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A Galois connection between distance functions and inequality relations


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A GALOIS CONNECTION BETWEEN DISTANCE FUNCTIONS
AND INEQUALITY RELATIONS

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Abstract. Following the ideas of R. DeMarr, we establish a Galois connection between
distance functions on a set $S$ and inequality relations on $X_S = S \times \mathbb{R}$. Moreover, we also
investigate a relationship between the functions of $S$ and $X_S$.

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nections, Lipschitz and monotone functions, fixed points

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INTRODUCTION

Extending and supplementing some of the results of R. DeMarr [6] we establish a
few consequences of the following definitions.

Let $S$ be a nonvoid set, and denote by $D_S$ the family of all functions $d$ on $S^2$ such
that $0 \leq d(p, q) \leq +\infty$ for all $p, q \in S$.

Moreover, let $X_S = S \times \mathbb{R}$, and denote by $E_S$ the family of all relations $\leq$ on $X_S$
such that $(p, \lambda) \leq (q, \mu)$ implies $\lambda \leq \mu$.

If $d \in D_S$, then for all $(p, \lambda), (q, \mu) \in X_S$ we define

$$(p, \lambda) \leq_d (q, \mu) \iff d(p, q) \leq \mu - \lambda.$$

While, if $\leq \in E_S$, then for all $p, q \in S$ we define

$$d_{\leq}(p, q) = \inf \{\mu - \lambda: (p, \lambda) \leq (q, \mu)\}.$$
Moreover, if $f$ is a function of $S$ into $S$ and $\alpha \in \mathbb{R}$, then for all $(p, \lambda) \in X_S$ we define

$$F(p, \lambda) = (f(p), \alpha \lambda).$$

Concerning the above definitions, for instance, we prove the following statements.

**Theorem 1.** The mappings

$$d \mapsto \leq_d \quad \text{and} \quad \leq \mapsto d \leq$$

establish a Galois connection between the posets $D_S$ and $E_S$ such that every element of $D_S$ is closed.

**Theorem 2.** The family $E_S^-$ of all closed elements of $E_S$ consists of all relations $\leq \in E_S$ such that for all $(p, \lambda), (q, \mu) \in X_S$

1. $(p, \lambda) \leq (q, \mu)$ implies $(p, \lambda + \omega) \leq (q, \mu + \omega)$ for all $\omega \in \mathbb{R}$;
2. $(p, \lambda) \leq (q, \mu)$ if and only if $(p, \lambda) \leq (q, \mu + \varepsilon)$ for all $\varepsilon > 0$.

**Theorem 3.** If $d \in D_S$, then $\leq_d$ is a partial order on $X_S$ if and only if $d$ is a quasi-metric on $S$ in the sense that

1. $d(p, p) = 0$ for all $p \in S$;
2. $d(p, q) = 0$ and $d(q, p) = 0$ imply $p = q$;
3. $d(p, r) \leq d(p, q) + d(q, r)$ for all $p, q, r \in S$.

**Theorem 4.** For the families of all fixed points of $f$ and $F$ we have

$$\text{Fix}(F) = \text{Fix}(f) \times \mathbb{R} \quad \text{if} \quad \alpha = 1 \quad \text{and} \quad \text{Fix}(F) = \text{Fix}(f) \times \{0\} \quad \text{if} \quad \alpha \neq 1.$$

**Theorem 5.** If $\alpha > 0$ and $d \in D_S$, then the following assertions are equivalent:

1. $d(f(p), f(q)) \leq \alpha d(p, q)$ for all $p, q \in S$;
2. $(p, \lambda) \leq_d (q, \mu)$ implies $F(p, \lambda) \leq_d F(q, \mu)$.

**Theorem 6.** If $0 < \alpha < 1$ and $d \in D_S$ is such that $d$ is finite valued, then for any $p, q \in S$ there exist $\lambda_0, \mu_0 \in \mathbb{R}$ with $\lambda_0 \leq 0 \leq \mu_0$ such that

$$(p, \lambda) \leq_d F(p, \lambda) \leq_d F(q, \mu) \leq_d (q, \mu)$$

for all $\lambda, \mu \in \mathbb{R}$ with $\lambda \leq \lambda_0$ and $\mu_0 \leq \mu$.

