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CHARACTERIZATION OF SEMIENTIRE GRAPHS WITH CROSSING NUMBER 2

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Abstract. The purpose of this paper is to give characterizations of graphs whose vertex-semientire graphs and edge-semientire graphs have crossing number 2. In addition, we establish necessary and sufficient conditions in terms of forbidden subgraphs for vertex-semientire graphs and edge-semientire graphs to have crossing number 2.

Keywords: semientire graph, vertex-semientire graph, edge-semientire graph, crossing number, forbidden subgraph, homeomorphic graphs

MSC 2000: 05C50, 05C99

1. Introduction

Graphs considered here are simple graphs (without loops and multiple edges). A graph is said to be embedded in a surface when it is drawn on $S$ so that no two edges intersect. A graph is planar if it can be embedded in the plane. By a plane graph we mean a graph embedded in the plane as opposed to a planar graph.

If there exists an edge $e_1 = uv$ in a plane graph $G$, we say that the vertices $u, v$ are adjacent to each other and both incident to the edge $e_1 = uv$. The edge $e_1 = uv$ is said to be adjacent to an edge $e_2$ if and only if $e_2 = uw$ or $e_2 = vw$, where $w$ is a vertex of $G$ distinct from $u$ and $v$. A region of $G$ is adjacent to the vertices and edges which are on its boundary, and two regions of $G$ are adjacent if their boundaries share a common edge. In this paper, vertices, edges and regions are called the elements of $G$.

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Kulli and Akka [2] introduced the concepts of a vertex-semientire graph and an edge-semientire graph of a graph. The vertex-semientire graph $e_v(G)$ of a plane graph $G$ is the graph whose vertex set is the union of the vertex set and the region set of $G$ and in which two vertices are adjacent if and only if the corresponding elements (two vertices, two regions or a vertex and a region) of $G$ are adjacent. The edge-semientire graph $e_e(G)$ of a plane graph $G$ is the graph whose vertex set is the union of the edge set and the region set of $G$ and in which two vertices are adjacent if and only if the corresponding elements (two edges, two regions or an edge and a region) of $G$ are adjacent. For other definitions see [1].

In [2], Kulli and Akka established characterizations of graphs whose vertex-semientire graphs and edge-semientire graphs are planar and outerplanar. Further, in [3], Kulli and Muddebihal established characterizations of graphs whose vertex-semientire graphs and edge-semientire graphs have crossing number one. In addition, they established necessary and sufficient conditions in terms of forbidden subgraphs for vertex-semientire graphs and edge-semientire graphs to have crossing number one.

The main results of this paper are characterizations of graphs whose vertex-semientire graphs and edge-semientire graphs have crossing number 2. In addition, we give characterizations in terms of forbidden subgraphs of graphs whose vertex-semientire graphs and edge-semientire graphs have crossing number 2.

The following will be useful for proving our theorems.

**Theorem A** [2]. Let $G$ be a connected plane graph. Then $e_v(G)$ is planar if and only if $G$ is a tree.

**Theorem B** [2]. Let $G$ be a connected plane graph. Then $e_e(G)$ is planar if and only if $∆(G) ≤ 3$ and $G$ is a tree.

**Theorem C** [3]. Let $G$ be a connected plane graph. Then $e_v(G)$ has crossing number 1 if and only if $G$ is unicyclic.

**Theorem D** [3]. The edge-semientire graph $e_e(G)$ of a connected plane graph $G$ has crossing number 1 if and only if (1) or (2) holds.
(1) $∆(G) = 3$, $G$ is unicyclic and such that at least one vertex of degree 2 is on the cycle.
(2) $∆(G) = 4$, $G$ is a tree and has exactly one vertex of degree 4.
2. Main results

In the next theorem, we present a characterization of graphs whose vertex-semientire graphs have crossing number 2.

**Theorem 1.** Let $G$ be a connected plane graph. Then $e_v(G)$ has crossing number 2 if and only if $G$ has exactly two cycles and these cycles are its blocks.

**Proof.** Suppose $e_v(G)$ has crossing number 2. Assume that $G$ is a tree. Then by Theorem A, $e_v(G)$ is planar, a contradiction.

Assume that $G$ has at least three cycles. Suppose each cycle is a block of $G$. Then by Theorem C, each block which is a cycle in $G$ gives at least one crossing in $e_v(G)$. Hence $e_v(G)$ has at least three crossings, a contradiction. Thus $G$ has exactly two cycles.

