Varaporn Saenpholphat; Futaba Okamoto; Ping Zhang
Measures of traceability in graphs

*Mathematica Bohemica*, Vol. 131 (2006), No. 1, 63--84

Persistent URL: [http://dml.cz/dmlcz/134076](http://dml.cz/dmlcz/134076)

**Terms of use:**

© Institute of Mathematics AS CR, 2006

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* [http://project.dml.cz](http://project.dml.cz)
MEASURES OF TRACEABILITY IN GRAPHS

VARAPORN SAENPHOLPHAT, Bangkok, FUTABA OKAMOTO,
PING ZHANG, Kalamazoo

(Received September 26, 2005)

Abstract. For a connected graph $G$ of order $n \geq 3$ and an ordering $s: v_1, v_2, \ldots, v_n$ of the vertices of $G$, $d(s) = \sum_{i=1}^{n-1} d(v_i, v_{i+1})$, where $d(v_i, v_{i+1})$ is the distance between $v_i$ and $v_{i+1}$. The traceable number $t(G)$ of $G$ is defined by $t(G) = \min \{d(s)\}$, where the minimum is taken over all sequences $s$ of the elements of $V(G)$. It is shown that if $G$ is a nontrivial connected graph of order $n$ such that $l$ is the length of a longest path in $G$ and $p$ is the maximum size of a spanning linear forest in $G$, then $2n-2-p \leq t(G) \leq 2n-2-l$ and both these bounds are sharp. We establish a formula for the traceable number of every tree in terms of its order and diameter. It is shown that if $G$ is a connected graph of order $n \geq 3$, then $t(G) \leq 2n-4$. We present characterizations of connected graphs of order $n$ having traceable number $2n-4$ or $2n-5$. The relationship between the traceable number and the Hamiltonian number (the minimum length of a closed spanning walk) of a connected graph is studied. The traceable number $t(v)$ of a vertex $v$ in a connected graph $G$ is defined by $t(v) = \min \{d(s)\}$, where the minimum is taken over all linear orderings $s$ of the vertices of $G$ whose first term is $v$. We establish a formula for the traceable number $t(v)$ of a vertex $v$ in a tree. The Hamiltonian-connected number $hcon(G)$ of a connected graph $G$ is defined by $hcon(G) = \sum_{v \in V(G)} t(v)$. We establish sharp bounds for $hcon(G)$ of a connected graph $G$ in terms of its order.

Keywords: traceable graph, Hamiltonian graph, Hamiltonian-connected graph

MSC 2000: 05C12, 05C45

1. Introduction

We refer to the book [6] for graph-theoretical notation and terminology not described in this paper. Hamiltonian graphs can be defined as those graphs of order $n \geq 3$ for which there is a cyclic ordering $v_1, v_2, \ldots, v_n$, $v_{n+1} = v_1$ of the vertices of $G$ such that $\sum_{i=1}^{n} d(v_i, v_{i+1}) = n$, where $d(v_i, v_{i+1})$ is the distance be-
tween $v_i$ and $v_{i+1}$. For a connected graph $G$ of order $n \geq 3$ and a cyclic ordering $s: v_1, v_2, \ldots, v_n, v_{n+1} = v_1$ of the vertices of $G$, the number $d(s)$ is defined in [5] as

$$d(s) = \sum_{i=1}^{n} d(v_i, v_{i+1}).$$

Therefore, $d(s) \geq n$ for each cyclic ordering $s$ of $V(G)$. The Hamiltonian number $h(G)$ of $G$ is defined in [5] by

$$h(G) = \min \{d(s)\},$$

where the minimum is taken over all cyclic orderings $s$ of the vertices of $G$. Therefore, $h(G) = n$ if and only if $G$ is Hamiltonian. To illustrate these concepts, consider the graph $G$ of Figure 1. For the cyclic orderings $s_1: v_1, v_2, v_3, v_4, v_5, v_1$ and $s_2: v_1, v_3, v_2, v_4, v_5, v_1$ of $V(G)$, we see that $d(s_1) = 8$ and $d(s_2) = 6$. Since $G$ is a non-Hamiltonian graph of order 5 and $d(s_2) = 6$, it follows that $h(G) = 6$.

In [8] Goodman and Hedetniemi introduced the concept of a Hamiltonian walk in a connected graph $G$, defined as a closed spanning walk of minimum length in $G$. They denoted the length of a Hamiltonian walk in $G$ by $h(G)$. It was shown in [5] that the Hamiltonian number of a connected graph $G$ is, in fact, the length of a Hamiltonian walk in $G$. Consequently, this result justifies using the notation $h(G)$ for both the Hamiltonian number of a graph $G$ and the length of a Hamiltonian walk in $G$. This concept was studied further in [4]. Hamiltonian walks were also studied by Asano, Nishizeki, and Watanabe [1], [2], [7], Bermond [3], Nebeský [9], and Vacek [11]. The following result appears in the papers [4], [5], [7], [8], [9].

**Theorem A.** For every connected graph $G$ of order $n \geq 2$,

$$n \leq h(G) \leq 2n - 2.$$ 

Moreover, $h(G) = 2n - 2$ if and only if $G$ is a tree.

In this paper, we study a natural related concept. A graph has been called traceable if it contains a Hamiltonian path. Therefore, every Hamiltonian graph is traceable.
The converse is not true of course. For a connected graph $G$ of order $n \geq 3$ and an ordering (also called a linear ordering) $s: v_1, v_2, \ldots, v_n$ of the vertices of $G$, the number $d(s)$ is defined as

$$d(s) = \sum_{i=1}^{n-1} d(v_i, v_{i+1}).$$

The traceable number $t(G)$ of $G$ is defined by

$$t(G) = \min \{d(s)\},$$

where the minimum is taken over all sequences $s$ of the elements of $V(G)$. Thus if $G$ is a connected graph of order $n \geq 2$, then $t(G) \geq n - 1$. Furthermore, $t(G) = n - 1$ if and only if $G$ is traceable. For example, since the graph $G$ of Figure 1 is traceable and has order 5, it follows that $t(G) = 4$.

As with Hamiltonian numbers of graphs, we now see that there is an alternative way to define the traceable number of a connected graph. Denote the length of a walk $W$ in a graph by $L(W)$.

**Proposition 1.1.** Let $G$ be a nontrivial connected graph. Then $t(G)$ is the minimum length of a spanning walk in $G$.

