact, we prove more. We prove an analogue of Theorem B for $P_3$-path graphs:

**Theorem 2.** Let $G$ be a graph and $n$ a number, $n \geq 1$, such that $G$ is isomorphic to $P^*_3(G)$. Then each component of $G$ is a cycle of length greater than or equal to 4.

We remark that the proof of Theorem C in [8] is analogous to the proof of Theorem A (in a weaker form) in [2], and since it is based on some counting arguments, it is not clear how to extend it to the proof of Theorem 2. In fact, our approach to the problem is completely different.

At present, we are not able to generalize Theorem 1 to $P_k$-path graphs for $k \geq 4$. The problem seems to be too complicated in general; for a solution for graphs in which every component contains a “large” cycle see Lemma 3. Thus, for $k \geq 4$ it may be useful to start with trees, see also Corollary 6. However, even for trees the problem appears to be hard. In this connection we pose the following
Problem. Does there exist a tree $T$ and a number $k$, $k \geq 4$, such that $P^i_k(T)$ is a nonempty forest for every $i \geq 0$?

If such a tree does not exist, then $P_k(G) \not\cong G$ for every number $k$ and a forest $G$. (We remark that the problem is trivial for $k = 1$; for $k = 2$ it is solved negatively in [5]; and for $k = 3$ it is solved negatively by Lemma 7.)

In the next two sections we prove Theorem 2. In Section 2 we present some general results for $P_k$-path graphs when $k \geq 2$, and Section 3 is devoted to $P_3$-path graphs of trees.

2. General results

We use standard graph-theoretic notation. Let $G$ be a graph. The vertex set and the edge set of $G$, respectively, are denoted by $V(G)$ and $E(G)$. If $v$ is a vertex of $G$ then $\deg_G(v)$ denotes the degree of $v$ in $G$. For two subgraphs, $H_1$ and $H_2$ of $G$, we denote by $H_1 \cup H_2$ the union of $H_1$ and $H_2$ in $G$, and by $H_1 \cap H_2$ their intersection. A path and a cycle, respectively, of length $l$ are denoted by $P_l$ and $C_l$.

For easier handling of paths of length $k$ in $G$ (i.e., the vertices of $P_k(G)$) we make the following agreement. We denote the vertices of $P_k(G)$ (as well as the vertices of $G$) by small letters $a$, $b$, ..., while the corresponding paths of length $k$ in $G$ are denoted by capital letters $A$, $B$, .... It means that if $A$ is a path of length $k$ in $G$ and $a$ is a vertex in $P_k(G)$, then $a$ must be the vertex corresponding to the path $A$.

Throughout this section, the symbol $k$ is used for the length of paths producing the path graph. I.e., we consider here only $P_k$-path graphs, $k \geq 2$. By a large cycle we mean a cycle of length greater than $k$. A cycle whose length does not exceed $k$ is a small cycle.

Lemma 3. Let $G$ be a graph such that $P^n_k(G) \cong G$ for some $n \geq 1$ and $k \geq 2$. Then every component of $G$ containing a large cycle is isomorphic to a single cycle.

Proof. If $C$ is a large cycle then $P_k(C) \cong C$, and hence $P^n_k(C) \cong C$, too. As $P^n_k(G) \cong G$, all large cycles in $P^n_k(G)$ are images of large cycles in $G$.

If $a$ and $b$ are adjacent vertices in $P_k(G)$, then $A$ and $B$ share a path of length $k - 1$. This implies that if $C$ is a large cycle in $G$, then $P_k(C)$ does not contain a chord in $P_k(G)$, and hence, $P^n_k(C)$ does not contain a chord in $P^n_k(G)$, either. Since the number of large cycles in $G$ is equal to the number of large cycles in $P^n_k(G)$, no large cycle contains a chord in $G$.

