Josef Král; Dagmar Medková
Essential norms of the Neumann operator of the arithmetical mean

Mathematica Bohemica, Vol. 126 (2001), No. 4, 669–690

Persistent URL: http://dml.cz/dmlcz/134114

Terms of use:

© Institute of Mathematics AS CR, 2001

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.
ESSENTIAL NORMS OF THE NEUMANN OPERATOR
OF THE ARITHMETICAL MEAN

JOSEF KRÁL, DAGMAR MEDKOVÁ, Praha

(Received October 6, 1999)

Abstract. Let $K \subset \mathbb{R}^m$ ($m \geq 2$) be a compact set; assume that each ball centered on the boundary $B$ of $K$ meets $K$ in a set of positive Lebesgue measure. Let $\mathcal{C}_0^{(1)}$ be the class of all continuously differentiable real-valued functions with compact support in $\mathbb{R}^m$ and denote by $\sigma_m$ the area of the unit sphere in $\mathbb{R}^m$. With each $\varphi \in \mathcal{C}_0^{(1)}$ we associate the function

$$W_K \varphi(z) = \frac{1}{\sigma_m} \int_{\mathbb{R}^m \setminus K} \text{grad} \varphi(x) \cdot \frac{z - x}{|z - x|^m} \, dx$$

of the variable $z \in K$ (which is continuous in $K$ and harmonic in $K \setminus B$). $W_K \varphi$ depends only on the restriction $\varphi|_B$ of $\varphi$ to the boundary $B$ of $K$. This gives rise to a linear operator $W_K$ acting from the space $\mathcal{C}^{(1)}(B) = \{ \varphi|_B; \varphi \in \mathcal{C}_0^{(1)} \}$ to the space $\mathcal{C}(B)$ of all continuous functions on $B$. The operator $T_K$ sending each $f \in \mathcal{C}^{(1)}(B)$ to $T_K f = 2W_K f - f \in \mathcal{C}(B)$ is called the Neumann operator of the arithmetical mean; it plays a significant role in connection with boundary value problems for harmonic functions. If $p$ is a norm on $\mathcal{C}(B) \supset \mathcal{C}^{(1)}(B)$ inducing the topology of uniform convergence and $\mathcal{G}$ is the space of all compact linear operators acting on $\mathcal{C}(B)$, then the associated $p$-essential norm of $T_K$ is given by

$$\omega_p T_K = \inf_{Q \in \mathcal{G}} \sup \{ p[(T_K - Q)f]; f \in \mathcal{C}^{(1)}(B), p(f) \leq 1 \}.$$

In the present paper estimates (from above and from below) of $\omega_p T_K$ are obtained resulting in precise evaluation of $\omega_p T_K$ in geometric terms connected only with $K$.

Keywords: double layer potential, Neumann’s operator of the arithmetical mean, essential norm

MSC 2000: 31B10, 45P05, 47A30
In what follows \( \mathbb{R}^m \) will be the Euclidean space of dimension \( m \geq 2 \). The Euclidean norm of a vector \( x = (x_1, \ldots, x_m) \in \mathbb{R}^m \) will be denoted by \( |x| \). If \( M \subset \mathbb{R}^m \), then the symbols \( \overline{M} \), \( M^o \) and \( \partial M \) will denote the closure, the interior and the boundary of \( M \), respectively. \( B_r(z) := \{ x \in \mathbb{R}^m ; |x - z| < r \} \) is the open ball of radius \( r > 0 \) centered at \( z \in \mathbb{R}^m \). The symbol \( \lambda_k \) will denote the outer \( k \)-dimensional Hausdorff measure with the usual normalization (so that \( \lambda_m \) coincides with the outer Lebesgue measure in \( \mathbb{R}^m \)). We put
\[
\sigma_m := \lambda_{m-1}(\partial B_1(0)) = \frac{2\pi^{m/2}}{\Gamma(m/2)},
\]
where \( \Gamma \) is the Euler gamma function. For fixed \( z \in \mathbb{R}^m \) the symbol \( h_z \) will denote the fundamental harmonic function with a pole at \( z \), whose values at any \( x \in \mathbb{R}^m \setminus \{z\} \) are given by
\[
h_z(x) := \begin{cases} 
\frac{1}{2\pi} \ln \frac{1}{|x - z|} & \text{if } m = 2, \\
\frac{1}{(m-2)\sigma_m} |x - z|^{2-m} & \text{if } m > 2;
\end{cases}
\]
we put \( h_z(z) = +\infty \). Let \( C_0^{(1)} \) be the space of all continuously differentiable compactly supported real-valued functions on \( \mathbb{R}^m \). We fix a compact set \( K \subset \mathbb{R}^m \) and put \( G = \mathbb{R}^m \setminus K, B = \partial K \). With any \( \varphi \in C_0^{(1)} \) we associate the function \( W_K \varphi \equiv W \varphi \) on \( K \) defined by
\[
W \varphi(z) = \int_G \nabla \varphi(x) \cdot \nabla h_z(x) \, d \lambda_m(x), \; z \in K.
\]
It is not difficult to verify that \( W \varphi \) is continuous in \( K \) and harmonic in \( K^\circ \); besides, \( W \varphi \) depends only on the restriction \( \varphi|_B \) of \( \varphi \in C_0^{(1)} \) to \( B \) (cf. §2 in [9]). Denote by
\[
C^{(1)}(B) := \{ \varphi|_B; \; \varphi \in C_0^{(1)} \}
\]
the vectorspace (over the reals) of all restrictions to \( B \) of functions in \( C_0^{(1)} \) and let \( C(K) \) be the vectorspace of all finite continuous real-valued functions in \( K \); then \( W \) gives rise to a linear operator acting from \( C^{(1)}(B) \) to \( C(K) \). In connection with boundary value problems it is natural to inquire about conditions on \( K \) guaranteeing the continuity of the operator \( W \) with respect to the topologies of uniform convergence in \( C^{(1)}(B) \) and in \( C(K) \) (compare [3], [15], [8], [9]). For simplicity, we will always assume that \( K \) is massive in the sense that
\[
\lambda_m(B_r(z) \cap K) > 0 \quad \text{for each } z \in K, \; r > 0,
\]
where \( \lambda_m \) is the \( m \)-dimensional Lebesgue measure.
which does not cause any lack of generality (cf. the observation on p. 27 in [9]).

Geometric conditions, which enable us to extend $W$ to a bounded linear operator from $C(B) \supset C^{(1)}(B)$ to $C(K)$ (equipped with the sup-norm), can be conveniently described in terms of the so-called essential boundary $\partial_e K \equiv B_e$ defined by

$$B_e := \left\{ x \in \mathbb{R}^m; \limsup_{r \downarrow 0} \lambda_m(B_r(x) \cap K) r^{-m} > 0, \limsup_{r \downarrow 0} \lambda_m(B_r(x) \cap G) r^{-m} > 0 \right\}$$

(cf. [4]). For any $z \in \mathbb{R}^m$ and $\theta \in \partial B_1(0)$ consider the half-line

$$H_z(\theta) := \{ z + t\theta; \ t > 0 \}$$

and denote by $n(z, \theta)$ ($0 \leq n(z, \theta) \leq +\infty$) the total number of points in $H_z(\theta) \cap B_e$.