Remark. From Theorems 3, 5 and 6, by writing $d_\leq$ instead of $d$, we can get some similar assertions for the relations $\leq \in E_S^-$. Namely, by Theorem 2, we have $\leq = \leq_\leq$ for all $\leq \in E_S^-$.  

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The only prerequisites for reading this paper is a knowledge of some basic facts on posets which will be briefly laid out in the next two preparatory sections. The proofs of most of those facts can be found in [10].

1. Closure operations on posets

If \( \leq \) is a reflexive, antisymmetric and transitive relation on a nonvoid set \( X \), then the relation \( \leq \) is called a partial order on \( X \), and the ordered pair \( X(\leq) = (X, \leq) \) is called a poset (partially ordered set).

If \( A \) is a subset of a poset \( X \), then \( \inf_X(A) \) and \( \sup_X(A) \) will denote the greatest lower bound and the least upper bound of \( A \) in \( X \), respectively. Further, the poset \( X \) is called complete if \( \inf_X(A) \) and \( \sup_X(A) \) exist for all \( A \subset X \).

The following useful characterization of infimum was already observed by Rennie [9]. However, despite this, it is not included in the standard textbooks.

**Lemma 1.1.** If \( X \) is a poset, and moreover \( A \subset X \) and \( \alpha \in X \), then the following assertions are equivalent:

1. \( \alpha = \inf_X(A) \);
2. for each \( u \in X \) we have \( u \leq \alpha \) if and only if \( u \leq x \) for all \( x \in A \).

Concerning the completeness of posets, according to Birkhoff [1, p. 112] we can at once state

**Theorem 1.2.** If \( X \) is a poset, then the following assertions are equivalent:

1. \( X \) is complete;
2. \( \inf_X(A) \) exists for all \( A \subset X \).

**Remark 1.3.** To obtain the corresponding results for supremum, one can observe that if \( X(\leq) \) is a partial ordered set, then its dual \( X(\geq) \) is also a partial ordered set. Moreover, we have \( \inf_{X(\geq)}(A) = \sup_{X(\leq)}(A) \) for all \( A \subset X \).

**Definition 1.4.** If \( - \) is a function of a poset \( X(\leq) \) into itself such that

1. \( x \leq y \) implies \( x^- \leq y^- \) for all \( x, y \in X \),
2. \( x \leq x^- \); and \( 3. x^- = x^-^- \) for all \( x \in X \),

then the function \( - \) is called a closure operation on \( X(\leq) \), and the ordered triple \( X(\leq, -) = (X, \leq, -) \) is called a closure space.

**Remark 1.5.** Note that the expansivity property (2) already implies that \( x^- \leq x^-^- \) for all \( x \in X \). Therefore, instead of the idempotency property (3), it suffices to assume only that \( x^-^- \leq x^- \) for all \( x \in X \).
The following useful characterization of closure operations was already observed by Everett [3]. However, despite this, it is not included in the standard textbooks.

**Lemma 1.6.** If $\Gamma$ is a function of a poset $X$ into itself, then the following assertions are equivalent:

1. the function $\Gamma$ is a closure operation on $X$;
2. for all $x, y \in X$ we have $x \leq y^\Gamma$ if and only if $x^\Gamma \leq y$.

If $X$ is a closure space, then the members of the family $X^\Gamma = \{x^\Gamma : x \in X\}$ may be called the closed elements of $X$. Namely, we have

**Theorem 1.7.** If $X$ is a closure space and $x \in X$, then the following assertions are equivalent:

1. $x^\Gamma \leq x$;
2. $x = x^\Gamma$;
3. $x \in X^\Gamma$.

**Remark 1.8.** Note that if $X$ is a closure space, then we have $x^\Gamma = \inf\{y \in X^- : x \leq y\}$ for all $x \in X$. Therefore, the closed elements of $X$ uniquely determine the closure operation of $X$.