Suppose that there is a large cycle in $G$ with a vertex incident to an edge outside this cycle. For every large cycle $C$, let $I_G(C)$ denote the total number of edges outside $C$ that are incident to a vertex of $C$. As $C$ does not contain a chord,
\(I_{P_k(G)}(P_k(C)) = 2 \cdot I_G(C)\), and \(I_{P^n_k(G)}(P^n_k(C)) = 2^n \cdot I_G(C)\). Let \(I(G)\) denote the maximum value of \(I_G(C)\), where \(C\) is a large cycle in \(G\). Then \(I(P^n_k(G)) = 2^n \cdot I(G)\). Since \(I(G) > 0\), \(P^n_k(G)\) is not isomorphic to \(G\). \(\square\)

**Lemma 4.** Every small cycle in \(P_k(G)\) has an even length.

**Proof.** Let \(C = (a_1, a_2, \ldots, a_l)\) be a small cycle of length \(l\) in \(P_k(G)\), \(l \leq k\). If \(u\) and \(v\) are adjacent vertices in \(P_k(G)\), then \(U\) and \(V\) share a path of length \(k - 1\). Since \(l \leq k\), \(A_1, A_2, \ldots, A_l\) share a path \(P\) of length \(t \geq k - (l - 1) = 1\) in \(G\). Let \(P = (p_0, p_1, \ldots, p_t)\).

Assume that \(A_1 = (a_{1,0}, a_{1,1}, \ldots, a_{1,k}), \ldots, A_l = (a_{l,0}, a_{l,1}, \ldots, a_{l,k})\) are denoted so that for every \(i, 1 \leq i \leq l\), we have \(i_0 < i_t\), where \(a_{i,i_0} = p_0\) and \(a_{i,i_t} = p_t\). Then \(i_0\) and \((i + 1)_0\) have different parity, \(1 \leq i < l\). As \((a_{1,1}, a_{2,1}, \ldots, a_{l,1})\) forms a cycle, \(l_0\) and \(l_0\) have different parity, too, and hence \(l\) is even. \(\square\)

We remark that Lemma 3 and Lemma 4 reduce the examination of graphs \(G\) for which \(P^n_k(G) \cong G\), to graphs without large cycles and without odd small cycles. Hence, to complete the proof of Theorem 2 it remains to study forests.

**Definition.** A tree \(T_t\) is obtained from a claw \(K_{1,3}\) subdividing each edge of \(K_{1,3}\) by \(t - 1\) vertices, see Figure 1 for \(T_3\). A tree \(K_{t_1,t_2}\) is obtained from two paths of length \(2t_1\) central vertices of which are joined by a path of length \(t_2\), see Figure 2 for \(K_{1,2}\) and Figure 3 for \(K_{2,1}\).

\[\begin{align*}
\Psi_3: & \quad \begin{align*}
\xymatrix{ & *+{a} \ar@{-}[dl] & \star \ar@{-}[l] & \star \ar@{-}[l] & \star \ar@{-}[l] & \star \ar@{-}[l] & \star \ar@{-}[l] & \star \ar@{-}[l] & \star \ar@{-}[l] & \star \ar@{-}[l] & \star \ar@{-}[l] & \star \ar@{-}[l] & \star \ar@{-}[l] & \star \ar@{-}[l] & \star \ar@{-}[l] \end{align*}
\end{align*}\]
Figure 1

\[\begin{align*}
\Psi_3: & \quad \begin{align*}
\xymatrix{ & *+{a} \ar@{-}[dl] & \star \ar@{-}[l] & \star \ar@{-}[l] & \star \ar@{-}[l] & \star \ar@{-}[l] & \star \ar@{-}[l] & \star \ar@{-}[l] & \star \ar@{-}[l] & \star \ar@{-}[l] & \star \ar@{-}[l] & \star \ar@{-}[l] & \star \ar@{-}[l] & \star \ar@{-}[l] & \star \ar@{-}[l] \end{align*}
\end{align*}\]
Figure 2

\[\begin{align*}
\Psi_3: & \quad \begin{align*}
\xymatrix{ & *+{a} \ar@{-}[dl] & \star \ar@{-}[l] & \star \ar@{-}[l] & \star \ar@{-}[l] & \star \ar@{-}[l] & \star \ar@{-}[l] & \star \ar@{-}[l] & \star \ar@{-}[l] & \star \ar@{-}[l] & \star \ar@{-}[l] & \star \ar@{-}[l] & \star \ar@{-}[l] & \star \ar@{-}[l] & \star \ar@{-}[l] \end{align*}
\end{align*}\]
Figure 3

It is easy to see that \(P_k(\Psi_k) = C_{3k}\) and \(P_k(\Psi_{t,k-t})\) contains \(C_{4t}\), \(1 \leq t < k\). Thus, \(P_k\)-path graphs of trees containing a copy of \(\Psi_k\) or \(\Psi_{t,k-t}\) contain cycles. The next lemma shows a converse.