It appears that, for fixed $z \in \mathbb{R}^m$, the function

$$\theta \mapsto n(z, \theta)$$

is $\lambda_{m-1}$-measurable on $\partial B_1(0)$ so that we may introduce the integral

$$v(z) := \frac{1}{\sigma_m} \int_{\partial B_1(0)} n(z, \theta) \, d\lambda_{m-1}(\theta)$$

(compare §2 in [9], Lemma 3 in [11] and [4]). With this notation

$$(2) \quad \sup_{z \in B} v(z) < +\infty$$

is a necessary and sufficient condition guaranteeing that for any uniformly convergent (on $B$) sequence $\varphi_n \in C^{(1)}(B)$, the corresponding sequence $W \varphi_n \in C(K)$ is uniformly convergent on $K$ (which is equivalent to continuous extendability of $W$, defined so far only on $C^{(1)}(B)$, to a bounded linear operator acting from $C(B) \supset C^{(1)}(B)$ to $C(K)$, where $C(B)$ and $C(K)$ are equipped with the usual maximum norm). In what follows we always assume (2), which implies that

$$\sup_{z \in \mathbb{R}^m} v(z) < +\infty$$

(cf. Theorem 2.16 in [9]) and guarantees the existence of a well-defined density

$$d_K(z) := \lim_{r \downarrow 0} \frac{\lambda_m(B_r(z) \cap K)}{\lambda_m(B_r(z))}$$
for any $z \in \mathbb{R}^m$ (cf. Lemma 2.1 in [9]). For any $f \in \mathcal{C}(B)$ the corresponding $Wf \in \mathcal{C}(K)$ is harmonic in $K^\circ$ and admits an integral representation reminding one of the classical double layer potential with momentum density $f$. For this purpose let us recall that a unit vector $n \in \partial B_1(0)$ is termed the exterior normal of $K$ at $y \in \mathbb{R}^m$ in the sense of Federer provided

$$
\lim_{r \searrow 0} r^{-m} \lambda_m(\{x \in B_r(y) \cap K; (x-y) \cdot n > 0\}) = 0, \\
\lim_{r \searrow 0} r^{-m} \lambda_m(\{x \in B_r(y) \cap G; (x-y) \cdot n < 0\}) = 0.
$$

For any fixed $y \in \mathbb{R}^m$ there exists at most one vector $n \in \partial B_1(0)$ with the property (3) and it will be denoted by $n^K(y) \equiv n$ provided it is available; if there is no such $n \in \partial B_1(0)$ with (3), then we put $n^K(y) = 0 \ (\in \mathbb{R}^m)$. The vector-valued function $y \mapsto n^K(y)$ is Borel measurable and

$$
\hat{B} \equiv \partial \hat{K} \equiv \{y \in \mathbb{R}^m; |n^K(y)| > 0\}
$$

is a Borel set which is termed the reduced boundary of $K$ (cf. [6]). Clearly,

$$
\hat{B} \subset \{y \in \mathbb{R}^m; d_K(y) = \frac{1}{2}\} \subset B_e
$$

and under our assumption (2) we have

$$
\lambda_{m-1}(B_e) < +\infty
$$

and

$$
\lambda_{m-1}(B_e \setminus \hat{B}) = 0
$$

(cf. Section 4.5 in [5], 5.6 in [17] and 2.12 in [9]). If $f \in \mathcal{C}(B)$, then $Wf$ can be represented by

$$
Wf(z) = \begin{cases}
  d_G(z)f(z) + \int_B f(y)n^K(y) \cdot \text{grad } h_z(y) \, d\lambda_{m-1}(y) & \text{for } z \in B \\
  \int_{\hat{B}} f(y)n^K(y) \cdot \text{grad } h_z(y) \, d\lambda_{m-1}(y) & \text{for } z \in K^\circ
\end{cases}
$$

where, of course, $d_G(z) = 1 - d_K(z)$ is the density of $G = \mathbb{R}^m \setminus K$ at $z$ (cf. [9], Proposition 2.8 and Lemmas 2.9, 2.15).

For $\alpha \in \mathbb{R}$ we denote by $W^\alpha$ the operator on $\mathcal{C}(B)$ sending $f \in \mathcal{C}(B)$ to $W^\alpha f \in \mathcal{C}(B)$ attaining the value $W^\alpha f(y) = Wf(y) - \alpha f(y)$ at any $y \in B$. Given a boundary condition $g \in \mathcal{C}(B)$ then an attempt to solve the corresponding Dirichlet problem for
$K^\circ$ (at least in the case $B \subset \overline{K^\circ}$) in the form of a $Wf$ with an unknown $f \in \mathcal{C}(B)$ leads to the equation

$$ (\alpha I + W^\alpha)f = g, $$

where $I$ denotes the identity operator on $\mathcal{C}(B)$.

The space $\mathcal{C}'(B)$ dual to $\mathcal{C}(B)$ can be identified with the space of all finite signed Borel measures with support contained in $B$. For any $\nu \in \mathcal{C}'(B)$ the potential

$$ U\nu(y) = \int_B h_y(x) \, d\nu(x), \quad y \in G $$

represents a harmonic function in $G$ whose weak normal derivative can be properly interpreted (cf. §1 in [9], [15]). Given a $\mu \in \mathcal{C}'(B)$ then an attempt to solve the corresponding Neumann problem for $G$ (with the Neumann boundary condition given by $\mu$) in the form of a potential (5) with an unknown $\nu \in \mathcal{C}'(B)$ leads to the equation

$$ (\alpha I + W^\alpha)'\nu = \mu $$

which is dual to (4).

Let us agree to denote by $\mathcal{G}$ the space of all compact linear operators acting on $\mathcal{C}(B)$. If $p$ is a norm on $\mathcal{C}(B)$ and $T$ is a bounded linear operator acting on $\mathcal{C}(B)$ then its norm $p(T)$ is defined in the usual way and the $p$-essential norm $\omega_p T$ is given by

$$ \omega_p T = \inf \{ p(T - Q); \, Q \in \mathcal{G} \}. $$

In connection with the applicability of the Fredholm-Radon theory to the pair of dual equations (4), (6) it is important to have estimates of the essential spectral radius of the operator $W^\alpha$. According to the theorem of Gohberg and Markus (cf. [7]), this radius coincides with

$$ \inf_p \omega_p W^\alpha, $$

where $p$ ranges over all equivalent norms on $\mathcal{C}(B)$ inducing the topology of uniform convergence in $\mathcal{C}(B)$. Let us recall that simple examples are known showing that for the usual maximum norm $p_1$, where $p_1(f) = \sup\{|f(y)|; y \in B\}$, $f \in \mathcal{C}(B)$, it may occur that

$$ \omega_{p_1} W^\alpha > |\alpha| \quad \text{for all } \alpha \neq 0, $$

while

$$ \omega_p W^{\frac{1}{2}} < \frac{1}{2} $$

for a suitable norm $p$ on $\mathcal{C}(B)$ topologically equivalent to $p_1$ (cf. [13], [1]; note that $2W^{\frac{1}{2}}$ is the so-called Neumann operator of the arithmetical mean as mentioned on
Accordingly, it is useful to investigate estimates of $\omega_p W^\alpha$ for general norms $p$ topologically equivalent to $p_1$, which is the subject of the present paper. Given such a norm $p$ on $C(B)$ inducing the topology of uniform convergence in $C(B)$ we put

\begin{equation}
\overline{p}(y) = \sup\{g(y); \ g \in C(B), \ p(g) \leq 1\}
\end{equation}

for $y \in B$. The function

$\overline{p}: y \mapsto \overline{p}(y)$

defined by (7) is lower-semicontinuous on $B$.