A closure space will be called complete if it is complete as a poset. Concerning the closed elements of complete closure spaces, according to Birkhoff [1, p. 112] we can also state

**Theorem 1.9.** If $X$ is a complete closure space, then $X^-\Gamma$ is a complete poset.

**Remark 1.10.** Note that if $A \subset X^\Gamma$, then we have $\inf_{X^\Gamma}(A) = \inf_X(A)$ and $\sup_{X^\Gamma}(A) = (\sup_X(A))^\Gamma$.

2. Galois connections between posets

**Definition 2.1.** If $X$ and $Y$ are posets and $\ast$ and $\#$ are functions of $X$ and $Y$ into $Y$ and $X$, respectively, such that

1. $x_1 \leq x_2$ implies $x_2^\ast \leq x_1^\ast$ for all $x_1, x_2 \in X$,
2. $y_1 \leq y_2$ implies $y_2^# \leq y_1^#$ for all $y_1, y_2 \in Y$,
3. $x \leq x^{\ast\#}$ for all $x \in X$,
4. $y \leq y^{\#\ast}$ for all $y \in Y$,

then we say that the functions $\ast$ and $\#$ establish a Galois connection between the posets $X$ and $Y$. 440
Remark 2.2. Galois connections between posets were first investigated by Ore [7] and Everett [3].

The following useful characterization of Galois connections was already observed by J. Schmidt [1, p. 124]. However, despite this, it is not included in the standard textbooks.

Lemma 2.3. If $X$ and $Y$ are posets and $*$ and $#$ are functions of $X$ and $Y$ into $Y$ and $X$, respectively, then the following assertions are equivalent:
(1) the functions $*$ and $#$ establish a Galois connection between $X$ and $Y$;
(2) for all $x \in X$ and $y \in Y$ we have $x \leq y#$ if and only if $y \leq x^*$.

The following basic theorem has already been established by Ore [7] and Everett [3].

Theorem 2.4. If the functions $*$ and $#$ establish a Galois connection between the posets $X$ and $Y$, then
(1) $x^* = x^{**}$ for all $x \in X$ and $y# = y^{**}$ for all $y \in Y$;
(2) the functions $**$ and $#$ are closure operations on $X$ and $Y$, respectively, such that $Y# = X**$ and $X^* = Y**$;
(3) the restrictions of the functions $*$ and $#$ to $Y#$ and $X^*$, respectively, are injective, and they are inverses of each other.

Remark 2.5. Note that actually $A = Y#$ is the largest subset of $X$ such that the restriction of the function $*$ to $A$ is injective and $A^*# \subset A$.

Definition 2.6. A Galois connection between posets $X$ and $Y$ established by the functions $*$ and $#$ will be called lower (upper) semiperfect if $x = x^*$ for all $x \in X$ ( $y = y#$ for all $y \in Y$).

Remark 2.7. Note that by Definition 2.1 we always have $x \leq x^*$ for all $x \in X$. Therefore, to define the lower semiperfectness of the above Galois connection it suffices to assume the reverse inequality.

The above definition and the following theorem are again due to Ore [7].

Theorem 2.8. A Galois connection between posets $X$ and $Y$ established by the functions $*$ and $#$ is lower semiperfect if and only if $X = Y#$, or equivalently the function $*$ is injective.

Remark 2.9. Note that if $X$ is a poset, then the Galois connection between the posets $\mathcal{P}(X)$ and $\mathcal{P}(X)$, established by the mappings

$$A \mapsto \text{lb}(A) \quad \text{and} \quad A \mapsto \text{ub}(A),$$

where $\text{lb}(A)$ and $\text{ub}(A)$ are the families of all lower and upper bounds of the set $A$ in $X$, respectively, is not, in general, lower or upper semiperfect.
The importance of this Galois connection lies mainly in the Dedekind-McNeille completion of the poset $X$ by the cuts $\text{lb}(\text{ub}(A))$ where $A \subset X$. (See, for instance, [1, p.126].)