**Lemma 5.** If \(T\) is a tree such that \(P_k(T)\) contains a cycle, then \(T\) contains either \(\Psi_k\) or \(\Psi_{t,k-t}\), \(1 \leq t < k\).

**Proof.** Only for this proof we define the notion of a turning path (see also [1]). Let \((x_0, x_1, \ldots, x_{l-1})\) be a closed walk of length \(l\) in \(P_k(G)\). Then for every \(i, 0 \leq i < l\), the edges of \(X_{i-1} \cap X_i\) form a path of length \(k - 1\) (the indices are modulo \(l\)). A path \(X_i\) is a turning path if and only if the edges of \(X_{i-1} \cap X_i \cap X_{i+1}\) form a path of length \(k - 1\), too.
Let $T$ be a tree, and let $C = (a_0, a_1, \ldots, a_{l-1})$ be a cycle of length $l$ in $P_k(T)$. Since $T$ is a tree, there are at least two turning paths among $A_0, A_1, \ldots, A_{l-1}$. Let $A_i$, and $A_j$ be turning paths such that $t = [(i_2 - i_1) \mod l]$ is the smallest possible. Then $A_i \cup A_{i+1} \cup \ldots \cup A_{i_2}$ forms a path of length $k + t$ (the indices are modulo $l$).

If $t \geq k$, then $A_{i-1} \cup A_{i-t+1} \cup \ldots \cup A_{i_2}$ forms another path of length $t + k$, since $A_{i-1}, A_{i-t+2}, \ldots, A_{i-1}$ are not turning paths. As $A_{i-1} \neq A_{i+1}, T$ contains $\Psi_k$.

Thus, suppose that $t < k$. Since $t < k$, $A_{i_1}$ and $A_{i_2}$ share a path $P = (p_0, p_1, \ldots, p_t)$ of length $l' = k - t$. Assume that $A_{i_1} = (b_t, b_{t-1}, \ldots, b_1, p_0, p_1, \ldots, p_t)$ and $A_{i_2} = (p_0, p_1, \ldots, p_t, c_1, c_2, \ldots, c_t)$. Then $A_{i_1} \cup A_{i_1+1} \cup \ldots \cup A_{i_2} = (b_t, b_{t-1}, \ldots, b_1, p_0, p_1, \ldots, p_t, c_1, c_2, \ldots, c_t)$. Since $t$ is the smallest possible distance between vertices of $C$ corresponding to turning paths, we have $A_{i_1} \cup A_{i_1-t+1} \cup \ldots \cup A_{i_2} = (b_t, b_{t-1}, \ldots, b_1, p_0, p_1, \ldots, p_t, d_1, d_2, \ldots, d_t)$, where $d_1 \neq c_1$, and $A_{i_2} \cup A_{i_2+1} \cup \ldots \cup A_{i_2+t} = (c_t, c_{t-1}, \ldots, c_1, p_0, p_1, \ldots, p_t, c_1, c_2, \ldots, c_t)$, where $e_1 \neq b_1$. As $T$ is a tree and $l' = k - t$, $T$ contains $K_{t,k-t}$. \(\square\)

By Lemma 5 and the note before it, we have

**Corollary 6.** Let $T$ be a tree. Then $P_k(T)$ contains a cycle if and only if $T$ contains either $\Psi_k$ or $K_{t,k-t}, 1 \leq t < k$.

### 3. $P_3$-Path Graphs

**Definition.** A *caterpillar* is a tree $T$ with a path $P = (v_0, v_2, \ldots, v_l)$, such that the eccentricity of $v_i$ is at most 2 in the component of $T - E(P)$ containing $v_i$, $0 \leq i \leq l$. A *3-caterpillar* $T$ is a caterpillar with a path $P = (v_0, v_2, \ldots, v_3r)$ such that $\deg_T(v_i) = 2$ if $0 < i < 3r$ and $i \neq 3j$. The vertices $v_{3j}$ are called the *basic vertices* of a 3-caterpillar, $0 \leq j \leq r$.