Given a bounded non-negative lower-semicontinuous function $\psi$ on $B$ we put for $z \in \mathbb{R}^m$, $r > 0$ and $\theta \in \partial B_1(0)$

\begin{equation}
\nu^\psi_r(z, \theta) = \sum_{\xi} \psi(\xi), \ \xi \in H_z(\theta) \cap B_\epsilon \cap B_r(z),
\end{equation}

the sum on the right-hand side of (8) counting, with the weight $\psi(\xi)$, all points $\xi$ in $B_\epsilon \cap \{z + \varrho \theta; 0 < \varrho < r\}$ ($0 \leq n^\psi_r(z, \theta) \leq +\infty$). We shall see that, for fixed $z \in \mathbb{R}^m$ and $r > 0$, the function $\theta \mapsto n^\psi_r(z, \theta)$ is $\lambda_{m-1}$-measurable on $\partial B_1(0)$, which justifies the definition

\begin{equation}
v^\psi_r(z) = \frac{1}{\sigma_m} \int_{\partial B_1(0)} n^\psi_r(z, \theta) \, d\lambda_{m-1}(\theta), \ z \in \mathbb{R}^m, \ 0 < r \leq \infty.
\end{equation}

(Observe that this quantity reduces to $\nu(z)$ in the case $r = \infty$ and $\psi \equiv 1$.) We are going to establish upper and lower estimates of $\omega_p W^\alpha$ with help of the functions

$y \mapsto v^\overline{p}_r(y), \ y \in B$.

In particular, for suitable weighted norms $p$ on $C(B)$ these estimates permit to prove the equality

$\omega_p W^\alpha = |\frac{1}{2} - \alpha| + \inf_{r > 0} \sup_{y \in B} \frac{v^\overline{p}_r(y)}{\overline{p}(y)}$,

extending Theorem 4.1 in [9].

**1. Lemma.** Let $p$ be a norm on $C(B)$ inducing the topology of uniform convergence and define the function $\overline{p}: B \to \mathbb{R}$ by (7). Then $\overline{p}$ is lower-semicontinuous on $B$ and there are constants $0 < k_p \leq K_p < \infty$ such that

\begin{equation}
k_p \leq \overline{p} \leq K_p
\end{equation}

on $B$. 674
The definition (7) shows that \( p \) is a (pointwise) supremum of a class of continuous functions on \( B \); hence \( p \) is lower-semicontinuous in \( B \). Since the identity operator acting from \( C(B) \) normed by \( p \) to \( C(B) \) normed by the maximum norm \( p_1 \) is bounded, there is a \( K_p \in (0, \infty) \) such that \( p \leq K_p \) on \( B \). Since also the identity operator acting inversely from \( (C(B), p_1) \) into \( (C(B), p) \) is bounded, there is a \( c \in (0, +\infty) \) such that the implication

\[
(g \in C(B), \ |g| \leq 1) \implies p\left(\frac{g}{c}\right) \leq 1
\]

is valid. This together with the definition of \( p \) shows that

\[
p(y) \geq \frac{1}{c}
\]

for any \( y \in B \), so that (10) holds with \( k_p = \frac{1}{c} \).

\[\Box\]

2. **Remark.** As a consequence of our assumption (1) we have

\[
\lambda_{m-1}(B_r(y) \cap \hat{B}) > 0, \quad \forall y \in B, \ \forall r > 0.
\]

This follows from the relative isoperimetric inequality concerning sets of locally finite perimeter (cf. Section 4.5 in [5] and p. 50 in [9]).

3. **Lemma.** If \( \psi \) is a non-negative \( \lambda_{m-1} \)-measurable function defined \( \lambda_{m-1} \)-a.e. on \( \hat{B} \) we denote by

\[
\hat{\psi}(y) := \lambda_{m-1}\text{-ess lim inf}_{x \to y, x \in \hat{B}} \psi(x)
\]

the \( \lambda_{m-1} \)-essential lower limit of \( \psi \) at \( y \in B \) which is defined as the least upper bound of all \( \gamma \in \mathbb{R} \) for which there is an \( r > 0 \) such that

\[
\lambda_{m-1}(\{x \in B_r(y) \cap \hat{B}; \ \psi(x) < \gamma\}) = 0.
\]

Then the function \( \hat{\psi}: y \mapsto \hat{\psi}(y) \) is lower-semicontinuous on \( B \) and

\[
\lambda_{m-1}(\{y \in \hat{B}; \ \psi(y) < \hat{\psi}(y)\}) = 0.
\]

**Proof.** For the sake of completeness we include the following argument occurring in [12] in connection with Lemma 8. Consider an arbitrary \( y \in B \) and \( c < \hat{\psi}(y) \). Then there are \( \gamma \in (c, \hat{\psi}(y)] \) and \( r > 0 \) such that (11) holds. If \( z \in B \cap B_{r/2}(y) \) then \( B_{r/2}(z) \subset B_r(y) \) and, consequently,

\[
\lambda_{m-1}(\{x \in B_{r/2}(z) \cap \hat{B}; \ \gamma(x) < \gamma\}) = 0,
\]
which shows that \( \hat{\psi}(z) \geq \gamma > c \). We have thus shown that, given \( c < \hat{\psi}(y) \), the inequality \( c < \psi(z) \) holds for all \( z \in B \) sufficiently close to \( y \) and the lower-semicontinuity of \( \hat{\psi} \) at \( y \) is established. Admitting

\[
\lambda_{m-1}(\{y \in \hat{B}; \psi(y) < \hat{\psi}(y)\}) > 0
\]

we get, by Lusin’s theorem, that there is a compact set \( C \subset \{y \in \hat{B}; \psi(y) < \hat{\psi}(y)\} \) with \( \lambda_{m-1}(C) > 0 \) such that the restriction \( \psi|_C \) is continuous. There is a \( z \in C \) such that

\[
\lambda_{m-1}(B_{\varrho}(z) \cap C) > 0, \quad \forall \varrho > 0.
\]

Since \( \psi(z) < \hat{\psi}(z) \), there are \( \gamma \in (\psi(z), \hat{\psi}(z)] \) and \( r > 0 \) such that

\[
\lambda_{m-1}(\{y \in B_{r}(z) \cap \hat{B}; \psi(y) < \gamma\}) = 0.
\]

Continuity of \( \psi|_C \) guarantees the validity of the implication

\[
y \in B_{\varrho}(z) \cap C \implies \psi(y) < \gamma
\]

for sufficiently small \( \varrho \in (0, r) \) which, in view of the inclusion \( B_{\varrho}(z) \cap C \subset B_{r}(z) \cap \hat{B} \), together with (12) contradicts (13). This completes the proof. \( \square \)

4. Lemma. If \( \psi \geq 0 \) is a lower-semicontinuous function on \( B \), then \( \hat{\psi} \) (defined as in Lemma 3) satisfies \( \hat{\psi} \geq \psi \) on \( B \); moreover, \( \hat{\psi} \) is the greatest lower-semicontinuous majorant of \( \psi \) on \( B \) coinciding with \( \psi \) almost everywhere \((\lambda_{m-1})\) on \( \hat{B} \).