3. A Galois connection between distance functions and inequality relations

**Definition 3.1.** Let $S$ be a nonvoid set, and denote by $\mathcal{D}_S$ the family of all functions $d$ on $S^2$ such that $0 \leq d(p, q) \leq +\infty$ for all $p, q \in S$.

Moreover, let $X_S = S \times \mathbb{R}$, and denote by $\mathcal{E}_S$ the family of all relations $\leq$ on $X_S$ such that $(p, \lambda) \leq (q, \mu)$ implies $\lambda \leq \mu$ for all $(p, \lambda), (q, \mu) \in X_S$.

**Remark 3.2.** The members of the families $\mathcal{D}_S$ and $\mathcal{E}_S$ will be called distance functions and inequality relations on $S$ and $X_S$, respectively.

The following theorems do not actually need the nonnegativity of distance functions on $S$ and the corresponding property of inequality relations on $X_S$.

**Theorem 3.3.** The families $\mathcal{D}_S$ and $\mathcal{E}_S$, equipped with the pointwise inequality and the ordinary set inclusion, respectively, are complete posets.

**Hint.** If $\mathcal{D} \subset \mathcal{D}_S$, then by defining $d_*(p, q) = \inf_{d \in \mathcal{D}} d(p, q)$ for all $p, q \in S$ we can see that $d_* = \inf(\mathcal{D})$.

On the other hand, if $\mathcal{E} \subset \mathcal{E}_S$, then by defining $\leq_* = \bigcap \mathcal{E}$ if $\mathcal{E} \neq \emptyset$ and $\leq_* = \bigcup \mathcal{E}_S$ if $\mathcal{E} = \emptyset$ we can see that $\leq_* = \inf(\mathcal{E})$. □

**Definition 3.4.** If $d \in \mathcal{D}_S$, then for all $(p, \lambda), (q, \mu) \in X_S$ we define

$$(p, \lambda) \leq_d (q, \mu) \iff d(p, q) \leq \mu - \lambda,$$

while if $\leq \in \mathcal{E}_S$, then for all $p, q \in S$ we define

$$d_<(p, q) = \inf\{\mu - \lambda : (p, \lambda) \leq (q, \mu)\}.$$

**Remark 3.5.** The relation $\leq_d$, for an ordinary metric $d$, has formerly been studied by DeMaar [6].

However, the function $d_<$ and the following theorem seem to be completely new.

**Theorem 3.6.** The mappings

$$d \longmapsto \leq_d \quad \text{and} \quad \leq \longmapsto d_\leq$$
establish a lower semiperfect Galois connection between the posets \( D_S \) and \( E_S \).

**Proof.** If \( d \in D_S \) and \( \leq \in E_S \), then by the corresponding definitions it is clear that \( \leq_d \in E_S \) and \( d_\leq \in D_S \). Therefore, by Lemma 2.3 and Remark 2.7, it suffices to prove only that \( d \leq d_\leq \) if and only if \( \leq \subseteq \leq_d \), and moreover \( d_\leq \leq d \).

If \( (p, \lambda), (q, \mu) \in X_S \) are such that \( (p, \lambda) \leq (q, \mu) \), then by the definition of \( d_\leq \) we have \( d_\leq (p, q) \leq \mu - \lambda \). Hence, if the inequality \( d \leq d_\leq \) holds, we can infer that \( d(p, q) \leq \mu - \lambda \). Thus, by the definition of \( \leq_d \), we also have \( (p, \lambda) \leq_d (q, \mu) \). Therefore, the inclusion \( \subseteq \subseteq \leq_d \) is also true.