![Figure 4](image_url)

In Figure 4 we have a 3-caterpillar $T$ with 4 basic vertices $v_0, v_3, v_6$ and $v_9$. We remark that usually, the term caterpillar is used for a bit different tree. However, in this paper we use this notion only for the graph defined in the preceding definition.

Let $G$ be a forest such that $P_3^n(G) \cong G$ for some $n \geq 1$. By Lemma 4, every cycle in $P_3(G)$ is a large cycle. Thus, $G$ does not contain $\Psi_3, K_{1,2}$ or $K_{2,1}$ by Corollary 6,
as otherwise $P^n_3(G)$ contains a large cycle. In particular, since $G$ does not contain $\Psi_3$, it is a disjoint union of caterpillars.

**Lemma 7.** Let $T$ be a caterpillar such that $P^j_3(T)$ is a forest for every $i \geq 0$. Then there is $j$ such that $P^j_3(T)$ is an empty graph.

**Proof.** Let $T$ be a caterpillar such that $P^j_3(T)$ does not contain a cycle for every $i \geq 0$. If the diameter of $T$ is at most 4, then $P_3(T)$ does not contain a path of length 3 (recall that $T$ does not contain $X_{1,2}$), so that $P^2_3(T)$ is an empty graph. Hence, assume that the diameter of $T$ is at least 5.

If $G$ is a tree, then at most one nontrivial component of $P_3(G)$ is different from a complete bipartite graph, see [4, Corollary 5]. Since the diameter of a complete bipartite graph is at most two, at most one nontrivial component of $P_3(T)$ is a caterpillar containing a path of length 3. Hence, in what follows it is enough to consider this unique “large” caterpillar of $P_3(G)$.

Let $T$ be a 3-caterpillar with a path $P = (v_0, v_1, \ldots, v_{3r})$ denoted as in the definition above. Then $T$ has exactly $r + 1$ basic vertices. Let $V'_i = (v_i, v_{i+1}, v_{i+2}, v_{i+3})$, $0 \leq i \leq 3r - 3$. Then the (large) caterpillar $T'$ of $P_3(T)$ is a 3-caterpillar with a path $P' = (v'_1, v'_2, \ldots, v'_{3r-3})$ with exactly $r$ basic vertices, see Figure 4. Hence, the caterpillar of $P^j_3(T)$ is a 3-caterpillar with a unique basic vertex, so that $P^{j+2}_3(T)$ is an empty graph.

To prove the lemma it is enough to concentrate on caterpillars that are not 3-caterpillars. These caterpillars $T^*$ contain a pair of vertices $u^*_1$ and $u^*_2$ at a distance either $3j + 1$ or $3j + 2$ such that $\deg_{T^*}(u^*_1) \geq 3$ and $\deg_{T^*}(u^*_2) \geq 3$. Moreover, if the distance from $u^*_1$ to $u^*_2$ is $3j + 1$ and $Q^*$ is a path joining $u^*_1$ with $u^*_2$, then the eccentricity of $u^*_i$ in the component of $T^* - E(Q^*)$ containing $u^*_i$ is at least 2, $i \in \{1, 2\}$. In what follows consider the caterpillar $T$ of $P^j_3(T^*)$. By our assumption, $T$ is not a 3-caterpillar and it contains two vertices $u_1$ and $u_2$ at a distance either 1 or 2 such that $\deg_T(u_1) \geq 3$ and $\deg_T(u_2) \geq 3$. Moreover, if $u_1u_2 \in E(G)$ then the eccentricity of $u_i$ in the component of $T - \{u_1u_2\}$ containing $u_i$ is at least 2, $i \in \{1, 2\}$.

If $u_1$ and $u_2$ have distance 2 then $T$ contains $X_{1,2}$, so that $P_3(T)$ contains a large cycle.

Now consider a caterpillar $T$ with $u_1u_2 \in E(T)$. Let $\text{Co}(u_1)$ and $\text{Co}(u_2)$ be components of $T - \{u_1u_2\}$ containing $u_1$ and $u_2$, respectively. As mentioned above, the eccentricities of $u_1$ and $u_2$ in these components are at least 2. However, if both of them exceed 2, then $P_3(T)$ contains $X_{1,2}$.