Proof. Let \( \tilde{\psi} \) be a lower-semicontinuous majorant of \( \psi \) coinciding with \( \psi \) almost everywhere \((\lambda_{m-1})\) on \( \hat{B} \). We are going to verify that \( \hat{\psi} \geq \tilde{\psi} \) on \( B \). Admit that there is a \( y \in B \) with \( \hat{\psi}(y) < \tilde{\psi}(y) \) and fix a \( c \in \mathbb{R} \) such that

\[
\hat{\psi}(y) < c < \tilde{\psi}(y).
\]

Since \( \tilde{\psi} \) is lower-semicontinuous, we have

\[
z \in B_{r}(y) \cap B \implies \tilde{\psi}(z) > c
\]

for sufficiently small \( r > 0 \), whence

\[
\lambda_{m-1}(\{z \in B_{r}(y) \cap \tilde{B}; \psi(z) \leq c\}) = 0,
\]

because \( \psi = \tilde{\psi} \) almost everywhere \((\lambda_{m-1})\) on \( \tilde{B} \). We conclude that \( \hat{\psi}(y) \geq c \), which contradicts (14). Letting \( \tilde{\psi} = \psi \) we get from Lemma 3 that \( \hat{\psi} = \psi \) almost everywhere \((\lambda_{m-1})\) on \( \hat{B} \) and the proof is complete. \( \square \)
5. Lemma. Let $\mathcal{C}^+(B)$ denote the class of all non-negative functions in $\mathcal{C}(B)$ and let $\mathcal{C}^\uparrow_+(B)$ denote the class of all non-negative lower-semicontinuous functions on $B$. Let $f \in \mathcal{C}^+(B)$, $\psi \in \mathcal{C}^\uparrow_+(B)$ and put $\varphi = f + \psi$. Then $\hat{\varphi} = f + \hat{\psi}$. In particular, $\hat{f} = f$ for each $f \in \mathcal{C}^+(B)$.

Proof. Observe that $f + \hat{\psi}$ is a lower-semicontinuous majorant of $\varphi$ on $B$ such that $f + \hat{\psi} = \varphi$ holds $\lambda_{m-1}$-a.e. in $\hat{B}$. By Lemma 4 we get $\hat{\varphi} \geq f + \hat{\psi}$. We see that $\hat{\varphi} - f \in \mathcal{C}^\uparrow_+(B)$ is a majorant of $\psi$ on $B$ coinciding with $\psi$ almost everywhere ($\lambda_{m-1}$) on $\hat{B}$. Using Lemma 4 again we arrive at the inequality $\hat{\varphi} - f \leq \hat{\psi}$, so that $\hat{\varphi} = f + \hat{\psi}$. Taking $\psi \equiv 0$ we get $\hat{f} = f$, $\forall f \in \mathcal{C}^+(B)$.

6. Lemma. Let $p$ be a norm on $\mathcal{C}(B)$ inducing the topology of uniform convergence in $\mathcal{C}(B)$ such that the implication

$$ |f| \leq |g| \implies p(f) \leq p(g) $$

holds for any $f, g \in \mathcal{C}(B)$. Then we have

$$ p(h) = \sup\{p(f); f \in \mathcal{C}(B), |f| \leq h\} $$

whenever $h \in \mathcal{C}^+(B)$, and (16) can be used to define $p(h)$ for any $h \in \mathcal{C}^\uparrow_+(B)$. Having extended $p$ from $\mathcal{C}^+(B)$ to $\mathcal{C}^\uparrow_+(B)$ in this way we get for any $\alpha \in [0, +\infty)$ and $\psi_j \in \mathcal{C}^\uparrow_+(B)$ ($j = 0, 1, 2$)

$$ p(\alpha \psi_0) = \alpha p(\psi_0), $$

$$ p(\psi_1 + \psi_2) \leq p(\psi_1) + p(\psi_2). $$

Proof. The implication (15) $\Rightarrow$ (16) is evident and if (15) is used to define $p(h)$ for any $h \in \mathcal{C}^\uparrow_+(B)$ then (17) obviously holds for $\alpha \in [0, +\infty)$ and $\psi_0 \in \mathcal{C}^\uparrow_+(B)$. It is easy to verify (18) assuming first that $\psi_1, \psi_2 \in \mathcal{C}^\uparrow_+(B)$ satisfy

$$ \psi_1 + \psi_2 > 0 \quad \text{on } B. $$

We then have

$$ p(\psi_1 + \psi_2) = \sup\{p(f); f \in \mathcal{C}(B), |f(y)| < \psi_1(y) + \psi_2(y), \forall y \in B\}. $$

Choose non-decreasing sequences $\{g^n_j\}_{n=1}^\infty$ in $\mathcal{C}^+(B)$ such that $g^n_j \nearrow \psi_j$ as $n \to \infty$ ($j = 1, 2$). Fix $f \in \mathcal{C}(B)$ such that $|f| < \psi_1 + \psi_2$. If the compact sets
$K_n = \{ x \in B; |f(x)| \geq g^n_1(x) + g^n_2(x) \}$ are nonempty then there is an $x \in \bigcap K_n$ and therefore $\psi_1(x) + \psi_2(x) \leq |f(x)|$, which is a contradiction. So, we have

$$|f| < g^n_1 + g^n_2$$

for all sufficiently large $n \in \mathbb{N}$. Defining for such $n$

$$f_j = f \frac{g^n_j}{g^n_1 + g^n_2} \quad (j = 1, 2)$$

we get

$$|f_j| \leq |f| \frac{g^n_j}{g^n_1 + g^n_2} < g^n_j \quad (j = 1, 2), \quad f_1 + f_2 = f,$$

whence

$$p(f) \leq p(f_1) + p(f_2) \leq p(\psi_1) + p(\psi_2).$$

Since $f \in \mathcal{C}(B)$ with $|f| < \psi_1 + \psi_2$ has been chosen arbitrarily, we get (18). It remains to observe that the additional assumption (19) can be omitted. Denote by $1_B \in \mathcal{C}(B)$ the constant function attaining the value 1 at any point in $B$. For any $\psi \in \mathcal{C}_+^1(B)$ and $\varepsilon > 0$ we then have

$$p(\psi) \leq p(\psi + \varepsilon 1_B) \leq p(\psi) + \varepsilon p(1_B),$$

so that

$$p(\psi + \varepsilon 1_B) \rightarrow p(\psi) \quad \text{as} \quad \varepsilon \downarrow 0.$$

Consequently, for any $\psi_j \in \mathcal{C}_+^1(B) \ (j = 1, 2)$ we get

$$p(\psi_1 + \psi_2) \leq p(\psi_1) + p(\psi_2 + \varepsilon 1_B) \rightarrow p(\psi_1) + p(\psi_2) \quad \text{as} \quad \varepsilon \downarrow 0$$

and (18) follows. \hfill \Box

7. Lemma. Let $\psi \geq 0$ be a bounded lower-semicontinuous function on $B$ and define for fixed $z \in \mathbb{R}^m$ and $r \in (0, \infty]$ the function $n^\psi_r(z, \theta)$ of the variable $\theta \in \partial B_1(0)$ by (8). This function is $\lambda_{m-1}$-integrable in $\partial B_1(0)$ and

$$\int_{\partial B_1(0)} n^\psi_r(z, \theta) \, d\lambda_{m-1}(\theta) = \int_{B \cap B_r(z)} \psi(x) |n^K(x) \cdot \text{grad} h_z(x)| \, d\lambda_{m-1}(x).$$

The function $v^\psi_r : z \mapsto v^\psi_r(z)$ defined by (9) is bounded and lower-semicontinuous on $\mathbb{R}^m$.

Proof. This is a consequence of Lemma 3 in [12]. \hfill \Box

678
8. Lemma. If
\[(x, y) \mapsto g_y(x)\]
is a continuous (real-valued) function on \(B \times B\) then, for each \(f \in C(B)\),
\[W(fg_y)(y) := f(y)g_y(y)d_G(y) + \int_B f(x)g_y(x)n^K(x) \cdot \text{grad} \ h_y(x) \, d\lambda_{m-1}(x)\]
represents a continuous function of the variable \(y \in B\).