Further, if \( p, q \in S \) and \( \beta \in \mathbb{R} \) are such that \( d_\leq (p, q) < \beta \), then by the definition of \( d_\leq \) there exist \( \lambda, \mu \in \mathbb{R} \) such that \( (p, \lambda) \leq (q, \mu) \) and \( \mu - \lambda < \beta \). Hence, if the inclusion \( \subseteq \subseteq \leq_d \) holds, we can infer that \( (p, \lambda) \leq_d (q, \mu) \). Thus, by the definition of \( \leq_d \), we also have \( d(p, q) \leq \mu - \lambda < \beta \). Hence, letting \( \beta \rightarrow d_\leq (p, q) \), we can infer that \( d(p, q) \leq d_\leq (p, q) \). Therefore, the inequality \( d \leq d_\leq \) is also true.

Finally, if \( p, q \in S \) and \( \beta \in \mathbb{R} \) are such that \( d(p, q) < \beta \), then by the definition of \( \leq_d \) we have \( (p, 0) \leq_d (q, \beta) \). Hence, by the definition of \( d_\leq \), it follows that \( d_\leq (p, q) \leq \beta \). Hence, letting \( \beta \rightarrow d(p, q) \), we can infer that \( d_\leq (p, q) \leq d(p, q) \). Therefore, the inequality \( d_\leq \leq d \) is also true. \( \square \)

**Remark 3.7.** Note that, by Theorem 3.6 and Definition 2.6, we actually have \( d = d_\leq \) for all \( d \in D_S \). Therefore, the mapping \( \leq \mapsto d_\leq \) is onto \( D_S \). Moreover, the mapping \( d \mapsto d_\leq \) is injective.

To briefly describe the range of the mapping \( d \mapsto \leq_d \) or that of the closure operation \( \leq \mapsto \leq_d \), we shall need the following

**Definition 3.8.** Denote by \( E_S^- \) the family of all relations \( \leq \in E_S \) such that for all \( (p, \lambda), (q, \mu) \in X_S \)

1. \( (p, \lambda) \leq (q, \mu) \) implies \( (p, \lambda + \omega) \leq (q, \mu + \omega) \) for all \( \omega \in \mathbb{R} \);
2. \( (p, \lambda) \leq (q, \mu) \) if and only if \( (p, \lambda) \leq (q, \mu + \varepsilon) \) for all \( \varepsilon > 0 \).

The appropriateness of the above definition is apparent from

**Theorem 3.9.** If \( \leq \in E_S \), then the following assertions are equivalent;

1. \( \leq \in E_S^- \);
2. \( \leq = \leq_d \);
3. \( \leq = \leq_d \) for some \( d \in D_S \).

**Proof.** Suppose that the assertion (1) holds, and \( (p, \lambda), (q, \mu) \in X_S \) are such that \( (p, \lambda) \leq_d (q, \mu) \). Then, by the definition of \( \leq_d \), we have \( d_\leq (p, q) \leq \mu - \lambda \). Therefore, by the definition of \( d_\leq \), for each \( \varepsilon > 0 \) there exist \( \omega, \tau \in \mathbb{R} \) such that \( (p, \omega) \leq (q, \tau) \) and \( \tau - \omega < \mu - \lambda + \varepsilon \). Hence, by the property 3.8 (2), it follows
that \((p, \omega) \leq (q, \mu - \lambda + \varepsilon + \omega)\). However, by the property 3.8 (1), this is equivalent to \((p, \lambda) \leq (q, \mu + \varepsilon)\). Hence, again by the property 3.8 (2), it follows that \((p, \lambda) \leq (q, \mu)\). Therefore, \(\leq_d \subseteq \leq\). And now, since the converse inclusion is automatic by Theorem 3.6, the assertion (2) also holds.

Now, since the implication (2) \(\implies\) (3) trivially holds, and the implication (3) \(\implies\) (1) follows immediately from the definition of \(\leq_d\), the proof is complete. \(\square\)

Remark 3.10. By Theorem 3.9, it is clear that the Galois connection established in Theorem 3.6 is not upper semiperfect, and the mapping \(d \mapsto \leq_d\) is only a partial inverse of the mapping \(\leq \mapsto d\).

4. Some further properties of the relations \(\leq_d\) and \(d\)

By using the definition of the relation \(\leq_d\) we can easily prove the following theorems.