**Proof.** Suppose that the minimum length of a spanning walk in a graph $G$ is $l$. Furthermore, let $s: v_1, v_2, \ldots, v_n$ be a sequence of the vertices of $G$ such that $d(s) = t(G)$. For each integer $i$ with $1 \leq i \leq n - 1$, let $P_i$ be a $v_i - v_{i+1}$ path of length $d(v_i, v_{i+1})$ in $G$. Let $W'$ be the $v_1 - v_n$ spanning walk of $G$ obtained by proceeding along the paths $P_1, P_2, \ldots, P_{n-1}$ in the given order. Thus the length of $W'$ is $L(W') = d(s) = t(G)$. Since $l \leq L(W')$, it follows that $l \leq t(G)$.

Next, let $W$ be a spanning walk of minimum length in $G$. Thus the length of $W$ is $l$. Suppose that $W: x_1, x_2, \ldots, x_{l+1}$, where then $l + 1 \geq n$. Define $u_1 = x_1$ and $u_2 = x_2$. For $3 \leq i \leq n$, define $u_i$ to be $x_{j_i}$, where $j_i$ is the smallest positive integer such that $x_{j_i} \notin \{u_1, u_2, \ldots, u_{i-1}\}$. Then $s: u_1, u_2, \ldots, u_n$ is an ordering of the vertices of $G$. For each integer $i$ with $1 \leq i \leq n - 1$, let $W_i$ be the $u_i - u_{i+1}$ subwalk of $W$ determined by the terms $u_i$ and $u_{i+1}$ in $s$. Thus $d(u_i, u_{i+1}) \leq L(W_i)$. Since

$$t(G) \leq d(s) = \sum_{i=1}^{n-1} d(u_i, u_{i+1}) \leq \sum_{i=1}^{n-1} L(W_i) = L(W) = l,$$

it follows that $t(G) \leq l$, giving the desired result. \qed
In Theorem A it is stated that for every connected graph $G$ of order $n \geq 2$, the Hamiltonian number $h(G) \leq 2n - 2$. As expected, there is a smaller upper bound for the traceable number of $G$.

**Theorem 2.1.** If $G$ is a nontrivial connected graph of order $n$ the length of whose longest path is $l$, then
\[
t(G) \leq 2n - 2 - l.
\]

**Proof.** To show that $t(G) \leq 2n - 2 - l$, we proceed by induction on $n$. Since it is straightforward to see that $t(G) = 2n - 2 - l$ for every connected graph $G$ of order $n$ with $2 \leq n \leq 4$, the inequality holds for every connected graph of order $n$ with $2 \leq n \leq 4$. Assume, for every connected graph $H$ of order $n - 1 \geq 4$ the length of whose longest path is $l'$, that $d(H) \leq 2n - 4 - l'$. Let $G$ be a connected graph of order $n$, the length of whose longest path is $l$. We show that $t(G) \leq 2n - 2 - l$. If $G$ contains a Hamiltonian path, then $l = n - 1$ and $t(G) = n - 1$; so $t(G) = 2n - 2 - l$. Hence we may assume that $G$ does not contain a Hamiltonian path. Let $P$ be a path of length $l < n - 1$ in $G$. Among the vertices of $G$ not on $P$, let $w$ be a vertex of $G$ such that the length of a path from $w$ to a vertex on $P$ is maximum. Thus $G - w$ has order $n - 1$, is connected, and the length of a longest path in $G - w$ is $l$. By the induction hypothesis, $t(G - w) \leq 2n - 4 - l$. Let $s: v_1, v_2, \ldots, v_{n-1}$ be a sequence of the vertices of $G - w$ for which $d(s) = t(G - w)$. Suppose that $w$ is adjacent to $v_i$ ($1 \leq i \leq n - 1$). If $i = n - 1$, then let $s': v_1, v_2, \ldots, v_{n-1}, w$. Thus
\[
t(G) \leq d(s') = d(s) + d(v_{n-1}, w) = d(s) + 1
\]
\[
= t(G - w) + 1 \leq (2n - 4 - l) + 1 < 2n - 2 - l.
\]

If $1 \leq i \leq n - 2$, then insert $w$ immediately after $v_i$ in $s$, producing the sequence
\[
s^* : v_1, v_2, \ldots, v_i, w, v_{i+1}, \ldots, v_{n-1}.
\]

Thus
\[
d(s^*) = d(s) - d(v_i, v_{i+1}) + d(v_i, w) + d(w, v_{i+1})
\]
\[
\leq d(s) - d(v_i, v_{i+1}) + d(v_i, w) + d(w, v_i) + d(v_i, v_{i+1})
\]
\[
= t(G - w) + 2 \leq (2n - 4 - l) + 2 = 2n - 2 - l.
\]

Since $t(G) \leq d(s^*)$, it follows that $t(G) \leq 2n - 2 - l$. \qed
A graph is a *linear forest* if each of its components is a path. The following result gives a lower bound for the traceable number of a connected graph in terms of its order and the maximum size of a spanning linear forest.

**Proposition 2.2.** If *G* is a nontrivial connected graph of order *n* such that the maximum size of a spanning linear forest in *G* is *p*, then

\[
t(G) \geq 2n - 2 - p.
\]

**Proof.** Let \(s : v_1, v_2, \ldots, v_n\) be an arbitrary sequence of the vertices of *G*. Since the maximum size of a spanning linear forest in *G* is *p*, at most *p* of the \(n - 1\) numbers \(d(v_i, v_{i+1})\) (\(1 \leq i \leq n - 1\)) are 1 and the remaining \(n - 1 - p\) numbers are at least 2. Thus

\[
d(s) \geq p \cdot 1 + (n - 1 - p) \cdot 2 = p + 2n - 2 - 2p = 2n - 2 - p.
\]

Therefore, \(t(G) \geq 2n - 2 - p\). \(\Box\)

The following corollary is an immediate consequence of Theorem 2.1 and Proposition 2.2.

**Corollary 2.3.** Let *G* be a nontrivial connected graph of order *n* such that *l* is the length of a longest path in *G* and *p* is the maximum size of a spanning linear forest in *G*. Then

\[
2n - 2 - p \leq t(G) \leq 2n - 2 - l.
\]

The graph *G* of Figure 2 has order *n* = 11. The length of a longest path in *G* is *l* = 6 and the maximum size of a spanning linear forest in *G* is *p* = 8. By Corollary 2.3, \(12 \leq t(G) \leq 14\). Actually, \(t(G) = 13\) and \(s : v_1, v_2, \ldots, v_{11}\) is a linear ordering of the vertices of *G* such that \(d(s) = 13\).