Proof. As mentioned above, our assumption (2) guarantees that the operator \(W\) sending each \(f \in C(B)\) to
\[Wf : y \mapsto f(y)d_G(y) + \int_B f(x)n^K(x) \cdot \text{grad} \ h_y(x) \, d\lambda_{m-1}(x), \quad y \in B\]
is continuous on \(C(B)\) with respect to the topology of the uniform convergence (cf. Proposition 2.8 and Lemmas 2.9, 2.15 in [9]). Let now \(\{y_n\}_{n=1}^{\infty}\) be an arbitrary convergent sequence of points in \(B\), \(\lim_{n \to \infty} y_n = y_0\). Then, for each \(f \in C(B)\), the sequence of functions \(\{fg_{y_n}\}_{n=1}^{\infty}\) converges uniformly on \(B\) to \(fg_{y_0} \in C(B)\) and \(\{W(fg_{y_n})\}_{n=1}^{\infty}\) converges uniformly on \(B\) to \(W(fg_{y_0})\) as \(n \to \infty\), whence
\[\lim_{n \to \infty} W(fg_{y_n})(y_n) = W(fg_{y_0})(y_0)\]
and the continuity of \(y \mapsto W(fg_y)(y)\) is established. □

9. Lemma. Let \(\psi \geq 0\) be a bounded lower-semicontinuous function on \(B\) and let
\[(x, y) \mapsto g_y(x)\]
be a continuous function on \(B \times B\) such that \(0 \leq g_y(x) \leq 1\). Then
\[F^\psi_g(y) := \psi(y)g_y(y) \left| d_G(y) - \frac{1}{2} \right| + \int_B \psi(x)g_y(x)|n^K(x) \cdot \text{grad} \ h_y(x)| \, d\lambda_{m-1}(x)\]
is a lower-semicontinuous function of the variable \(y\) on \(B\).

Proof. It follows from Lemma 8 that
\[H^f_g(y) := (W - \frac{1}{2} I)(fg_y)(y) = f(y)g_y(y)[d_G(y) - \frac{1}{2}] + \int_B f(x)g_y(x)n^K(x) \cdot \text{grad} \ h_y(x) \, d\lambda_{m-1}(x)\]
is a continuous function of the variable $y$ on $B$ for each $f \in C(B)$. It is therefore sufficient to verify that $F_{\psi}^g$ is the (pointwise) supremum of the class

$$\mathcal{F} := \{ H_f^g ; f \in C(B), \ |f| \leq \psi \} \subset C(B).$$

Clearly, any function in $\mathcal{F}$ is majorized by $F_{\psi}^g$. Fix now an arbitrary $\xi \in B$ and $\varepsilon > 0$. Since

$$\sup \left\{ \int_B f(x)g_\xi(x)nK(x) \cdot \text{grad} h_\xi(x) \, d\lambda_{m-1}(x) ; \ f \in C(B), \ |f| \leq \psi, \ \text{spt} \ f \subset B \setminus \{ \xi \} \right\}$$

there is an $f_0 \in C(B)$ such that $|f_0| \leq \psi$, $f_0 = 0$ on $B_\varrho(\xi) \cap B$ for sufficiently small $\varrho > 0$ and

$$\int_B f_0(x)g_\xi(x)nK(x) \cdot \text{grad} h_\xi(x) \, d\lambda_{m-1}(x) > \int_B \psi(x)g_\xi(x)|nK(x) \cdot \text{grad} h_\xi(x)| \, d\lambda_{m-1}(x) - \varepsilon. \quad (21)$$

Since

$$\int_B \psi(x)g_\xi(x)|nK(x) \cdot \text{grad} h_\xi(x)| \, d\lambda_{m-1}(x) \leq v^\psi(\xi) < \infty,$$

we can assume that $\varrho > 0$ has been chosen small enough to have

$$\int_{B \cap B_\varrho(\xi)} \psi(x)g_\xi(x)|nK(x) \cdot \text{grad} h_\xi(x)| \, d\lambda_{m-1}(x) < \varepsilon. \quad (22)$$

Consider first the case when

$$\psi(\xi)g_\xi(\xi)|dG(\xi) - \frac{1}{2}| > 0.$$

Clearly, we can assume that $0 < \varepsilon < \psi(\xi)$. Choose $f_1 \in C(B)$ with spt $f_1 \subset B_\varrho(\xi) \cap B$ such that $|f_1| \leq \psi$ and

$$|f_1(\xi)| > \psi(\xi) - \varepsilon, \ \ \text{sign} \ f_1(\xi) = \text{sign}[dG(\xi) - \frac{1}{2}].$$
Letting \( f = f_0 + f_1 \) we have \( |f| \leq \psi \),

\[
H^f_g(\xi) = f_1(\xi)g_\xi(\xi)[d_G(\xi) - \frac{1}{2}] + \int_{B_\varepsilon(\xi) \cap B} f_1(x)g_\xi(x)n^K(x) \cdot \text{grad} \, h_\xi(x) \, d\lambda_{m-1}(x)
\]

\[
+ \int_{B \setminus B_\varepsilon(\xi)} f_0(x)g_\xi(x)n^K(x) \cdot \text{grad} \, h_\xi(x) \, d\lambda_{m-1}(x)
\]

\[
\geq \psi(\xi)g_\xi(\xi)|d_G(\xi) - \frac{1}{2}| - \varepsilon
\]

\[- \int_{B \cap B_\varepsilon(\xi)} \psi g_\xi |n^K \cdot \text{grad} \, h_\xi| \, d\lambda_{m-1} + \int_B \psi g_\xi |n^K \cdot \text{grad} \, h_\xi| \, d\lambda_{m-1} - \varepsilon
\]

\[
> \psi(\xi)g_\xi(\xi)|d_G(\xi) - \frac{1}{2}| + \int_B \psi g_\xi |n^K \cdot \text{grad} \, h_\xi| \, d\lambda_{m-1} - 3\varepsilon
\]

by (21), (22). The inequality

\[
H^f_g(\xi) > F^\psi_g(\xi) - 3\varepsilon
\]

with arbitrarily small \( \varepsilon > 0 \) shows that

\[
(23) \quad F^\psi_g(\xi) = \sup\{h(\xi); \, h \in \mathcal{F}\}.
\]

If

\[
\psi(\xi)g_\xi(\xi)|d_G(\xi) - \frac{1}{2}| = 0,
\]

then (21) yields

\[
H^{f_0}_g(\xi) > F^\psi_g(\xi) - \varepsilon
\]

and (23) holds again. Since \( \xi \in B \) was arbitrary, the proof is complete. \( \square \)

10. **Corollary.** Let \( \psi \geq 0 \) be a bounded lower-semicontinuous function on \( B \), \( r \in (0, \infty) \) and define

\[
V^\psi_r(y) = \psi(y)[d_G(y) - \frac{1}{2}] + v^\psi_r(y), \quad y \in B.
\]

Then \( V^\psi_r : y \mapsto V^\psi_r(y) \) is lower-semicontinuous on \( B \).