Suppose two cycles lie in a block. Then $G$ has a subgraph homeomorphic to $K_4 - x$. $G$ has two interior regions $r_1$ and $r_2$ and the exterior region $R$. In $e_v(G)$, the vertices $r_1$, $r_2$ and $R$ are mutually adjacent, since the regions $r_1$, $r_2$ and $R$ are mutually adjacent in $G$. Then in each adjacency there exists at least one crossing. Hence $e_v(G)$ has at least 3 crossings, a contradiction. Thus we conclude that $G$ has exactly two cycles as blocks.

Conversely, assume that $G$ has exactly two cycles $C_i$, $i = 1, 2$, which are both blocks. Also, let each edge which is not on $C_i$ be a block of $G$. Let $r_i$, $i = 1, 2$ be two interior regions of $C_i$ and $R$ the exterior region of $G$. In $e_v(G)$, the vertex $r_i$ is adjacent to each vertex of $C_i$ without crossings, the vertex $R$ is adjacent to each vertex of $G$ without crossings and the vertex $R$ is adjacent to $r_i$ with two crossings.

Thus $e_v(G)$ has crossing number 2. This completes the proof of the theorem.

In the next theorem, we obtain a characterization of graphs whose edge-semientire graphs have crossing number 2.

**Theorem 2.** The edge-semientire graph $e_e(G)$ of a connected plane graph $G$ has crossing number 2 if and only if

1) $\deg v \leq 4$ for every vertex $v$ of $G$, and $G$ is a tree and has exactly two vertices of degree 4, or $G$ is not a tree and has exactly one cutvertex of degree 4 and exactly one cycle such that at least one vertex of degree 2 is on the cycle or

2) $\deg v \leq 3$ for every vertex $v$ of $G$ and $G$ has exactly two cycles and these cycles are its blocks in which at least one vertex of degree 2 lies on each cycle, or $G$ is unicyclic and such that no vertex of degree 2 is on the cycle.
Proof. Suppose the edge-semientire graph $e_e(G)$ of a connected plane graph $G$ has crossing number 2. Then it is nonplanar. By Theorem B or D, $G$ is a tree with $\Delta(G) \geq 4$ or $G$ is not a tree and $\Delta(G) \leq 3$.

Suppose $G$ is a tree with $\deg \geq 4$ for some vertex $v$ of $G$. We consider the following cases.

Case 1. Suppose $\deg v \geq 5$ for some vertex $v$ of the tree $G$. Then clearly $c(e_e(G)) > 2$, a contradiction. Hence $\Delta(G) \leq 4$.

Case 2. Suppose $\deg v = 4$ for some vertex $v$ of $G$. Assume $G$ has at least 3 vertices of degree 4. Then $L(G)$ has at least 3 subgraphs isomorphic $K_4$. By the definition of $e_e(G)$, $L(G)$ is a subgraph of $e_e(G)$. The vertex $R$ in $e_e(G)$ which corresponds to the exterior region is adjacent to every vertex of $L(G)$, which gives at least 3 subgraphs isomorphic $K_5$ in $e_e(G)$. Hence $c(e_e(G)) > 2$, a contradiction. Thus $G$ has at most two vertices of degree 4.

Suppose $G$ is not a tree and assume $\deg v = 4$ for some vertex $v$ of $G$. We consider 2 cases.

Case 1. Assume $G$ has at least two vertices of degree 4 and at least one cycle $C$. Then $L(G)$ has at least 2 subgraphs isomorphic to $K_4$ and at least one subgraph $L(C)$. By the definition of $e_e(G)$, $L(G) \subset e_e(G)$. The vertex $r$ in $e_e(G)$ (which corresponds to an interior region of $C$) is adjacent to every vertex of $L(C)$. This gives one wheel $W$. The vertex $R$ in $e_e(G)$ is adjacent to every vertex of two $K_4$ and $W$ of $L(G)$. This gives at least 3 subgraphs isomorphic to $K_5$ in $e_e(G)$. Thus $c(e_e(G)) \geq 3$, a contradiction.