Figure 2. A graph *G* with \(2n - 2 - p < t(G) < 2n - 2 - l\)
Proposition 2.4. If $G$ is a nontrivial connected graph of order $n$ and diameter 2 such that the maximum size of a spanning linear forest in $G$ is $p$, then

$$t(G) = 2n - 2 - p.$$ 

Proof. Since the maximum size of a spanning linear forest in $G$ is $p$, there exists a sequence $s: v_1, v_2, \ldots, v_n$ of the vertices of $G$ such that $p$ of the $n - 1$ distances $d(v_i, v_{i+1})$ $(1 \leq i \leq n - 1)$ are 1 and the remaining $n - 1 - p$ numbers are 2. Thus $d(s) = p \cdot 1 + (n - 1 - p) \cdot 2 = p + 2n - 2 - 2p = 2n - 2 - p$. Hence $t(G) \leq 2n - 2 - p$. Since $t(G) \geq 2n - 2 - p$ by Proposition 2.2, it follows that $t(G) = 2n - 2 - p$. \hfill \square

Each of the graphs $G_1$ and $G_2$ of Figure 3 has order $n = 10$ and the maximum size of a spanning linear forest of each graph is $p = 7$. Such a spanning linear forest $F_i$ of $G_i$ $(i = 1, 2)$ is also shown in Figure 3.

![Graphs and spanning linear forests](image)

Figure 3. The graphs $G_1$ and $G_2$ and a spanning linear forest in each

By Proposition 2.2, $t(G_i) \geq 2n - 2 - p = 11$ for $i = 1, 2$. While $t(G_1) = 11$, it turns out that $t(G_2) = 12$. In the sequence $s_1: v_1, v_2, \ldots, v_{10}$ of the vertices of $G_1$, exactly $p = 7$ of the 9 distances $d(v_i, v_{i+1})$ $(1 \leq i \leq 9)$ are 1 and the other distances are 2. On the other hand, there is no sequence of the vertices of $G_1$ with this property and so $t(G_2) \geq 12$. Because $d(s_2) = 12$ for the sequence $s_2: u_1, u_2, \ldots, u_{10}$, it follows that $t(G_2) = 12$. 

68
The following lemma establishes expected upper and lower bounds for \( h(G) - t(G) \) for a nontrivial connected graph \( G \). The diameter \( \text{diam}(G) \) of a connected graph \( G \) is the largest distance between two vertices in \( G \).

**Lemma 2.5.** For every nontrivial connected graph \( G \),

\[
1 \leq h(G) - t(G) \leq \text{diam}(G).
\]

**Proof.** The lower bound is immediate. To verify the upper bound, let \( s : v_1, v_2, \ldots, v_n \) be an ordering of the vertices of \( G \) such that \( d(s) = t(G) \) and let \( s_c : v_1, v_2, \ldots, v_n, v_1 \) be the cyclic ordering of the vertices of \( G \) obtained from \( s \). Then

\[
h(G) \leq d(s_c) = d(s) + d(v_n, v_1) \leq t(G) + \text{diam}(G).
\]

Therefore, \( h(G) - t(G) \leq \text{diam}(G) \). \( \square \)

We now determine all connected graphs \( G \) for which \( h(G) - t(G) = 1 \).

**Proposition 2.6.** For a nontrivial connected graph \( G \),

\[
h(G) - t(G) = 1 \text{ if and only if } G \text{ is Hamiltonian.}
\]

**Proof.** Observe first that if \( G \) is a Hamiltonian graph of order \( n \), then \( h(G) = n \) and \( t(G) = n - 1 \); so \( h(G) - t(G) = 1 \). For the converse, assume that \( G \) is a connected graph such that \( h(G) - t(G) = 1 \). Let \( s_c : v_1, v_2, \ldots, v_n, v_{n+1} = v_1 \) be a cyclic ordering of the vertices of \( G \) with \( d(s_c) = h(G) \). We show that \( d_G(v_i, v_{i+1}) = 1 \) for \( 1 \leq i \leq n \), which implies that \( v_1, v_2, \ldots, v_n, v_1 \) is a Hamiltonian cycle of \( G \). Consider the linear ordering \( s_l : v_1, v_2, \ldots, v_n \) of the vertices of \( G \) obtained from \( s_c \). Since

\[
d(s_l) = d(s_c) - d(v_1, v_n) = h(G) - d(v_1, v_n),
\]

it follows that \( t(G) \leq d(s_l) = h(G) - d(v_1, v_n) \) and so \( 1 \leq d(v_1, v_n) \leq h(G) - t(G) = 1 \). Thus \( d(v_1, v_n) = 1 \). Consequently, \( d(v_{i-1}, v_i) = 1 \) for \( 2 \leq i \leq n \) as well. Therefore, \( v_1, v_2, \ldots, v_n, v_1 \) is a Hamiltonian cycle of \( G \) and so \( G \) is Hamiltonian. \( \square \)
3. Traceable numbers of trees

If \( G \) is a connected graph and \( H \) is a connected spanning subgraph of \( G \), then 
\[
d_G(u, v) \leq d_H(u, v)
\]
for all \( u, v \in V(G) = V(H) \). Thus for every linear ordering 
\( s: v_1, v_2, \ldots, v_n \) of the vertices of \( G \) (or \( H \)),
\[
d_G(s) = \sum_{i=1}^{n-1} d_G(v_i, v_{i+1}) \leq \sum_{i=1}^{n-1} d_H(v_i, v_{i+1}) = d_H(s)
\]
and so \( t(G) \leq t(H) \). We state this useful observation below.

**Observation 3.1.** If \( G \) is a connected graph and \( H \) is a connected spanning subgraph of \( G \), then 
\( t(G) \leq t(H) \). In particular, if \( G \) is a connected graph and \( T \) is a spanning tree of \( G \), then 
\( t(G) \leq t(T) \).

Observation 3.1 suggests the usefulness of knowing the traceable numbers of trees. Since a tree \( T \) is traceable if and only if \( T \) is a path, it follows for a tree \( T \) of order \( n \) that \( t(T) = n - 1 \) if and only if \( T = P_n \) and so \( t(T) \geq n \) if \( T \neq P_n \).

Since the length of a longest path in \( T \) is the diameter of \( T \), we have the following consequence of Corollary 2.3.

**Corollary 3.2.** If \( T \) is a nontrivial tree of order \( n \) such that the maximum size of a spanning linear forest in \( T \) is \( p \), then 
\[
2n - 2 - p \leq t(T) \leq 2n - 2 - \text{diam}(T).
\]

A caterpillar is a tree \( T \) the removal of whose end-vertices is a path. The trees \( T_1 \) and \( T_2 \) of Figure 4 are caterpillars of the same order \( n = 10 \). While the maximum size of a spanning linear forest of \( T_1 \) is \( \text{diam}(T_1) \), the maximum size of a spanning linear forest of \( T_2 \) is \( \text{diam}(T_2) + 1 \). In Figure 4, \( F_i \) is a spanning linear forest of maximum size in \( T_i \) for \( i = 1, 2 \).