**Proof.** Let \( h^n \geq 0 \) be a nondecreasing sequence of continuous functions on \([0, \infty)\) such that

\[
\lim_{n \to \infty} h^n(t) = \begin{cases} 1 & \text{for } t \in [0, r), \\ 0 & \text{elsewhere on } [0, \infty) \end{cases}
\]

681
and put
\[ g^n(y) = h^n(|x - y|), \quad x, y \in B. \]

Then
\[ F_{g^n}(y) \not\nearrow \psi(y)|dG(y) - \frac{1}{2} | \int_{B \cap B_r(y)} \psi(x)|nK(x) \cdot \text{grad} h(x)| \text{d} \lambda_{m-1}(x) = V_{r\psi}(y) \]
as \( n \to \infty \). Since the functions \( F_{g^n} \) are all lower-semicontinuous on \( B \), the same holds of \( V_{r\psi} \). \( \square \)

11. Definition. Let \( p \) be a norm on \( C(B) \) with the property (15), inducing the topology of uniform convergence; extend \( p \) to \( C^\uparrow_+(B) \) by (16) and for any \( h \in C^\uparrow_+(B) \) put
\[ \tilde{p}(h) = p(\tilde{h}), \quad h \in C^\uparrow_+(B), \]
where \( \tilde{h} \) is defined by Lemma 3. Combining this definition with Lemmas 5 and 6 we arrive at

12. Remark. If \( \varphi = f + \psi \), where \( f \in C_+(B) \) and \( \psi \in C^\uparrow_+(B) \), then \( \tilde{p}(\varphi) \leq p(f) + \tilde{p}(\psi) \). In particular, \( \tilde{p}(f) = p(f) \) whenever \( f \in C_+(B) \).

13. Theorem. Let \( p \) be a norm on \( C(B) \) with (15) inducing the topology of uniform convergence, define \( \overline{p}: y \mapsto \overline{p}(y) \) by (7) and for \( r \in (0, \infty) \) put
\[ v^\overline{p}_r: y \mapsto v^\overline{p}_r(y), \quad y \in B, \]
\[ V^\overline{p}_r: y \mapsto \overline{p}(y)|\frac{1}{2} - dG(y)| + v^\overline{p}_r(y), \quad y \in B. \]

Then for each \( \alpha \in \mathbb{R} \)
\[ \omega_p(W^\alpha) \leq |\alpha - \frac{1}{2}| + \inf_{r > 0} \tilde{p}(v^\overline{p}_r) = |\alpha - \frac{1}{2}| + \inf_{r > 0} \tilde{p}(V^\overline{p}_r). \]

Proof. Fix \( r > 0 \) and construct a function \( g^r \) on \( \mathbb{R}^m \) satisfying the Lipschitz condition
\[ x^1, x^2 \in \mathbb{R}^m \implies |g^r(x^1) - g^r(x^2)| \leq \frac{1}{r}|x^1 - x^2| \]
and such that
\[ 0 \leq g^r \leq 1, \quad g^r(B_r(0)) = \{1\}, \quad g^r(\mathbb{R}^m \setminus B_{2r}(0)) = \{0\}. \]

Put
\[ g_y(x) = g^r(x - y), \quad x, y \in \mathbb{R}^m \]
and define an operator $V$ on $\mathcal{C}(B)$ sending each $f \in \mathcal{C}(B)$ to $Vf$ given by

$$Vf(y) = \int_B f(x)[1 - g_y(x)]nK(x) \cdot \text{grad} h_y(x) \, d\lambda_{m-1}(x), \quad y \in B.$$ 

Elementary reasoning (described in detail in the proof of Theorem 4.1 in [9], pp. 104–111) shows that $V$ is a compact linear operator acting in $\mathcal{C}(B)$. We are going to estimate $p(W^\alpha - V)$. Let $f \in \mathcal{C}(B)$, $p(f) \leq 1$. Consequently, $|f| \leq \overline{p}$ on $B$. By Proposition 2.8 and Lemmas 2.9 and 2.15 in [9] we have

$$(W^\alpha - V)f(y) = f(y)[d_G(y) - \alpha] + \int_B f(x)g_y(x)nK(x) \cdot \text{grad} h_y(x) \, d\lambda_{m-1}(x), \quad y \in B.$$ 

Hence

$$|(W^\alpha - V)f(y)| \leq |(\frac{1}{2} - \alpha)f(y)| + \overline{p}(y)|d_G(y) - \frac{1}{2}| + \int_B \overline{p}(x)g^r(x - y)n^K(x) \cdot \text{grad} h_y(x) \, d\lambda_{m-1}(x) = |(\frac{1}{2} - \alpha)f(y)| + F^\overline{p}_g(y),$$

where $F^\overline{p}_g$ is the lower-semicontinuous function on $B$ defined in Lemma 9. Since $p(f) \leq 1$ implies $p(|f|) \leq 1$, in view of Remark 12 we get

$$p[(W^\alpha - V)f] \leq |\frac{1}{2} - \alpha|p(|f|) + \widehat{p}(F^\overline{p}_g) \leq |\frac{1}{2} - \alpha| + \widehat{p}(F^\overline{p}_g).$$

Observe that $F^\overline{p}_g \leq V^\overline{p}_{2r}$, where $V^\overline{p}_{2r}$ is a lower-semicontinuous function on $B$ coinciding with $v^\overline{p}_{2r}$ on $\hat{B}$, so that $\widehat{p}(V^\overline{p}_{2r}) = \widehat{p}(v^\overline{p}_{2r})$. Since $r > 0$ was arbitrary, we arrive at

$$p(W^\alpha - V) \leq |\frac{1}{2} - \alpha| + \widehat{p}(V^\overline{p}_{2r}),$$

$$\omega_p(W^\alpha) \leq |\frac{1}{2} - \alpha| + \inf_{r > 0} \widehat{p}(V^\overline{p}_{2r}) = |\frac{1}{2} - \alpha| + \inf_{r > 0} \widehat{p}(v^\overline{p}_{2r})$$

and (24) is established. 

14. **Corollary.** Let $q > 0$ be a bounded lower-semicontinuous function on $B$ such that

$$(25) \quad q(y) \geq \lambda_{m-1}\text{-ess lim inf}_{x \in B, x \to y} q(x), \quad \forall y \in B.$$ 

For $f \in \mathcal{C}(B)$ define

$$(26) \quad p_q(f) := \sup_{y \in B} \frac{|f(y)|}{q(y)}.$$
Then $p_q$ is a norm on $C(B)$ inducing the topology of uniform convergence and for each $\alpha \in \mathbb{R}$ we have

$$\omega_{p_q} W^\alpha \leq |\alpha - \frac{1}{2}| + \inf_{r>0} \sup_{y \in \hat{B}} \frac{v_q^q(y)}{q(y)}.$$ 

**Proof.** Let $\underline{p}_q$ correspond to $p_q$ in the sense of Lemma 1. It is easy to see from (26) that $\underline{p}_q = q$ on $B$. In view of Theorem 13 it suffices to verify

$$\widehat{p}_q(v_r^q) = \sup_{x \in \hat{B}} \frac{v_r^q(x)}{q(x)}$$

for any $r > 0$. Recalling Definition 11 we get

$$\widehat{p}_q(v_r^q) = \sup \{p_q(f); f \in C(B), |f| \leq \widehat{v}_r^q \} = \sup_{y \in \hat{B}} \frac{\widehat{v}_r^q(y)}{q(y)} \geq \sup_{x \in \hat{B}} \frac{v_r^q(x)}{q(x)}.$$ 

In order to obtain the desired inequality

$$\sup_{y \in \hat{B}} \frac{\widehat{v}_r^q(y)}{q(y)} \leq \sup_{x \in \hat{B}} \frac{v_r^q(x)}{q(x)},$$

consider an arbitrary $y \in B$ with $\widehat{v}_r^q(y) > 0$ and choose $\varepsilon \in (0, \widehat{v}_r^q(y))$. There is a $\varrho > 0$ such that

$$v_r^q(x) \geq \widehat{v}_r^q(y) - \varepsilon \quad \text{for } \lambda_{m-1}\text{-a.e. } x \in B_\varrho(y) \cap \hat{B}.$$ 