Case 2. Assume $G$ has at least one vertex of degree 4, at least two cycles $C_i$, $i = 1, 2$ as blocks and let $r_i$ be the interior regions of $C_i$. Then $L(G)$ has at least one subgraph isomorphic to $K_4$ and at least two subgraphs $L(C_i)$. In $e_e(G)$, $r_i$ is adjacent to every vertex of $L(C_i)$, which gives a wheel $W_i$. Since $L(G) \subset e_e(G)$, the vertex $R$ in $e_e(G)$ which corresponds to the exterior region is adjacent to every vertex of $L(G)$ and $r_i$. This gives at least 3 subgraphs isomorphic to $K_5$ in $e_e(G)$. Hence $c(e_e(G)) > 2$, a contradiction.

From cases 1 and 2 we conclude that $G$ has exactly one vertex of degree 4 and exactly one cycle.

Suppose $G$ has exactly one vertex $v$ of degree 4 and a cycle $C$. Assume that every vertex of $C$ has degree at least three. Let $e_i$, $i = 1, 2, 3$ and 4 be edges adjacent to $v$. Then $L(G)$ has exactly one subgraph isomorphic to $K_4$ and exactly one cycle $L(C)$. Let $r$ be the interior region of $C$ and $R$ the exterior region of $G$. In $e_e(G)$, the vertex $r$ is adjacent to every vertex of $L(C)$ without crossing, which gives $e_e(G) - R$. We get two wheels $L(C) + r$ and $K_3 + e_i (= K_4)$, $i = 1, 2, 3$ or 4 in $e_e(G) - R$. In $e_e(G) - \{r, R, e_i\}$, the vertex $R$ is adjacent to every vertex of $e_e(G) - \{r, e_i\}$ without crossings. In $e_e(G)$ it is easy to see that the edges $Re_i$ and $rR$ cross respectively at
least one edge and at least 2 edges of $e_e(G) - \{rR, re_i\}$. Thus $e_e(G)$ has at least 3 crossings, a contradiction. This proves (1).

Assume $G$ is not a tree and $\deg v \leq 3$ for every vertex $v$ of $G$. We consider three cases.

**Case 1.** Assume $G$ has at least 3 cycles. Suppose each cycle has at least one vertex of degree two and each cycle is a block of $G$. Let $R$ and $r_i$, $i = 1, 2, 3$ be vertices in $e_e(G)$ which correspond to the exterior and interior regions of $G$. Then $e_e(G) - R$ has at least 3 blocks each of which is a wheel. In $e_e(G)$, $R$ is adjacent to each wheel. We get at least 1 crossing in each case. It is clear that $e_e(G)$ has at least 3 crossings, a contradiction.

**Case 2.** Suppose $G$ has at least two cycles in a block. Then $G$ has a subgraph homeomorphic to $K_4 - x$. Obviously $G$ has 2 interior regions, say $r_1$ and $r_2$, and the exterior region $R$. Clearly $e_e(G) - R$ has a block in which the edge joining the vertices $r_1$ and $r_2$ has two crossings. Also in $e_e(G)$, the vertex $R$ is adjacent to $r_1$ and $r_2$, which makes two more crossings. Thus $c(e_e(G)) \geq 4$, a contradiction.

From the above cases, we conclude that $G$ has at most two cycles $C_i$ as blocks.

Assume $G$ has no vertex of degree 2 on each cycle $C_i$. The interior regions $r_1$ and $r_2$ are adjacent respectively to every vertex of $C_1$ and $C_2$ without crossings and this gives $e_e(G) - R$ where $R$ is the exterior region. The vertex $R$ is adjacent to each vertex of $e_e(G) - \{r_1, r_2\}$ without crossings. In $e_e(G)$, $r_1R$ and $r_2R$ are edges. Clearly each $r_iR$ crosses at least 2 edges in $e_e(G) - \{r_1R, r_2R\}$. Thus $c(e_e(G)) \geq 4$, a contradiction.

Suppose $G$ is unicyclic and all vertices of the cycle $C$ are of degree less than 3. Assume that at least one vertex of the cycle $C$ of $G$ has degree 2. Then by condition (1) of Theorem D, $e_e(G)$ has exactly one crossing, a contradiction. This proves (2). Conversely, suppose $G$ is a graph satisfying conditions (1) or (2). Then by Theorem B or D, $e_e(G)$ has crossing number at least 2. We now show that its crossing number is at most 2. Assume first that $G$ satisfies condition (1). We consider 3 cases.

**Case 1.** Suppose $G$ is a tree and has exactly two vertices of degree 4. Then clearly $e_e(G)$ has exactly two subgraphs, each isomorphic to $K_5$, and hence $e_e(G)$ can be drawn with exactly two crossings.

