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CHARACTERIZATIONS OF THE 0-DISTRIBUTIVE SEMILATTICE

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Abstract. The 0-distributive semilattice is characterized in terms of semiideals, ideals and filters. Some sufficient conditions and some necessary conditions for 0-distributivity are obtained. Counterexamples are given to prove that certain conditions are not necessary and certain conditions are not sufficient.

Keywords: semilattice, prime ideal, filter

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1. Introduction and preliminaries

The 0-distributive lattice and the 0-distributive semilattice have been studied by Varlet [7], [8], Pawar and Thakare [4], [5], Jayaram [3] and Balasubramani and Venkatanarasimhan [1]. In this paper we obtain some characterizations of the 0-distributive semilattice. For the lattice theoretic concepts which have now become commonplace the reader is referred to Szasz [6] and Grätzer [2].

A semilattice is a partially ordered set in which any two elements have a greatest lower bound. Let $S$ be a semilattice. A semiideal of $S$ is a nonempty subset $A$ of $S$ such that $a \in A$, $b \leq a$ ($b \in S$) $\Rightarrow b \in A$. An ideal of $S$ is a semiideal $A$ of $S$ such that the join of any finite number of elements of $A$, whenever it exists, belongs to $A$. If $a \in S$, then $\{x \in S; x \leq a\}$ is an ideal. It is called the principal ideal generated by $a$ and is denoted by $(a)$. A filter of $S$ is a nonempty subset $F$ of $S$ such that (i) $a \in F$, $b \geq a$ ($b \in S$) $\Rightarrow b \in F$ and (ii) $a, b \in F \Rightarrow a \land b \in F$. The dual of a principal ideal is called a principal filter. The principal filter generated by $a$ is denoted by $[a]$. A maximal ideal (filter) of $S$ is a proper ideal (filter) which is not contained in any other proper ideal (filter). A prime semiideal (ideal) is a proper semiideal (ideal)
A such that \( a \land b \in A \Rightarrow a \in A \) or \( b \in A \). A minimal prime semiideal (ideal) is a prime semiideal (ideal) which does not contain any other prime semiideal (ideal). Let \( F(S) \) denote the set of filters of \( S \). A prime filter of \( S \) is a filter \( A \) such that \( B, C \in F(S), B \cap C \subseteq A, B \cap C \neq \emptyset \Rightarrow B \subseteq A \) or \( C \subseteq A \). If \( A \) is a prime filter of \( S \) and \( A_1, \ldots, A_n \in F(S), A_1 \cap \ldots \cap A_n \subseteq A, A_1 \cap \ldots \cap A_n \neq \emptyset \), then \( A_i \subseteq A \) for some \( i \in \{1, \ldots, n\} \).

Let \( A \) be a nonempty subset of a semilattice \( S \) with \( 0, A^* = \{x \in S; a \land x = 0 \) for all \( a \in A \} \) and \( A^0 = \{x \in S; a \land x = 0 \) for some \( a \in A \} \). Then \( A^* \) is called the annihilator of \( A \) and \( A^0 \) is called the pseudoannihilator of \( A \). If \( a \in S \), we write \( \langle a \rangle^* \) for \( \{a\}^* \) and \( \langle a \rangle^0 \) for \( \{a\}^0 \). We say that \( a \) is dense if \( (a)^* = \{0\} \). If \( \sup(a)^* \in (a)^* \), it is called the pseudocomplement of \( a \) and is denoted by \( a^* \). A pseudocomplemented semilattice is a semilattice with \( 0 \) in which every element has a pseudocomplement. An ideal (semiideal) \( A \) of a semilattice \( S \) with \( 0 \) is said to be normal if \( A^{**} = A \).

The following five lemmas are contained in Venkatanarasimhan [9].

**Lemma 1.1.** The set \( I(S) \) of all ideals of a semilattice \( S \) forms a lattice under set inclusion as the partial ordering relation. The meet in \( I(S) \) coincides with the set intersection.

**Lemma 1.2.** Let \( S \) be a semilattice and \( \{a_i; i \in I\} \) any subset of \( S \). Then \( \land a_i (\lor a_i) \) exists if and only if \( \cap(a_i) (\cup(a_i)) \) is a principal ideal (principal filter). Whenever \( \land a_i (\lor a_i) \) exists then \( \cap(a_i) = (\land a_i) (\cup(a_i)) \).

**Lemma 1.3.** Let \( S \) be a semilattice. Then for \( a_1, \ldots, a_n \in S, a_1 \lor \ldots \lor a_n \) exists if and only if \( (a_1) \lor \ldots \lor (a_n) \) is a principal ideal. Whenever \( a_1 \lor \ldots \lor a_n \) exists then \( (a_1) \lor \ldots \lor (a_n) = (a_1 \lor \ldots \lor a_n) \).

**Lemma 1.4.** If \( \{A_i; i \in I\} \) is a family of ideals of a semilattice, then \( \lor A_i = \{x; (x) \subseteq (a_{i1}) \lor \ldots \lor (a_{in}); a_{i1}, \ldots, a_{in} \in \bigcup A_i\} \).

**Lemma 1.5.** Every proper filter of a semilattice with \( 0 \) is contained in a maximal filter.

The following lemma is easily proved.

**Lemma 1.6.** Let \( A \) be a nonempty subset of a semilattice \( S \) with \( 0 \) and \( x \in S \). Then \( A^* \) and \( A^0 \) are semiideals of \( S \) and \( (x)^* = [x]^0 = (x)^0 = (x)^* \).

The following four lemmas are contained in Venkatanarasimhan [10].
Lemma 1.7. Let $A$ be a nonempty proper subset of a semilattice $S$ with 0. Then $A$ is a filter if and only if $S - A$ is a prime semiideal.