Thus, suppose that the eccentricity of $u_1$ in $\text{Co}(u_1)$ is exactly 2. Then $\text{Co}(u_1)$ consists of at most 2 paths of length 2 rooted in $u_1$ and of some edges incident
with \( u_1 \). Otherwise \( T \) contains \( \mathcal{K}_{1,2} \) or \( P_3(T) \) contains \( \Psi_3 \). Moreover, as iterated \( P_3 \)-path graphs of \( T \) do not contain \( \mathcal{K}_{1,2} \), the distance from \( u_2 \) to \( u \) is \( 3j \) if \( u \in V(\text{Co}(u_2)) \) and \( \text{deg}_T(u) \geq 3 \).

We distinguish two cases.

Case 1: There are exactly two paths of length 2 rooted in \( u_1 \).

If the eccentricity of \( u_2 \) is at least 5 in \( \text{Co}(u_2) \), then \( P_3(T) \) contains \( \Psi_3 \). On the other hand, if the eccentricity of \( u_2 \) is at most 4 in \( \text{Co}(u_2) \), then the caterpillar \( T' \) of \( P_3(T) \) is a 3-caterpillar, or \( P_3(T) \) contains \( \mathcal{K}_{1,2} \), see Figure 5. (We remark that \( U' = (u_1, u_2, u_3, u_4) \). Extra edges that are possibly in \( T \) or in \( T' \), are represented by halfedges in Figure 5.)

![Figure 5](image)

Figure 5

![Figure 6](image)

Figure 6

Case 2: There is exactly one path of length 2 rooted in \( u_1 \).

Let \((u_2, u_3, \ldots, u_k)\) be a longest path of \( \text{Co}(u_2) \) rooted in \( u_2 \). Denote by \( T' \) the caterpillar of \( P_3(T) \), \( U'_2 = (u_1, u_2, u_3, u_4) \) and \( U'_4 = (u_2, u_3, u_4, u_5) \) (if \( u_5 \) exists), see Figure 6. Let \( \text{Co}(u'_1) \) and \( \text{Co}(u'_2) \) be defined analogously to \( \text{Co}(u_1) \) and \( \text{Co}(u_2) \) above.

If the eccentricity of \( u_2 \) is at least 6 in \( \text{Co}(u_2) \) (i.e., if \( k \geq 8 \)), then the eccentricities of \( u'_1 \) and \( u'_2 \) are greater than 2 in \( \text{Co}(u'_1) \) and \( \text{Co}(u'_2) \), respectively. Hence, \( P^2_3(T) \) contains \( \mathcal{K}_{1,2} \), a contradiction.

If the eccentricity of \( u_2 \) is at most 2 in \( \text{Co}(u_2) \) (i.e., if \( k = 4 \)), then \( T' \) is a 3-caterpillar (recall that \( T' \) does not contain \( \mathcal{K}_{1,2} \)).

Let \( u \) be a vertex adjacent to \( u_2 \), \( u \neq u_3 \), such that \( \text{deg}_T(u) = 2 \), see Figure 6. If \( k = 7 \) then \( T' \) is of the type already solved in Case 1, and if \( 5 \leq k \leq 6 \) then \( T' \) is of the type from Case 2 and all (but one) neighbours of \( u_2 \) have degree 1 in \( \text{Co}(u'_2) \). Hence, we may assume that all neighbours of \( u_2 \) (except \( u_3 \)) have degree 1 in \( \text{Co}(u'_2) \).

Now if \( k = 7 \) then \( T' \) is of the type from Case 2 with the eccentricity of \( u'_2 \) exactly 3 in \( \text{Co}(u'_2) \), and if \( 5 \leq k \leq 6 \) then \( T' \) is a 3-caterpillar.

By Lemma 7, for every caterpillar \( T \) there is a number \( t \) such that \( P^t_3(T) \) is an empty graph or \( P^t_3(G) \) contains a large cycle. Thus, for every \( n, n \geq 1 \), we have \( P^n_3(G) \not\cong G \) if \( G \) is a forest, which completes the proof of Theorem 2.
References


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