![Figure 4. Spanning linear forests of maximum size in caterpillars](image.png)
Since the maximum size of a spanning linear forest of $T_1$ is $\text{diam}(T_1)$, it follows by Corollary 3.2 that $t(T_1) = 2n - 2 - \text{diam}(T_1)$. In fact, $s_1: u_1, u_2, u_3, u_8, u_7, u_4, u_9, u_5, u_6, u_{10}$ is a linear ordering of the vertices of $T_1$ for which $d(s_1) = t(T_1)$. For the caterpillar $T_2$, however, the maximum size $p$ of a spanning linear forest is $\text{diam}(T_2)+1$. Consequently, by Corollary 3.2 either $t(T_2) = 2n - 2 - \text{diam}(T_2)$ or $t(T_2) = 2n - 3 - \text{diam}(T_2)$. The linear ordering $s_2: v_7, v_1, v_2, v_8, v_9, v_3, v_4, v_{10}, v_5, v_6$ of the vertices of $T_2$ has the property that $d(s_2) = 2n - 2 - \text{diam}(T_2)$. A total of $p$ of the $n - 1$ terms in the sum $d(s_2)$ are 1. All of the remaining terms in $d(s_2)$ are 2, except for one which is 3. If fewer than $p$ terms in the sum $d(s')$ for a linear ordering $s'$ of the vertices of $T_2$ are 1, then $d(s') \geq 2n - 3 - \text{diam}(T_2)$. Hence if there is a linear ordering $s$ of the vertices of $T_2$ for which $d(s) = 2n - 3 - \text{diam}(T_2)$, then there must be $p$ terms in $d(s)$ equal to 1. We may assume that both $v_1, v_2, v_8$ (or $v_8, v_2, v_1$) and $v_9, v_3, v_4$ (or $v_4, v_3, v_9$) are subsequences of $s$. Assume, without loss of generality, that the vertices $v_1, v_2, v_8$ occur before $v_9, v_3, v_4$. Then the first vertex in $s$ that follows the last vertex of $v_1, v_2, v_8$ or the last vertex of $v_1, v_2, v_8, v_7$ is a vertex whose distance is at least 3 from that vertex. Hence $d(s) \geq 2n - 2 - \text{diam}(T_2)$ and so $t(T_2) = 2n - 2 - \text{diam}(T_2)$. Proceeding in a similar manner for every caterpillar gives us the following result.

**Corollary 3.3.** If $T$ is a caterpillar of order $n$, then

$$t(T) = 2n - 2 - \text{diam}(T).$$

We now show that the formula presented in Corollary 3.3 for the traceable number of a caterpillar holds in fact for all trees.

**Theorem 3.4.** If $T$ is a nontrivial tree of order $n$, then

$$t(T) = 2n - 2 - \text{diam}(T).$$

**Proof.** Since $h(T) = 2n - 2$ for every tree $T$ of order $n$, it follows by Lemma 2.5 that $t(T) \geq 2(n - 1) - \text{diam}(T)$. Furthermore, since the length of a longest path in $T$ is $\text{diam}(T)$, it follows by Theorem 2.1 that $t(T) \leq 2(n - 1) - \text{diam}(T)$, giving the desired result. \hfill $\square$

If $T$ is a tree of order $n \geq 3$, then $2 \leq \text{diam}(T) \leq n - 1$. Therefore, by Theorem 3.4, if $T$ is a tree of order $n \geq 3$, then

$$n - 1 \leq t(T) \leq 2n - 4.$$

We saw that $t(T) = n - 1$ if and only if $T = P_n$. Furthermore, only stars have diameter 2. So $t(T) = 2n - 4$ if and only if $T = K_{1,n-1}$ by Theorem 3.4. More generality, we have the following the realization result.
Proposition 3.5. For each pair \( k, n \) of integers with \( 3 \leq n - 1 \leq k \leq 2n - 4 \), there exists a tree \( T \) of order \( n \) with \( t(T) = k \).

Proof. Let \( P: v_1, v_2, \ldots, v_{2n-1-k} \) be a path of length \( 2n - 2 - k \). A tree \( T \) is constructed by adding \( k + 1 - n \) new vertices \( w_1, w_2, \ldots, w_{k+1-n} \) and joining all of these vertices to \( v_2 \). Since \( \text{diam}(T) = 2n - 2 - k \), it follows by Theorem 3.4 that \( t(T) = 2n - 2 - (2n - 2 - k) = k \).

With the aid of Theorem 3.4, it is straightforward to determine those nontrivial trees \( T \) of order \( n \) such that \( t(T) = n \).

Proposition 3.6. Let \( T \) be a tree of order \( n \geq 4 \). Then \( t(T) = n \) if and only if \( T \) is a caterpillar with maximum degree \( \Delta(T) = 3 \) and having exactly one vertex of degree 3.

Proof. By Theorem 3.4, \( t(T) = n \) if and only if \( 2n - 2 - \text{diam}(T) = n \) and so \( \text{diam}(T) = n - 2 \). Hence \( T \) contains a path \( P: v_1, v_2, \ldots, v_{n-1} \) of length \( n - 2 \) and a vertex \( w \) not on \( P \) that is adjacent to some vertex \( v_i \) with \( 2 \leq i \leq n - 2 \).

By (1) and Observation 3.1, if \( G \) is a connected graph of order \( n \geq 3 \), then

\[
(2) \quad n - 1 \leq t(G) \leq 2n - 4.
\]

We now determine all those connected graphs \( G \) of order \( n \) such that \( t(G) = 2n - 4 \) or \( t(G) = 2n - 5 \).

Proposition 3.7. Let \( G \) be a connected graph of order \( n \geq 3 \). Then

\[
t(G) = 2n - 4 \text{ if and only if } G = K_3 \text{ or } G = K_{1,n-1}.
\]

Proof. Let \( G \) be a connected graph of order \( n \geq 3 \) such that \( t(G) = 2n - 4 \). If \( G \) contains a path of length 3 or more, then it follows by Theorem 2.1 that \( t(G) \leq 2n - 5 \). Hence the length of a longest path in \( G \) is 2. This implies that \( \Delta(G) = n - 1 \) and so \( G = K_3 \) or \( G = K_{1,n-1} \). Furthermore, note that \( t(K_3) = 2n - 4 = n - 1 \) and \( t(K_{1,n-1}) = 2n - 4 \).

A tree \( T \) is a double star if \( T \) contains exactly two vertices that are not end-vertices, necessarily these vertices are adjacent in \( T \). For integers \( a, b \geq 2 \), let \( S_{a,b} \) denote the double star whose two vertices that are not end-vertices have degrees \( a \) and \( b \).
Proposition 3.8. Let $G$ be a connected graph of order $n \geq 4$. Then $t(G) = 2n - 5$ if and only if (1) $n = 4$ and $G \neq K_{1,3}$ and (2) $n \geq 5$ and $G = K_{1,n-1} + e$ or $G = S_{a,b}$ for some positive integers $a$ and $b$ with $a + b = n$.