Our assumption (25) guarantees that

$$\lambda_{m-1}\left(\{x \in B_\varrho(y) \cap \hat{B}; q(y) + \varepsilon > q(x)\}\right) > 0$$

(for otherwise we would have $\lambda_{m-1}\text{-ess lim inf}_{x \in \hat{B}, x \rightarrow y} q(x) \geq q(y) + \varepsilon > q(y)$). As $\lambda_{m-1}(B_\varrho(y) \cap \hat{B}) > 0$ (cf. Remark 2), there are $x \in B_\varrho(y) \cap \hat{B}$ for which we have, simultaneously,

$$v_r^q(x) \geq \widehat{v}_r^q(y) - \varepsilon, \quad q(x) < q(y) + \varepsilon,$$

so that

$$\frac{\widehat{v}_r^q(y) - \varepsilon}{q(y) + \varepsilon} \leq \frac{v_r^q(x)}{q(x)} \leq \sup_{x \in \hat{B}} \frac{v_r^q(x)}{q(x)}.$$ 

Making $\varepsilon \downarrow 0$ we get (28), which completes the proof. □
15. Remark. Since $q$ is lower-semicontinuous, we have

$$\lambda_{m-1}-\text{ess lim inf}_{x \in \hat{B}, x \to y} q(x) \geq \liminf_{x \in \hat{B}, x \to y} q(x) \geq q(y),$$

which combined with (25) yields

$$q(y) = \lambda_{m-1}-\text{ess lim inf}_{x \in \hat{B}, x \to y} q(x) = \liminf_{x \in \hat{B}, x \to y} q(x), \quad y \in B.$$ 

16. Lemma. Let $p$ be a norm defining the topology of uniform convergence in $C(B)$ and define $p$ by (7). Suppose that $q \geq 0$ is a bounded lower-semicontinuous function on $B$ such that for each $\mu \in C'(B),$

\begin{equation}
\sup \left\{ \int_{B} f \, d\mu; \; f \in C(B), \; p(f) \leq 1 \right\} \geq \int_{B} q \, d|\mu|,
\end{equation}

where $|\mu|$ is the indefinite total variation of $\mu$. Then

\begin{equation}
\omega_{p}W^{\alpha} \geq \inf_{r > 0} \sup_{y \in B} \left[ \frac{1}{2} - \alpha \bar{q}(y) + v^{q}_{r}(y) \right] / \bar{p}(y) \quad \text{for } \alpha \in \mathbb{R}.
\end{equation}

If

\begin{equation}
\bar{p}(y) = \liminf_{x \in \hat{B} \setminus \{ y \}, x \to y} \bar{p}(x) \quad \text{for each } y \in \hat{B},
\end{equation}

then

\begin{equation}
\omega_{p}W^{\alpha} \geq \inf_{r > 0} \sup_{y \in B} \left[ \frac{1}{2} - \alpha \bar{q}(y) + v^{q}_{r}(y) \right] / \bar{p}(y).
\end{equation}

Proof. Fix an $\varepsilon > 0$ and denote by $\langle f, \nu \rangle$ ($\equiv \int_{B} f \, d\nu$) the pairing between $f \in C(B)$ and $\nu \in C'(B)$. As explained in [9], pp.107–108, there are $\varphi_{1}, \ldots, \varphi_{n} \in C(B)$ and $\nu_{1}, \ldots, \nu_{n} \in C'(B)$ such that

$$D := \left\{ y \in B; \; \sum_{k=1}^{n} |\nu_{k}|(y) > 0 \right\}$$

is finite and the finite-dimensional operator $V$ sending $f \in C(B)$ to

$$Vf := \sum_{k=1}^{n} \langle f, \nu_{k} \rangle \varphi_{k}$$
satisfies

\[ p(W^\alpha - V) \leq \omega_p W^\alpha + \varepsilon. \]

For any \( y \in B \) denote by \( \delta_y \in \mathcal{C}'(B) \) the Dirac measure concentrated at \( y \) and by \( \lambda_y \in \mathcal{C}'(B) \) the representing measure of the functional

\[
 f \mapsto W f(y) = \int_B f(x) \, d\lambda_y(x).
\]

According to (20) (33)

\[
 d\lambda_y(x) = d_G(y) \, d\delta_y(x) + n^K(x) \cdot \text{grad} \, h_y(x) \, d\lambda_{m-1}(x).
\]

Observing that

\[
 p(g) \geq \sup_{y \in B} \frac{|g(y)|}{p(y)}, \quad \forall g \in \mathcal{C}(B),
\]

we get

(34) \[ p(W^\alpha - V) = \sup_{p(f) \leq 1, f \in \mathcal{C}(B)} p((W^\alpha - V)f) \]

\[
 \geq \sup_{p(f) \leq 1} \sup_{y \in B \setminus D} \frac{1}{p(y)} \left| \int_B f \, d\left( \lambda_y - \alpha \delta_y - \sum_{k=1}^n \varphi_k(y) \nu_k \right) \right|.
\]

Now we decompose each \( \nu_k \) into a continuous part \( \nu^1_k \) (not charging singletons) and a finite combination of the Dirac measures; we thus have \( \nu_k = \nu^1_k + \nu^2_k \) and

\[
 \nu^1_k(M) = \nu_k(M \setminus D), \nu^2_k(M) = \nu_k(M \cap D)
\]

for each Borel set \( M \). By virtue of (34) we obtain

\[
 \omega_p(W^\alpha) + \varepsilon \geq \sup_{y \in B \setminus D} \frac{1}{p(y)} \sup_{p(f) \leq 1} \left| \int_B f \, d\left( \lambda_y - \alpha \delta_y - \sum_{k=1}^n \varphi_k(y) \nu_k \right) \right|
\]

\[
 \geq \sup_{y \in B \setminus D} \frac{1}{p(y)} \int_B q \, d\left| \lambda_y - \alpha \delta_y - \sum_{k=1}^n \varphi_k(y) \nu_k \right|
\]

\[
 = \sup_{y \in B \setminus D} \frac{1}{p(y)} \left[ \int_B q \, d\left| \lambda_y - \alpha \delta_y - \sum_{k=1}^n \varphi_k(y) \nu^1_k \right| + \int_B q \, d\left| \sum_{k=1}^n \varphi_k(y) \nu^2_k \right| \right]
\]

\[
 \geq \sup_{y \in B \setminus D} \frac{1}{p(y)} \int_{B \cap B_r(y)} q \, d\lambda_y - \alpha \delta_y
\]

\[
 - \sum_{k=1}^n \max_{x \in B} |\varphi_k(x)| \sup_{z \in B} q(z) |\nu^1_k|(B \cap B_r(y)) \]