**Case 2.** Suppose $G$ is not a tree and has exactly one vertex of degree 4 and exactly one cycle $C$ such that at least one vertex of degree 2 is on the cycle. Then it is easy to see that $e_e(G)$ has exactly two crossings.

Now assume (2). Then $G$ has exactly two cycles $C_i$ as blocks in which at least one vertex of degree 2 lies on each cycle. Let $r_i$, $i = 1, 2$ be the interior regions of two circles $C_i$ of $G$. The vertex $r_i$ is adjacent to every vertex of $L(C_i)$ without crossings, which gives $e_e(G) - R$ where $R$ is the exterior region of $G$. Obviously $e_e(G) - R$
has at least two blocks each of which is a wheel with at least one boundary edge. In $e_e(G) - \{r_1R, r_2R\}$ the vertex $R$ is adjacent to every vertex of $e_e(G) - \{r_1, r_2\}$ without crossings. By the definition of $e_e(G)$, $r_1R$ and $r_2R$ are edges. Hence either of $r_1R$ and $r_2R$ crosses exactly one edge of $e_e(G) - \{r_1R, r_2R\}$ and gives $e_e(G)$. Hence $e_e(G)$ has exactly two crossings.

Suppose $G$ is unicyclic in which no vertex of degree 2 is on the cycle $C$. Let the vertices $r$ and $R$ correspond to the interior and exterior regions of $G$, respectively. The vertex $r$ is adjacent to every vertex of $L(C)$ and gives one wheel together with a triangle on each side (in $e_e(G) - R$) without crossings. In $e_e(G) - rR$, the vertex $R$ is adjacent to every vertex of $e_e(G) - rR$ without crossings. Thus the edge $rR$ crosses exactly two boundary edges of $e_e(G) - rR$ and gives $e_e(G)$. Hence $c(e_e(G)) = 2$. This completes the proof of the theorem. □

### 3. Forbidden subgraphs

With help of Theorems 1 and 2 we now characterize graphs whose semientire graphs have crossing number 2, in terms of forbidden subgraphs.

**Theorem 3.** Suppose a connected plane graph $G$ has at least two cycles as blocks. The vertex-semientire graph $e_v(G)$ has crossing number 2 if and only if it has no subgraph homeomorphic to $G_i$, $i = 12, 13, 14, 16, \ldots, 19$ or 20 (Fig. 1).

**Proof.** Assume a connected plane graph $G$ has at least two cycles. Suppose $c(e_v(G)) = 2$. Then by Theorem 1, $G$ has at most two cycles as blocks. It follows that $G$ has no subgraph homeomorphic to $G_{12}, G_{13}, G_{14}, G_{16}, G_{17}, G_{18}, G_{19}$ or $G_{20}$.

Conversely, suppose $G$ has at least two cycles as blocks and has no subgraph homeomorphic to $G_{12}, G_{13}, G_{14}, G_{16}, G_{17}, G_{18}, G_{19}$ or $G_{20}$.

Suppose $G$ has at least 3 cycles each of them being a block of $G$. Then $G$ has a subgraph homeomorphic to $G_{12}, G_{13}, G_{14}, G_{16}, G_{17}, G_{18}, G_{19}$ or $G_{20}$, a contradiction.

Suppose $G$ has a block which contains at least two cycles. Then $G$ has a subgraph homeomorphic to $G_{14}$, a contradiction.

In each case we have arrived at a contradiction. Thus Theorem 1 implies that $c(e_v(G)) = 2$. This completes proof. □

**Theorem 4.** The edge-semientire graph $e_e(G)$ of a connected plane graph $G$ (with at least 5 vertices and 5 edges and $\Delta(G) \leq 4$) has crossing number 2 if and only if $G$ has no subgraph homeomorphic to $G_i$, $i = 1, 2, \ldots, 14$ or 15 (Fig. 1).
Assume $G$ is a connected plane graph whose edge-semiter graph $e_e(G)$ has crossing number 2. We prove that all graphs homeomorphic to $G_i$, $i = 1, 2, \ldots, 14$ or 15 have $c(e_e(G_i)) > 2$. By Theorem 2, we have (1) $\deg v \leqslant 4$ for every vertex $v$ of $G$ and $G$ is a tree and has exactly two vertices of degree 4 or $G$ is not a tree and has exactly one vertex of degree 4 and exactly one cycle such that at least one vertex of degree 2 is on the cycle. Or (2) $\deg v \leqslant 3$ for every vertex $v$ of $G$ and $G$ has exactly two cycles as blocks in which at least one vertex of degree 2 is on each cycle.