Proof. Let $G$ be a connected graph of order $n \geq 4$ such that $t(G) = 2n - 5$. From Theorem 2.1, it follows that the length of a longest path in $G$ is 3. This implies that (1) $n = 4$ and $G \neq K_{1,3}$, (2) $n \geq 5$, $\Delta(G) = n - 1$, and $G = K_{1,n-1} + e$, or (3) $n \geq 5$, $\Delta(G) \leq n - 2$ and $G$ is a double star. The converse is straightforward.

4. Traceable numbers of vertices

Let $G$ be a connected graph of order $n$. For $v \in V(G)$, the traceable number $t(v)$ of $v$ is defined by

$$t(v) = \min \{d(s)\},$$

where the minimum is taken over all linear orderings $s$ of the vertices of $G$ whose first term is $v$. Thus $t(v) \geq n - 1$ for every vertex $v$ of $G$. Furthermore, $t(v) = n - 1$ if and only if $G$ contains a Hamiltonian path with initial vertex $v$. Observe that

$$t(G) = \min \{t(v) : v \in V(G)\}.$$ 

Using an argument similar to that used in the proof of Proposition 1.1, we have the following.

Proposition 4.1. Let $G$ be a nontrivial connected graph and let $v \in V(G)$. Then $t(v)$ is the minimum length of a spanning walk in $G$ whose initial vertex is $v$.

We present a result concerning the traceable number of adjacent vertices in a connected graph.

Proposition 4.2. Let $G$ be a connected graph and let $u$ and $v$ be adjacent vertices of $G$. Then

$$|t(u) - t(v)| \leq 1.$$ 

Proof. Let $s : v = v_1, v_2, \ldots, v_n$ be a linear ordering of the vertices of $G$ such that $d(s) = t(v)$. Thus $u = v_i$ for some integer $i$ with $2 \leq i \leq n$. We consider two cases.

Case 1. $u = v_i$, where $2 \leq i \leq n - 1$. Let

$$s' : u = v_i, v_{i-1}, \ldots, v_2, v_1 = v, v_{i+1}, v_{i+2}, \ldots, v_n.$$
Then
\[
    t(u) \leq d(s') = d(s) - d(u, v_{i+1}) + d(v, v_{i+1}) \\
    \leq d(s) - d(u, v_{i+1}) + d(v, u) + d(u, v_{i+1}) = d(s) + 1 = t(v) + 1.
\]

Thus \(t(u) - t(v) \leq 1\).

**Case 2.** \(u = v_n\). Consider the sequence
\[
s'' : u = v_n, v_{n-1}, \ldots, v_2, v_1 = v.
\]

Then \(t(u) \leq d(s'') = d(s) = t(v)\) and so \(t(u) - t(v) \leq 0\).

In either case, \(t(u) - t(v) \leq 1\). Applying a similar argument to that given above, we have \(t(v) - t(u) \leq 1\) as well and so \(|t(u) - t(v)| \leq 1\). \(\square\)

For a connected graph \(G\), let
\[
t^+(G) = \max\{t(v) : v \in V(G)\}.
\]

Obviously, \(t(G) \leq t^+(G)\) for every connected graph \(G\). The following is a consequence of Proposition 4.2.

**Corollary 4.3.** Let \(G\) be a connected graph and let \(k\) be an integer such that \(t(G) \leq k \leq t^+(G)\). Then there exists a vertex \(w\) of \(G\) such that \(t(w) = k\).

**Proof.** The statement is obvious if \(k = t(G)\) or \(k = t^+(G)\). Hence we may assume that \(t(G) < k < t^+(G)\). Let \(u\) be a vertex such that \(t(u) = t(G)\) and let \(v\) be a vertex such that \(t(v) = t^+(G)\). Since \(G\) is connected, \(G\) contains a \(u - v\) path \(P: u = u_1, u_2, \ldots, u_s = v\). By Proposition 4.2, \(|t(u_i) - t(u_{i+1})| \leq 1\) for all \(i\) with \(1 \leq i \leq s - 1\). Let \(j\) be the largest integer with \(1 \leq j < s\) such that \(t(u_j) \leq k\). Then \(t(u_j) = k\); for otherwise, \(t(u_j) \leq k - 1\) and so \(t(u_{j+1}) \leq 1 + (k - 1) = k\), producing a contradiction. \(\square\)

For a vertex \(v\) in a connected graph \(G\), the **eccentricity** \(e(v)\) of \(v\) is the largest distance between \(v\) and a vertex of \(G\).

**Theorem 4.4.** If \(T\) is a nontrivial tree of order \(n\) and let \(v\) be a vertex of \(T\), then
\[
t(v) = 2(n - 1) - e(v).
\]

**Proof.** First, we show that \(t(v) \geq 2(n - 1) - e(v)\). Let \(s: v = v_1, v_2, \ldots, v_n\) be a linear ordering of the vertices of \(T\) such that \(d(s) = t(v)\), and let
\[
s' : v = v_1, v_2, \ldots, v_n, v_1
\]
be the cyclic ordering of the vertices of \( T \) obtained by adding \( v_1 = v \) at the end of \( s \). Then
\[
2(n - 1) = h(T) \leq d(s') = d(s) + d(v_n, v_1) \leq t(v) + e(v)
\]
and so \( t(v) \geq 2(n - 1) - e(v) \).

Next, we show that \( t(v) \leq 2(n - 1) - e(v) \) for each vertex \( v \) in a nontrivial tree of order \( n \). We proceed by induction on \( n \). This is certainly true for a tree of order 2. Assume, for every tree \( T' \) of order \( n - 1 \), where \( n - 1 \geq 2 \), and every vertex \( u \) of \( T' \), that \( t(u) \leq 2(n - 2) - e(u) \). We show that if \( T \) is a nontrivial tree of order \( n \) and \( v \) is a vertex of \( T \), then
\[
t(v) \leq 2(n - 1) - e(v).
\]
This is certainly the case if \( T \) is the path \( P_n \) and \( v \) is an end-vertex of \( P_n \). Hence we may assume that this is not the case. Let \( P \) be a longest path in \( T \) with initial vertex \( v \), say \( P \) is a \( v - w \) path. Then \( d(v, w) = e(v) \). Hence there exists an end-vertex \( x \) of \( T \) such that \( x \) does not lies on \( P \). Let \( y \) be the vertex of \( T \) that is adjacent to \( x \). Thus \( T - x \) is a tree of order \( n - 1 \) such that \( v \in V(T - x) \) and \( e_{T - x}(v) = e_T(v) \). By the induction hypothesis,
\[
t_{T - x}(v) \leq 2(n - 2) - e_{T - x}(v) = 2(n - 2) - e_T(v).
\]
Let \( s_1 : v = u_1, u_2, \ldots, u_{n-1} \) be a linear ordering of the vertices of \( T - x \) such that \( d(s_1) = t_{T - x}(v) \). Then \( y = u_i \) for some \( i \) with \( 2 \leq i \leq n - 1 \). Let \( z \) be the vertex of \( T - x \) that immediately follows or immediately precedes \( y \) in \( s_1 \), say \( z \) immediately follows \( y \) in \( s_1 \). Thus \( z = u_{i+1} \). Let \( s \) be the linear ordering of the vertices of \( T \) obtained by inserting \( x \) between \( y \) and \( z \). Then
\[
d(s) = d(s_1) - d(y, z) + d(y, x) + d(x, z) \leq d(s_1) - d(y, z) + 1 + 1 + d(y, z)
\]
\[
= d(s_1) + 2 = t_{T - x}(v) + 2 \leq 2(n - 2) - e_T(v) + 2.
\]
Therefore, \( t_T(v) \leq d(s) \leq 2(n - 1) - e_T(v) \). Hence \( t(v) = 2(n - 1) - e(v) \). \( \square \)