**Theorem 4.1.** If \(d \in D_S\), then the following assertions are equivalent:
1. \(\leq_d\) is reflexive on \(X_S\);
2. \(d(p, p) = 0\) for all \(p \in S\).

Remark 4.2. More generally, we can also easily see that a relation \(\leq \in E_S\) is reflexive on \(X_S\) if and only if \(d_\leq(p, p) = 0\) for all \(p \in S\).

**Theorem 4.3.** If \(d \in D_S\), then the following assertions are equivalent:
1. \(\leq_d\) is antisymmetric;
2. \(d(p, q) = 0\) and \(d(q, p) = 0\) imply \(p = q\).

**Hint.** If \((p, \lambda) \leq_d (q, \mu)\) and \((q, \mu) \leq_d (p, \lambda)\), then by the definition of \(\leq_d\) we have \(d(p, q) \leq \mu - \lambda\) and \(d(q, p) \leq \lambda - \mu\). Hence, by using the nonnegativity of \(d\), we can infer that \(\lambda = \mu\). Therefore, we actually have \(d(p, q) = 0\) and \(d(q, p) = 0\). Hence, if the assertion (2) holds, we can infer that \(p = q\). Therefore, \((p, \lambda) = (q, \mu)\), and thus the assertion (1) also holds. \(\square\)

Remark 4.4. Note that the relation \(\leq_d\) is reflexive (antisymmetric) if and only if its restriction to \(S \times \{0\}\) is reflexive (antisymmetric).

**Theorem 4.5.** If \(d \in D_S\), then the following assertions are equivalent:
1. \(\leq_d\) is transitive;
2. \(d(p, r) \leq d(p, q) + d(q, r)\) for all \(p, q, r \in S\).
Hint. If \(d(p, q) < +\infty\) and \(d(q, r) < +\infty\), then by the definition of \(\leq_d\) we have
\[(p, 0) \leq_d (q, d(p, q)) \quad \text{and} \quad (q, d(p, q)) \leq_d (r, d(p, q) + d(q, r)).\]
Hence, if the assertion (1) holds, we can infer that
\[(p, 0) \leq_d (r, d(p, q) + d(q, r)).\]
Therefore, by the definition of \(\leq_d\), we also have \(d(p, r) \leq d(p, q) + d(q, r)\), and thus the assertion (2) also holds.

Remark 4.6. Now, by using a reasonable modification of the usual definition of quasi-metrics [4, p. 3], we can also state that a function \(d \in D_S\) is a quasi-metric on \(S\) if and only if the relation \(\leq_d\) is a partial order on \(X_S\).

**Theorem 4.7.** If \(d \in D_S\), then the following assertions are equivalent:

1. \(d(p, q) = d(q, p)\) for all \(p, q \in S\);
2. \((p, \lambda) \leq_d (q, \mu)\) implies \((q, \lambda) \leq_d (p, \mu)\).

Hint. If \(d(p, q) < +\infty\), then by the definition of \(\leq_d\) we have
\[(p, 0) \leq_d (q, d(p, q)).\]
Hence, if the assertion (2) holds, we can infer that \((q, 0) \leq_d (p, d(p, q))\). Therefore, by the definition of \(\leq_d\), we also have \(d(q, p) \leq d(p, q)\). Hence, by changing the roles of \(p\) and \(q\), we can see that the converse inequality is also true. Therefore, the assertion (1) also holds.

Remark 4.8. The latter theorem shows that symmetry is a less natural property of distance functions than the properties considered in the previous three theorems. This may be another reason why quasi-pseudo-metrics are more natural objects than pseudo-metrics.

Note that if \(d\) is only an extended real-valued quasi-pseudo-metric on \(S\), then by identifying \(p\) with \((p, 0)\) for all \(p \in S\) we can already get a natural preorder \(\leq_d\) on \(S\) such that for all \(p, q \in S\) we have \(p \leq_d q\) if and only if \(d(p, q) = 0\).