Fig. 1
cycle or $G$ is unicyclic and such that no vertex of degree 2 is on the cycle. From (1) or (2) it follows that $G$ has no subgraph homeomorphic to any one of the graphs $G_i$, $i = 1, 2, \ldots, 15$.

Conversely, assume that $G$ is a connected plane graph and does not contain a subgraph homeomorphic to any one of the graphs $G_i$, $i = 1, \ldots, 15$. We shall show that $G$ satisfies (1) or (2) and hence by Theorem 2, $e_e(G)$ has crossing number 2. Suppose $\deg v \geq 5$ for some vertex $v$ of $G$. Then $G$ contains a subgraph homeomorphic to $G_1$, a contradiction. Hence $\deg v \leq 4$ for every vertex $v$ of $G$. We consider the following two cases.

**Case 1.** Suppose $G$ is a tree. Assume there exist at least three vertices of degree 4. Then $G$ has a subgraph homeomorphic to $G_2$ or $G_3$, a contradiction. Hence $G$ has exactly two vertices of degree 4.

**Case 2.** Suppose $G$ is not a tree. Then we consider two subcases.

**Subcase 2.1.** Suppose $G$ is unicyclic $C$. Assume $G$ has exactly two vertices $v_1$ and $v_2$ of degree 4. Then we consider 3 possibilities.

a) If $v_1, v_2 \in C$, then $G$ has a subgraph homeomorphic to $G_4$.

b) If $v_1$ or $v_2 \in C$, then $G$ has a subgraph homeomorphic to $G_5$.

c) If $v_1, v_2 \notin C$, then $G$ has a subgraph homeomorphic to $G_6$.

In each case we have a contradiction. Thus $G$ has exactly one vertex of degree 4 and exactly one cycle.

Suppose $G$ has exactly one vertex $v$ of degree 4 and exactly one cycle $C$ such that no vertex of degree 2 is on the cycle. Then we consider two possibilities.

a) If $v \in C$, then $G$ has a subgraph homeomorphic to $G_7$, a contradiction.

b) If $v \notin C$, then $G$ has a subgraph homeomorphic to $G_8$, a contradiction.

Thus $G$ has exactly one vertex of degree 4 and exactly one cycle such that at least one vertex of degree 2 is on the cycle, or $G$ is unicyclic with every vertex of degree 3 on the cycle.

**Subcase 2.2.** Assume $G$ is not a unicyclic graph. Suppose $G$ has exactly one vertex $v$ of degree 4 and at least two cycles $C_1$ and $C_2$, each of which has at least one vertex of degree 2. We consider the following three possibilities.

a) If $v \in C_1$ and $C_2$, then $G$ has a subgraph homeomorphic to $G_9$.

b) If $v \in C_1$ or $C_2$, then $G$ has a subgraph homeomorphic to $G_{10}$.

c) If $v \notin C_1$ and $C_2$, then $G$ has a subgraph homeomorphic to $G_{11}$.

In each case we have a contradiction. Thus $G$ has at least 2 cycles each of which has at least one vertex of degree 2. Assume $\deg v \leq 3$ for every vertex $v$ of $G$. Then we consider 3 cases.

**Case 1.** Suppose $G$ has at least 3 cycles as blocks such that each block has at least one vertex of degree two. Then $G$ has a subgraph homeomorphic to $G_{12}$ or $G_{13}$, a contradiction.
Case 2. Suppose $G$ has a block which contains at least two cycles. Then $G$ has a subgraph homeomorphic to $G_{14}$, a contradiction.

Thus $G$ has at most two cycles as blocks.

Case 3. Suppose $G$ has exactly two cycles as blocks such that one block has no vertex of degree 2. Then $G$ has a subgraph homeomorphic to $G_{15}$, a contradiction. Thus $G$ has exactly two cycles such that each cycle has at least one vertex of degree 2, or $G$ has exactly one cycle such that each vertex on the cycle is of degree 3.

We have exhausted all possibilities. In each case we found that $G$ contains a subgraph homeomorphic to some of the forbidden subgraphs $G_i$, $i = 1, \ldots, 15$. Hence by Theorem 2, $e_e(G)$ has crossing number 2. This completes the proof of the theorem.

\[ \square \]

References


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