686
for any $r > 0$. Since $|\nu_k^j|$ does not charge singletons, we have
\[
\lim_{r \downarrow 0} |\nu_k^j|(B_r(y) \cap B) = 0 \quad \text{uniformly with respect to } y \in B.
\]
We can thus choose an $r_0 > 0$ small enough to ensure the validity of the implication
\[
0 < r < r_0 \implies \sum_{k=1}^n \max |\varphi_k|(B) \sup q(B)|\nu_k^j|(B_r(y) \cap B) < \varepsilon, \forall y \in B.
\]
Hence we get
\[
\omega_p(W^\alpha) + 2\varepsilon \geq \sup_{y \in B \setminus D} \frac{1}{p(y)} \int_{B \cap B_r(y)} q d|\lambda_y - \alpha \delta_y| \geq \sup_{y \in B \setminus D} \frac{1}{p(y)} [q(y)|\frac{1}{2} - \alpha| + v_r^q(y)]
\]
for any $r \in (0, r_0)$ by Lemma 3 in [12]. Recall that
\[
H := \{x \in B; \hat{q}(x) \neq q(x)\} \cup D
\]
has vanishing $\lambda_{m-1}$-measure. By Remark 2 we get for each $x \in B$ a sequence $x_n \in \hat{B} \setminus H$ such that
\[
x_n \to x \quad \text{and} \quad \overline{p}(x_n) \to \overline{p}(x) \quad \text{as} \quad n \to \infty.
\]
Noting that the functions $v_r^q = v_r^q$ (cf. Remark 4 in [12]) and $\hat{q}$ are lower-semicontinuous, we obtain
\[
\frac{1}{\overline{p}(x)} [\hat{q}(x)|\frac{1}{2} - \alpha| + v_r^q(x)] \leq \liminf_{n \to \infty} \frac{1}{\overline{p}(x_n)} [q(x_n)|\frac{1}{2} - \alpha| + v_r^q(x_n)] \leq \omega_p W^\alpha + 2\varepsilon.
\]
We have thus shown
\[
\omega_p W^\alpha + 2\varepsilon \geq \sup_{x \in B} \frac{1}{\overline{p}(x)} [\hat{q}(x)|\frac{1}{2} - \alpha| + v_r^q(x)]
\]
for any $r \in (0, r_0)$, which proves (30), because $\varepsilon > 0$ was arbitrary. Assuming (31) and noting that $D$ is finite we get for any $x \in \hat{B}$ a sequence $x_n \in \hat{B} \setminus D$ such that
\[
x_n \to x \quad \text{and} \quad \overline{p}(x_n) \to \overline{p}(x) \quad \text{as} \quad n \to \infty.
\]
Hence
\[
\frac{1}{\overline{p}(x)} [q(x)|\frac{1}{2} - \alpha| + v_r^q(x)] \leq \liminf_{n \to \infty} \frac{1}{\overline{p}(x_n)} [q(x_n)|\frac{1}{2} - \alpha| + v_r^q(x_n)] \leq \omega_p W^\alpha + 2\varepsilon,
\]
so that
\[
\sup_{x \in B} \frac{1}{\overline{p}(x)} [q(x)|\frac{1}{2} - \alpha| + v_r^q(x)] \leq \omega_p W^\alpha + 2\varepsilon
\]
and (32) follows. \qed
17. Lemma. Let \( \mu \) be a finite signed Borel measure with support in \( B \). Let \( q > 0 \) be a bounded lower-semicontinuous function on \( B \) and define the norm \( p_q \) on \( C(B) \) by (26). Then

\[
\sup \left\{ \int_B f \, d\mu; \ f \in C(B), \ p_q(f) \leq 1 \right\} = \int_B q \, d|\mu|.
\]

Proof. If \( f \in C(B) \), then \( p_q(f) \leq 1 \) means that \( |f| \leq q \) on \( B \), so that

\[
\int_B f \, d\mu \leq \int_B q \, d|\mu| \text{ and } \sup \left\{ \int_B f \, d\mu; \ f \in C(B), \ p_q(f) \leq 1 \right\} \leq \int_B q \, d|\mu|.
\]

In order to prove the converse inequality we fix an arbitrary \( \varepsilon > 0 \) and consider a nondecreasing sequence \( f_n \in C_+(B) \) such that \( f_n \nearrow q \) as \( n \to \infty \). Since

\[
\lim_{n \to \infty} \int_B f_n \, d|\mu| = \int_B q \, d|\mu|
\]

we can fix \( n \in N \) large enough to have

\[
(35) \quad \int_B f_n \, d|\mu| > \int_B q \, d|\mu| - \varepsilon.
\]

Consider the Hahn decomposition (cf. [14])

\[
B = B_+ \cup B_-
\]

corresponding to the signed measure \( \mu \) formed by disjoint Borel sets \( B_+ \), \( B_- \) such that

\[
\mu(B_+ \cap M) = |\mu|(B_+ \cap M), \mu(B_- \cap M) = -|\mu|(B_- \cap M)
\]

for each Borel set \( M \). Choose compact sets \( Q_+ \subset B_+ \) and \( Q_- \subset B_- \) such that

\[
(36) \quad \int_S q \, d|\mu| < \varepsilon,
\]

where \( S = (B_+ \setminus Q_+) \cup (B_- \setminus Q_-) \). Construct a \( \varphi \in C(B) \) satisfying the conditions

\[
\varphi(Q_+) = \{1\}, \ \varphi(Q_-) = \{-1\}, \ |\varphi| \leq 1
\]

and put \( f = \varphi f_n \), so that

\[
f \in C(B), \ p_q(f) \leq 1.
\]
We then have
\[ \int_B f \, d\mu = \int_{Q_+} f_n \, d|\mu| + \int_{Q_-} f_n \, d|\mu| + \int_S \varphi f_n \, d\mu = \int_B f_n \, d|\mu| - \int_S f_n \, d|\mu| + \int_S \varphi f_n \, d\mu. \]

Noting that
\[ \left| \int_S f_n \, d|\mu| \right| \leq \int_S q \, d|\mu| \]
and
\[ \left| \int_S \varphi f_n \, d\mu \right| \leq \int_S q \, d\mu \]
we conclude from (36), (35) that
\[ \int_B f \, d\mu > \int_B q \, d|\mu| - 3\varepsilon. \]

Since \( \varepsilon > 0 \) was arbitrary, we arrive at
\[ \sup \left\{ \int_B f \, d\mu; \ f \in \mathcal{C}(B), \ p_q(f) \leq 1 \right\} \geq \int_B q \, d|\mu|, \]
which completes the proof. \( \square \)

18. **Theorem.** Let \( q > 0 \) be a bounded lower-semicontinuous function on \( B \) satisfying (25) and define the norm \( p_q \) on \( \mathcal{C}(B) \) by (26). Then \( p_q \) induces the topology of uniform convergence in \( \mathcal{C}(B) \) and, for each \( \alpha \in \mathbb{R} \),
\[
\omega_{p_q} W^\alpha = |\alpha - \frac{1}{2}| + \inf_{r > 0} \sup_{y \in B} \frac{v_q^\alpha(y)}{q(y)}.
\]

**Proof.** This follows from Corollary 14 and Lemma 16 combined with (27) together with Lemma 17. \( \square \)

19. **Remark.** Theorem 18 shows that, for the norm \( p_q \) defined on \( \mathcal{C}(B) \) by (26), the optimal choice of the parameter \( \alpha \) in the equation (4) is \( \alpha = \frac{1}{2} \) (compare also 4.2 in [9]), which leads to the Neumann operator \( T = 2W^{1/2} \). Simple examples of domains “built of bricks” in \( \mathbb{R}^3 \) demonstrate that \( \omega_{p_1} T > 1 \) may occur for the maximum norm \( p_1 \) while, as shown in [1], [13], for such domains an elementary construction of another norm \( p \) topologically equivalent to \( p_1 \) such that \( \omega_p T < 1 \) is always possible.
References


Authors’ addresses: Josef Král, Mathematical Institute, Czech Academy of Sciences, Žitná 25, 115 67 Praha 1, Czech Republic, Dagmar Medková, Mathematical Institute, Czech Academy of Sciences, Žitná 25, 115 67 Praha 1, Czech Republic, e-mail: medkova@math.cas.cz.