**Theorem 4.9.** If \(d \in D_S\), then the following assertions are equivalent:

1. \(\leq_d\) is symmetric;
2. \(d(p, q) = +\infty\) for all \(p, q \in S\).

Hint. If \(p, q \in S\) are such that \(d(p, q) < +\infty\), then by defining \(\mu = d(p, q) + 1\) we have \((p, 0) \leq_d (q, \mu)\). Hence, if the assertion (1) holds we can infer that \((q, \mu) \leq_d (p, 0)\). Therefore, we also have \(d(q, p) \leq -\mu\). Hence, by using the nonnegativity of \(d\), we can infer that \(0 < -1\). Therefore, the implication (1) \(\implies\) (2) is true.
Remark 4.10. Hence, it is clear that the relation $\leq_d$ is symmetric if and only if $\leq_d = \emptyset$.

5. A relationship between the functions of $S$ and $X_S$

Definition 5.1. Let $f$ be a function of $S$ into itself, $\alpha \in \mathbb{R}$, and

$$F(p, \lambda) = (f(p), \alpha \lambda)$$

for all $(p, \lambda) \in X_S$.

Remark 5.2. The relationships between the functions $f$ and $F$ have formerly been studied by DeMarr [6].

The following theorems will only extend and supplement some of the observations of the above mentioned author.

Theorem 5.3. For the families of all fixed points of $f$ and $F$ we have

$$\text{Fix}(F) = \text{Fix}(f) \times \mathbb{R} \quad \text{if} \quad \alpha = 1 \quad \text{and} \quad \text{Fix}(F) = \text{Fix}(f) \times \{0\} \quad \text{if} \quad \alpha \neq 1.$$ 

Proof. By the corresponding definitions, for any $(p, \lambda) \in X_S$ we have

$$(p, \lambda) \in \text{Fix}(F) \iff F(p, \lambda) = (p, \lambda) \iff (f(p), \alpha \lambda) = (p, \lambda) \iff f(p) = p \text{ and } \alpha \lambda = \lambda \iff p \in \text{Fix}(f) \text{ and } (\alpha - 1)\lambda = 0.$$ 

Consequently, the assertions of the theorem are immediate. \qed

Under the notation of Definition 5.1, we can also easily prove the following theorems.

Theorem 5.4. If $\alpha > 0$ and $d \in \mathcal{D}_S$, then the following assertions are equivalent:

1. $d(f(p), f(q)) \leq \alpha d(p, q)$ for all $p, q \in S$;
2. $p, \lambda \leq_d (q, \mu)$ implies $F(p, \lambda) \leq_d F(q, \mu)$.

Proof. If $(p, \lambda), (q, \mu) \in X_S$ are such that $(p, \lambda) \leq_d (q, \mu)$, then by the definition of $\leq_d$ we have $d(p, q) \leq \mu - \lambda$. Hence, if the assertion (1) holds, we can infer that $d(f(p), f(q)) \leq \alpha \mu - \alpha \lambda$. Therefore, by the definition $\leq_d$, we also have $(f(p), \alpha \lambda) \leq_d (f(q), \alpha \mu)$. Hence, by the definition of $F$, it follows that $F(p, \lambda) \leq_d F(q, \mu)$. Therefore, the assertion (2) also holds.

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On the other hand, if \( p, q \in S \) are such that \( d(p, q) < +\infty \), then by the definition of \( \leq_d \) we have \( (p, 0) \leq_d (q, d(p, q)) \). Hence, if the assertion (2) holds, we can infer that \( F(p, 0) \leq_d F(q, d(p, q)) \). Therefore, by the definition of \( F \), we also have \( (f(p), 0) \leq_d (f(q), \alpha d(p, q)) \). Hence, again by the definition of \( \leq_d \), it follows that \( d(f(p), f(q)) \leq \alpha d(p, q) \). Therefore, the assertion (1) also holds. \( \square \)

**Theorem 5.5.** If \( 0 \leq \alpha \leq 1 \) and \( d \in D_S \) is such that \( d(p, p) = 0 \) for all \( p \in S \), then

\[
(p, \lambda) \leq_d F(p, \lambda) \leq_d F(p, \mu) \leq_d (p, \mu)
\]

for all \( p \in \text{Fix}(f) \) and \( \lambda, \mu \in \mathbb{R} \) with \( \lambda \leq 0 \leq \mu \).