By Theorem 4.4,
\[
t(v) = h(T) - e(v)
\]
for every tree \( T \) and every vertex \( v \) of \( T \). Since \( t(T) = \min \{ t(v) : v \in V(G) \} \), it follows that
\[
t(T) = h(T) - \max \{ e(v) : v \in V(T) \} = 2n - 2 - \text{diam}(T),
\]
which provides us with an alternative proof of Theorem 3.4.
Observe that Theorem 4.4 is not true in general for connected graphs that are not trees. Consider the graphs $G$ and $H$ in Figure 5. Each vertex of $G$ and $H$ is labeled with its traceable number. The Hamiltonian number of graph $G$ is $h(G) = 7$. Since $e(u) = e(y) = 3$ and $e(v) = e(w) = e(x) = 3$, it follows that $t(z) = h(G) - e(z)$ for every vertex $z$ of $G$. On the other hand, for the graph $H$, $h(H) = 6$. While $t(z) = h(H) - e(z)$ for $z = w$ and $z = x$, this is not true otherwise.

![Figure 5. The graphs $G$ and $H$](image)

5. **Graphs with Prescribed Hamiltonian and Traceable Numbers**

We have seen in Lemma 2.5 that for every nontrivial connected graph $G$,

$$1 \leq h(G) - t(G) \leq \text{diam}(G).$$

Furthermore, by Proposition 2.6, Hamiltonian graphs are the only connected graphs $G$ for which $h(G) - t(G) = 1$. By Theorems A and 3.4, if $T$ is a tree then $h(T) - t(T) = \text{diam}(T)$. However, trees are not the only connected graphs with this property. In fact, there are other classes of connected graphs with this property. For example, if $G = K_{n_1,n_2,\ldots,n_k}$ is a complete $k$-partite graph, where $k \geq 2$, $n_1 \leq n_2 \leq \ldots \leq n_k$, and $n_1 + n_2 + \ldots + n_{k-1} < n_k$, then $h(G) - t(G) = 2 = \text{diam}(G)$. Next, we show that for each pair $k, d$ of integers with $1 \leq k \leq d$, there exists a connected graph $G$ with $\text{diam}(G) = d$ such that $h(G) - t(G) = k$. In order to do this, we first state a useful lemma that appeared in [5].

**Lemma B.** Let $G$ be a connected graph having blocks $B_1, B_2, \ldots, B_k$. Then

$$h(G) = \sum_{i=1}^{k} h(B_i).$$
Proposition 5.1. For each pair $k, d$ of integers with $1 \leq k \leq d$, there exists a connected graph $G$ with diameter $d$ such that $h(G) - t(G) = k$.

Proof. If $k = d$, let $G$ be a tree with diam$(G) = d$. It then follows by Theorem A and Theorem 3.4 that $h(G) - t(G) = (2n - 2) - (2n - d - 2) = d$. Thus, we may assume that $k < d$. For $k = 1$, the cycle $C_{2d}$ of order $2d$ has the desired property. For $k \geq 2$, let $G$ be the graph obtained from the cycle $C_{2(d-k+1)}$: $u_1, u_2, \ldots, u_{2(d-k+1)}$, $u_1$ and the path $P_{k-1}$: $v_1, v_2, \ldots, v_{k-1}$ by joining $u_{d-k+1}$ and $v_{k-1}$. Then the order of $G$ is $n = 2d - k + 1$ and its diameter is diam$(G) = d$. By Lemma B,

$$h(G) = h(C_{2(d-k+1)}) + (k - 1)h(P_2) = 2(d - k + 1) + 2(k - 1) = 2d.$$ 

Since $G$ is traceable, $t(G) = n - 1 = 2d - k$. Therefore, $h(G) - t(G) = k$. \hfill \Box

Since $h(G) \leq t(G) +$ diam$(G)$ for every nontrivial connected graph $G$ and, trivially, $t(G) \geq$ diam$(G)$, it follows that $t(G) < h(G) \leq 2t(G)$. Thus if $G$ is a connected graph with $t(G) = a$ and $h(G) = b$, then $a < b \leq 2a$. Next, we show that every pair $a, b$ of positive integers with $a < b \leq 2a$ is realizable as the traceable number and the Hamiltonian number of some connected graph, respectively.

Proposition 5.2. For every pair $a, b$ of positive integers with $a < b \leq 2a$, there is a connected graph $G$ with $t(G) = a$ and $h(G) = b$.

Proof. If $b = 2a$, then $G = P_{a+1}$ has the desired properties. Hence we may assume that $a < b < 2a$. Let $k = b - a$. Thus $k < a$. Let $G$ be the graph obtained from the path $P$: $u_1, u_2, \ldots, u_a, u_{a+1}$ by joining $u_{a+1}$ and $u_k$. By Lemma B,

$$h(G) = h(C_{a-k+2}) + (k - 1)h(P_2) = (a - k + 2) + 2(k - 1) = b.$$ 

Since $G$ is traceable, $t(G) = (a + 1) - 1 = a$. \hfill \Box

By Theorem A, Lemma 2.5, and (2), if $G$ is a connected graph of order $n \geq 3$ with $t(G) = a$ and $h(G) = b$, then

$$1 \leq n - 1 \leq a < b \leq 2n - 2.$$ 

Next we determine all triples $(a, b, n)$ of positive integers satisfying (3) that can be realized as the traceable number, Hamiltonian number, and order, respectively, of some connected graph.
**Theorem 5.3.** For each triple \((a, b, n)\) of positive integers with \(1 \leq n - 1 \leq a < b < 2n - 2\) and \(n \geq 3\), there is a connected graph \(G\) of order \(n\) such that \(t(G) = a\) and \(h(G) = b\) if and only if (1) \(b = a + 1 = n\) or (2) \(b \geq a + 2\).

