DISTRIBUTIVITY OF LATTICES OF BINARY RELATIONS

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Abstract. We present a formal scheme which whenever satisfied by relations of a given relational lattice $L$ containing only reflexive and transitive relations ensures distributivity of $L$.

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Distributivity of lattices of binary relations was treated by several authors, see e.g. [1] for lattices of tolerances and [2], [3] for lattices of congruences. H.-P. Gumm developed in [4] two schemes (the so called Shifting Lemma and Shifting Principle) to characterize modularity of congruence lattices in algebras and varieties. A certain scheme characterizing distributivity of congruence lattices can be found in [2]. The aim of this short note is to present a suitable scheme for characterizing distributivity in a more general case.

Let $\alpha$ be a binary relation on a set $A$. The fact that $\langle x, y \rangle \in \alpha$ will be visualized by an arrow going from $x$ to $y$ (where $x, y$ are depicted by points in a plane) which is valuated by $\alpha$, see Fig. 1.

![Fig. 1.](image)

**Definition.** Let $L$ be a lattice of binary relations on a set $A \neq \emptyset$. We say that $L$ satisfies the Corner Scheme if for any $\alpha, \beta, \gamma \in L$ the following condition is satisfied:

if $\alpha \cap \beta \subseteq \gamma$ and $\langle z, y \rangle \in \beta$, $\langle a, x \rangle \in \alpha$ and $\langle x, y \rangle \in \alpha \lor \gamma$, then $\langle z, y \rangle \in \gamma$.

**Remark.** In our graphical convention, the Corner Scheme can be visualized as shown in Fig. 2.
Lemma 1. Let $L$ be a lattice of transitive binary relations on a set $A \neq \emptyset$. If $L$ is distributive than it satisfies the Corner Scheme.

Proof. Let $L$ be distributive, $\alpha, \beta, \gamma \in L$ and $\alpha \cap \beta \subseteq \gamma$. Suppose $\langle z, y \rangle \in \beta$, $\langle z, x \rangle \in \alpha$ and $\langle x, y \rangle \in \alpha \lor \gamma$. Due to transitivity, we have $\langle z, y \rangle \in \alpha \cdot (\alpha \lor \gamma) \subseteq (\alpha \lor \gamma) \cdot (\alpha \lor \gamma) \subseteq \alpha \lor \gamma$, thus also

$$\langle a, y \rangle \in \beta \cap (\alpha \lor \gamma) = (\beta \cap \alpha) \lor (\beta \cap \gamma) \subseteq \gamma \lor (\beta \cap \gamma) = \gamma,$$

so $L$ satisfies the Corner Scheme.

Lemma 2. Let $L$ be a lattice of reflexive binary relations on a set $A \neq \emptyset$. If $L$ satisfies the Corner Scheme then it is distributive.

Proof. Let $L$ satisfy the Corner Scheme and suppose that it is not distributive. Then $L$ contains a sublattice isomorphic to $M_3$ or $N_5$ as shown in Fig. 3.

Of, course, we have $\alpha \cap \beta \subseteq \gamma$ in the both cases. Suppose $\langle z, y \rangle \in \beta$. Then $\langle z, y \rangle \in \alpha \lor \gamma$ and, due to reflexivity and the property $\alpha \subseteq \alpha \lor \gamma$, also $\langle z, y \rangle \in \alpha \cdot (\alpha \lor \gamma)$. Thus there is $x \in A$ with $\langle z, x \rangle \in \alpha$ and $\langle x, y \rangle \in \alpha \lor \gamma$. By the Corner Scheme we conclude $\langle z, y \rangle \in \gamma$. We have shown $\beta \subseteq \gamma$ which contradicts $\beta \parallel \gamma$ in $M_3$ or $\gamma \subset \beta$ in $N_5$. \qed
Theorem. Let $L$ be a lattice of reflexive and transitive binary relations on a set $A \neq \emptyset$. Then $L$ is distributive if and only if $L$ satisfies the Corner Scheme.

This is an immediate consequence of Lemma 1 and Lemma 2. Since $\beta \cap \gamma \subseteq \beta \cap (\alpha \lor \gamma)$ for any $\alpha, \beta, \gamma$ of any lattice $L$, we can state the following conclusion of the Corner Scheme.

**Corollary 1.** Any lattice of reflexive and transitive relations on a set $A \neq \emptyset$ is distributive if and only if it satisfies the quasiidentity:

$$\alpha \cap \beta \subseteq \gamma \Rightarrow \beta \cap \gamma = \beta \cap (\alpha \lor \gamma).$$

**Remark.** It is well-known and easy to check that in any lattice $L$ of reflexive and transitive relations we have

$$\alpha \lor \gamma = \bigcup \{\alpha \cdot \gamma \cdot \alpha \cdot \ldots \text{ (n factors)}; \ n \in \mathbb{N}\}. $$

Denote by $\Lambda_n = \gamma \cdot \alpha \cdot \gamma \cdot \ldots \text{ (n factors)}$ for $n \in \mathbb{N}_0$ (if $n = 0$ then $\Lambda_0$ is the identity relation on $A$). Then our Corner Scheme can be reformulated as follows:

**Corollary 2.** Let $L$ be any lattice of reflexive and transitive binary relations on a set $A \neq \emptyset$. Then $L$ is distributive if and only if it satisfies the following scheme for all $n \in \mathbb{N}_0$.

Let us note that the last scheme was first used for characterizing distributivity of congruence lattices in [2] under the name of Triangular Scheme.

We say that the lattice $L$ of binary relations on a set $A$ is *permutable* if

$$\alpha \cdot \gamma = \gamma \cdot \alpha$$

for every $\alpha, \gamma \in L$.

Of course, if $L$ is a permutable lattice of reflexive and transitive relations on a set $A \neq \emptyset$ then $\alpha \lor \gamma = \alpha \cdot \gamma$. 

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Hence, we can take $n = 1$ in Corollary 2 to prove the last result:

**Corollary 3.** Let $L$ be a permutable lattice of reflexive and transitive binary relations on a set $A \neq \emptyset$. Then $L$ is distributive if and only if it satisfies the following scheme:

\[
\begin{align*}
&\alpha \\ &\alpha \cap \beta \subseteq \gamma \\
&\Rightarrow \\
&\alpha \\ &\gamma
\end{align*}
\]

**References**


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