**Proof.** Under the above conditions, we have

\[
d(p, f(p)) \leq \alpha \lambda - \lambda; \quad d(f(p), f(p)) \leq \alpha \mu - \alpha \lambda; \quad d(f(p), p) \leq \mu - \alpha \mu.
\]

Hence, by the definition of \( \leq_d \), it follows that

\[
(p, \lambda) \leq_d (f(p), \alpha \lambda) \leq_d (f(p), \alpha \mu) \leq_d (p, \mu).
\]

Therefore, by the definition of \( F \), the required equalities are also true. \( \square \)

**Theorem 5.6.** If \( 0 < \alpha < 1 \) and \( d \in D_S \) is such that \( d \) is finite valued, then for any \( p, q \in S \) there exist \( \lambda_0, \mu_0 \in \mathbb{R} \) with \( \lambda_0 \leq 0 \leq \mu_0 \) such that

\[
(p, \lambda) \leq_d F(p, \lambda) \leq_d F(q, \mu) \leq_d (q, \mu)
\]

for all \( \lambda, \mu \in \mathbb{R} \) with \( \lambda \leq \lambda_0 \) and \( \mu_0 \leq \mu \).

**Proof.** Let \( p, q \in S \), and define

\[
\lambda_0 = \frac{d(p, f(p))}{(\alpha - 1)} \quad \text{and} \quad \mu_0 = \max \left\{ \frac{d(f(p), f(q))}{\alpha}, \frac{d(f(q), q)}{1 - \alpha} \right\}.
\]

Then, by our assumptions on \( d \) and \( \alpha \), it is clear that \( \lambda_0, \mu_0 \in \mathbb{R} \) are such that \( \lambda_0 \leq 0 \leq \mu_0 \). Moreover, we can easily see that, for all \( \lambda, \mu \in \mathbb{R} \) with \( \lambda \leq \lambda_0 \) and \( \mu_0 \leq \mu \), we have

\[
d(p, f(p)) \leq \alpha \lambda - \lambda; \quad d(f(p), f(q)) \leq \alpha \mu - \alpha \lambda; \quad d(f(q), q) \leq \mu - \alpha \mu.
\]

Hence, by the definitions of \( \leq_d \) and \( F \), it is clear that the required inequalities are also true. \( \square \)
Theorem 5.7. If $\alpha > 1$, $d \in D_S$ and $(p, \lambda), (q, \mu) \in X_S$ are such that

$$(p, \lambda) \leq_d F(p, \lambda) \leq_d F(q, \mu) \leq_d (q, \mu),$$

then $\lambda = \mu = d(p, f(p)) = d(f(p), f(q)) = d(f(q), q) = 0$.

Proof. Again by the definitions of $F$ and $\leq_d$, it is clear that

$$d(p, f(p)) \leq \alpha \lambda - \lambda; \quad d(f(p), f(q)) \leq \alpha \mu - \alpha \lambda; \quad d(f(q), q) \leq \mu - \alpha \mu.$$

Hence, by using our assumptions on $d$ and $\alpha$, we can easily see that

$$0 \leq \frac{d(p, f(p))}{(\alpha - 1)} \leq \lambda \leq \frac{d(f(q), q)}{(1 - \alpha)} \leq 0.$$

Therefore, $\lambda = \mu = 0$, and thus the required equalities are also true. \qed

Remark 5.8. Note that, by writing $d_\leq$ instead of $d$ in the results of Sections 4 and 5, we can get some similar assertions for the relations $\leq \in \mathcal{E}_S^-$. Namely, by Theorem 3.9 we have $\leq = \leq_d \leq_\leq$ for all $\leq \in \mathcal{E}_S^-$. 

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References


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