**Proof.** Let \(G\) be a connected graph of order \(n\) such that \(t(G) = a\) and \(h(G) = b\). If \(b = a + 1\), then \(h(G) - t(G) = 1\). By Proposition 2.6, \(G\) is Hamiltonian. Thus \(t(G) = n - 1\) and \(h(G) = n\). Thus \(b = a + 1 = n\). If \(b \neq a + 1\), then \(b \geq a + 2\) by Lemma 2.5.

For the converse, let \((a, b, n)\) be a triple of positive integers with \(1 \leq n - 1 \leq a < b < 2n - 2\) such that \(b = a + 1 = n\) or \(b \geq a + 2\). If \(b = a + 1 = n\), then any Hamiltonian graph of order \(n\) has the desired property. Thus, we may assume that \(b \geq a + 2\). Observe that \(b - a - 1 \geq 1\) and \(2n - b \geq 2\). We consider two cases.

**Case 1.** \(a = n - 1\). Let \(G_1\) be the graph obtained from the path \(P_{b-a-1} = u_1, u_2, \ldots, u_{b-a-1}\) of order \(b - a - 1\) and the complete graph \(K_{2n-b}\) with \(V(K_{2n-b}) = \{v_1, v_2, \ldots, v_{2n-b}\}\) by joining \(u_{b-a-1}\) to \(v_1\). Then the order of \(G_1\) is \(n = (b - a - 1) + (2n - b) = n\). By Lemma B,

\[
h(G_1) = (b - a - 1)h(P_2) + h(K_{2n-b}) = 2(b - a - 1) + (2n - b) = b.
\]

Since \(G_1\) is traceable, \(t(G_1) = n - 1 = a\).

**Case 2.** \(a \geq n\). Let \(G_2\) be the graph obtained from the graph \(G_1\) in Case 1 by adding \(a - n + 1\) new vertices \(w_1, w_2, \ldots, w_{a-n+1}\) and joining \(w_i\) to \(v_1\) for \(1 \leq i \leq a - n + 1\). Then the order of \(G_2\) is \(n = (b - a - 1) + (2n - b) + (a - n + 1) = n\) and \(\text{diam}(G_2) = b - a\). By Lemma B,

\[
h(G_2) = (b - a - 1)h(P_2) + h(K_{2n-b}) + (a - n + 1)h(P_2)
= 2(b - a - 1) + (2n - b) + 2(a - n + 1) = b.
\]

It remains to show that \(t(G_2) = a\). By Lemma 2.5,

\[
t(G_2) \leq h(G_2) - \text{diam}(G_2) = b - (b - a) = a.
\]

Since the maximum size of a spanning linear forest in \(G_2\) is \(p = 2n - a - 2\), it follows by Proposition 2.2 that \(t(G_2) \geq 2n - 2 - p = a\). Thus \(t(G_2) = a\). \(\square\)
6. Hamiltonian-connected numbers of graphs

For a connected graph $G$ of order $n$, the Hamiltonian-connected number $hcon(G)$ of $G$ is defined by

$$hcon(G) = \sum_{v \in V(G)} t(v).$$

Since $t(v) \geq n - 1$ for every vertex $v$ of $G$, it follows that $hcon(G) \geq n(n - 1)$. Furthermore, $hcon(G) = n(n - 1)$ if and only if $G$ is Hamiltonian-connected. Therefore, the Hamiltonian-connected number of a connected graph $G$ of order $n$ can be considered as a measure of how close $G$ is to being Hamiltonian-connected—the closer $hcon(G)$ is to $n(n - 1)$, the closer $G$ is to being Hamiltonian-connected.

Consider the graphs $H_1$ and $H_2$ in Figure 6, where $H_1$ is obtained from the complete graph $K_{n-1}$ by adding a pendant edge and $H_2 \cong 2K_1 + (K_{n-4} \cup 2K_1)$. For the graph $H_1$, every vertex of $H_1$ has traceable number $n - 1$, except for the vertex $v$ which has traceable number $n$. Thus $hcon(H_1) = n(n-1) + 1$. Every vertex of the graph $H_2$ has traceable number $n - 1$, except for $v_1$ and $v_2$, which have traceable number $n$. Thus $hcon(H_2) = n(n-1) + 2$.

Next consider the graphs $G_1$ and $G_2$ in Figure 7, where $G_1$ is obtained from the complete graph $K_{n-2}$ $(n \geq 5)$ by adding two pendant edges and $G_2$ is obtained from the cycle $C_{n-1}$ $(n \geq 4)$ by adding a pendant edge. The graph $G_1$ of order $n$ in Figure 7 contains exactly two vertices with traceable number $n - 1$, namely $t(u) = t(v) = n - 1$. All other vertices of $G_1$ have traceable number $n$. Thus $hcon(G_1) = n(n-1) + (n-2)$. The graph $G_2$ of order $n$ in Figure 7 contains exactly three vertices with traceable number $n - 1$, namely $t(u) = t(v) = t(w) = n - 1$. All other vertices of $G_2$ have traceable number $n$. Thus $hcon(G_2) = n(n-1) + (n-3)$. Therefore, the graphs $H_1$ and $H_2$ in Figure 6 are closer to being Hamiltonian-connected than are the graphs $G_1$ and $G_2$ of Figure 7.

The minimum eccentricity among the vertices of $G$ is its radius, which is denoted by $\text{rad}(G)$. A vertex $v$ in $G$ is a central vertex if $e(v) = \text{rad}(G)$ and the subgraph

![Figure 6. The graphs $H_1$ and $H_2$](image-url)
induced by the central vertices of $G$ is the center of $G$. Next, we establish upper and lower bounds for the Hamiltonian-connected number of a connected graph in terms of its order, beginning with trees.

**Theorem 6.1.** For every tree $T$ of order $n \geq 3$,

$$n(n - 1) + \left\lfloor \left(\frac{n - 1}{2}\right)^2 \right\rfloor \leq \text{hcon}(T) \leq n(n - 1) + (n^2 - 3n + 1).$$

**Proof.** For a tree $T$, it is known (see [10]) that there exists at least one vertex $v$ with $e(v) = \text{rad}(T)$ and there exist at least two vertices $v$ with $e(v) = k$ for every integer $k$ with $\text{rad}(T) < k \leq \text{diam}(T)$. Furthermore, it is well-known that for every tree $T$, either

$$\text{diam}(T) = 2 \text{rad}(T) \text{ or } \text{diam}(T) = 2 \text{rad}(T) - 1$$

where the center of $T$ contains exactly one vertex in the first case and exactly two vertices in the second case. Since $\text{diam}(T) \leq n - 1$ for every tree $T$ of order $n$, the largest possible radius of a tree $T$ having odd order is $(n - 1)/2$, while the largest possible radius of a tree $T$ having even order is $n/2$. We consider the cases when $n$ is odd or $n$ is even separately.

**Case 1.** $n$ is odd. In this case,

$$\sum_{v \in V(T)} e(v) \leq \frac{n - 1}{2} + 2\left[\frac{n + 1}{2} + \frac{n + 3}{2} + \ldots + (n - 1)\right]$$

$$= \frac{n - 1}{2} + (n + 1) + (n + 3) + \ldots + 2(n - 1)$$

$$= \frac{n - 1}{2} + \frac{n(n - 1)}{2} + \left(\frac{n - 1}{2}\right)^2 = \frac{n^2 - 1}{2} + \left(\frac{n - 1}{2}\right)^2.$$
It then follows by Theorem 4.4 that
\[
\text{hcon}(T) = \sum_{v \in V(T)} t(v) = \sum_{v \in V(T)} (2n - 2 - e(v)) = n(2n - 2) - \sum_{v \in V(T)} e(v)
\]
\[
\geq n(2n - 2) - \left[ \frac{n^2 - 1}{2} + \left( \frac{n - 1}{2} \right)^2 \right] = n(n - 1) + \left( \frac{n - 1}{2} \right)^2.
\]

Case 2. \(n\) is even. In this case,
\[
\sum_{v \in V(T)} e(v) \leq 2 \left[ \frac{n}{2} + \frac{n + 2}{2} + \ldots + (n - 1) \right]
\]
\[
= n + (n + 2) + \ldots + 2(n - 1) = \frac{n^2}{2} + \frac{n^2 - 2n}{4}.
\]

It then follows by Theorem 4.4 that
\[
\text{hcon}(T) = \sum_{v \in V(T)} t(v) = \sum_{v \in V(T)} (2n - 2 - e(v)) = n(2n - 2) - \sum_{v \in V(T)} e(v)
\]
\[
\geq n(2n - 2) - \left( \frac{n^2}{2} + \frac{n^2 - 2n}{4} \right) = n(n - 1) + \frac{n^2 - 2n}{4}.
\]

Therefore, \(\text{hcon}(T) \geq n(n - 1) + \left\lfloor \left( \frac{n - 1}{2} \right)^2 \right\rfloor\) for every tree \(T\) of order \(n \geq 3\).

If a tree \(T\) of order \(n \geq 3\) contains a vertex with eccentricity 1, then \(T\) is a star and all other vertices have eccentricity 2. If the minimum eccentricity of a vertex of \(T\) is 2, then at most two vertices of \(T\) have eccentricity 2, with all other vertices have eccentricity 3 or 4. In any case,
\[
\sum_{v \in V(T)} e(v) \geq 1 + (n - 1) \cdot 2 = 2n - 1.
\]

Consequently,
\[
\text{hcon}(T) = \sum_{v \in V(T)} t(v) = \sum_{v \in V(T)} (2n - 2 - e(v)) = n(2n - 2) - \sum_{v \in V(T)} e(v)
\]
\[
\leq n(2n - 2) - (2n - 1) = n(n - 1) + (n^2 - 3n + 1).
\]

Therefore, \(\text{hcon}(T) \leq n(n - 1) + (n^2 - 3n + 1)\) for every tree \(T\) of order \(n \geq 3\). \(\square\)

Since \(\text{hcon}(P_n) = n(n - 1) + \left\lfloor \left( \frac{n - 1}{2} \right)^2 \right\rfloor\) and \(\text{hcon}(K_{1,n-1}) = n(n - 1) + (n^2 - 3n + 1)\), the lower and upper bounds in Theorem 6.1 are both sharp.
Corollary 6.2. For a nontrivial connected graph $G$ of order $n$,

$$n(n - 1) \leq h_{\text{con}}(G) \leq n(n - 1) + (n^2 - 3n + 1).$$

Proof. We have already noted that $h_{\text{con}}(G) \geq n(n - 1)$, so it remains only to show that $h_{\text{con}}(G) \leq n(n - 1) + (n^2 - 3n + 1)$. For every connected spanning subgraph $H$ of $G$ and every two vertices $x$ and $y$ of $G$, $d_G(x, y) \leq d_H(x, y)$. Therefore, for every vertex $v$ of $G$, $t_G(v) \leq t_H(v)$. Hence if $T$ is a spanning tree of $G$, then $t_G(v) \leq t_T(v)$ for every vertex $v$ of $G$. This implies that among all connected graphs $G$ of order $n$, the maximum value of $h_{\text{con}}(G)$ occurs when $G$ is a tree. The result then follows by Theorem 6.1. □

We now show that for every integer $n \geq 3$ and integer $k$ with $2 \leq k \leq n$, there exists a connected graph $G$ of order $n$ containing $k$ vertices $v$ with $t(v) = n - 1$ such that $h_{\text{con}}(G) = n(n - 1) + (n - k)$.

Proposition 6.3. For every integer $n \geq 3$ and integer $k$ with $2 \leq k \leq n$, there exists a connected graph of order $n$ containing $k$ vertices with traceable number $n - 1$ and $n - k$ vertices with traceable number $n$.

Proof. Since every Hamiltonian-connected graph has the desired properties for $k = n$, we restrict our attention to those integers $k$ for which $2 \leq k \leq n - 1$. For $3 \leq n \leq 5$, the graphs $G_{k,n}$ of Figure 8 have the desired properties.

For $n \geq 6$, the graphs $G_{k,n}$ of Figure 9 have the appropriate properties. □

There is no graph of order $n$ containing exactly one vertex with traceable number $n - 1$. We know of no example of a nontrivial connected graph of order $n$, every vertex of which has traceable number $n$, that is, of a non-traceable graph $G$ of order $n$ for which $h_{\text{con}}(G) = n^2$. 82
Figure 9. Graphs $G_{k,n}$ where $2 \leq k \leq n - 1$ and $n \geq 6$

Acknowledgments. We are grateful to Professor Gary Chartrand for suggesting the concepts of traceable number and Hamiltonian-connected number to us and kindly providing useful information on this topic. Also, we are grateful to Professor Ladislav Nebeský whose valuable suggestions resulted in an improved paper.

References


Authors’ addresses: Varaporn Saenpholphat, Department of Mathematics, Srinakharinwirot University, Sukhumvit Soi 23, Bangkok, 10110, Thailand; Futaba Okamoto, Ping Zhang, Department of Mathematics, Western Michigan University, Kalamazoo, MI 49008, USA, e-mail: ping.zhang@wmich.edu.