Lemma 1.8. Let $A$ be a nonempty subset of a semilattice $S$ with 0. Then $A$ is a maximal filter if and only if $S - A$ is a minimal prime semiideal.

Lemma 1.9. Any prime semiideal of a semilattice with 0 contains a minimal prime semiideal.

Lemma 1.10. Let $A$ be a nonempty subset of a semilattice $S$ with 0. Then $A^*$ is the intersection of all minimal prime semiideals not containing $A$.

The following lemma is contained in Pawar and Thakare [4].

Lemma 1.11. Let $A$ be a proper filter of a semilattice $S$ with 0. Then $A$ is maximal if and only if for each $x$ in $S - A$, there is some $a$ in $A$ such that $a \land x = 0$.

Lemma 1.12. Let $A$ and $B$ be filters of a semilattice $S$ with 0 such that $A$ and $B^0$ are disjoint. Then there is a minimal prime semiideal containing $B^0$ and disjoint from $A$.

Proof. It is easily seen that $A \lor B$ is a proper filter of $S$. Hence by Lemma 1.5, $A \lor B \subseteq M$ for some maximal filter $M$. Now $B \subseteq M$ and so $M \cap B^0 = \emptyset$. By Lemma 1.8, $S - M$ is a minimal prime semiideal. Clearly $B^0 \subseteq S - M$ and $(S - M) \cap A = \emptyset$. $\Box$

Lemma 1.13. Let $A$ be a filter of a semilattice $S$ with 0. Then $A^0$ is the intersection of all minimal prime semiideals disjoint from $A$.

Proof. Let $N$ be any minimal prime semiideal disjoint from $A$. If $x \in A^0$, then $x \land a = 0$ for some $a \in A$ and so $x \in N$.

Let $y \in S - A^0$. Then $a \land y \neq 0$ for all $a \in A$. Hence $A \lor [y] \neq S$. By Lemma 1.5, $A \lor [y] \subseteq M$ for some maximal filter $M$. By Lemma 1.8, $S - M$ is a minimal prime semiideal. Clearly $(S - M) \cap A = \emptyset$ and $y \notin S - M$. $\Box$

Lemma 1.14. Let $S$ be a semilattice with 0. Then the set complement of a prime filter is a prime ideal. If $S$ is finite, then the set complement of a prime ideal is a prime filter.

Proof. Let $A$ be a prime filter of $S$. By Lemma 1.7, $S - A$ is a prime semiideal. Let $x_1, \ldots, x_n \in S - A$ and suppose $x_1 \lor \ldots \lor x_n$ exists. Since $A$ is prime it follows that $x_1 \lor \ldots \lor x_n \in S - A$. Thus $S - A$ is a prime ideal. $\Box$
Let $S$ be finite and let $A$ be any prime ideal of $S$. By Lemma 1.7, $S - A$ is a filter. Since $S$ is finite, every filter of $S$ is principal. Let $a, b \in A$ be such that $[a] \cap [b] \neq \emptyset$. Let $[a] \cap [b] = \{c_1, \ldots, c_n\}$ and $c = c_1 \land \ldots \land c_n$. Then $c \geq a, b$. If $d \geq a, b$ then $d = c_j$ for some $j$ and so $d \geq c$. Thus $c = a \lor b \in A$. Hence $(a) \cap (b) \not\subseteq S - A$ proving $S - A$ is prime.

2. Definition and characterizations

Definition 2.1. A 0-distributive lattice is a lattice with 0 in which $a \land b = 0 = a \land c$ implies $a \land (b \lor c) = 0$.

Varlet [7], has proved that a lattice $L$ bounded below is 0-distributive if and only if the ideal lattice $I(L)$ is pseudocomplemented. He also observed that for an ideal lattice, the two notions of pseudocomplementedness and 0-distributivity are equivalent. These results motivate the following definition.

Definition 2.2. A 0-distributive semilattice is a semilattice $S$ with 0 such that $I(S)$, the lattice of ideals of $S$, is 0-distributive.

Theorem 2.3. Let $S$ be a semilattice with 0. Then the following statements are equivalent:

1. $S$ is 0-distributive.
2. If $A, A_1, \ldots, A_n$ are ideals of $S$ such that $A \cap A_1 = \ldots = A \cap A_n = (0)$, then $A \cap (A_1 \lor \ldots \lor A_n) = (0)$.
3. If $a, a_1, \ldots, a_n$ are elements of $S$ such that $(a) \cap (a_1) = \ldots = (a) \cap (a_n) = (0)$, then $(a) \cap ((a_1) \lor \ldots \lor (a_n)) = (0)$.
4. If $M$ is a maximal filter of $S$, then $S - M$ is a minimal prime ideal.
5. Every minimal prime semiideal of $S$ is a minimal prime ideal.
6. Every prime semiideal of $S$ contains a minimal prime ideal.
7. Every proper filter of $S$ is disjoint from a minimal prime ideal.
8. For each nonzero element $a$ of $S$, there is a minimal prime ideal not containing $a$.
9. For each nonzero element $a$ of $S$, there is a prime ideal not containing $a$.

Proof. 1 $\Rightarrow$ 2: Suppose 1 holds and let $A, A_1, \ldots, A_n \in I(S)$ be such that $A \cap A_1 = \ldots = A \cap A_n = (0)$. By 1, $I(S)$ is 0-distributive. Hence $A \cap (A_1 \lor A_2) = (0)$. Assume $A \cap (A_1 \lor \ldots \lor A_{k-1}) = (0)$ for $2 < k \leq n$. Then $A \cap (A_1 \lor \ldots \lor A_{k-1} \lor A_k) = A \cap (B \lor A_k)$ where $B = A_1 \lor \ldots \lor A_{k-1}$. By our induction hypothesis $A \cap B = (0)$. Also $A \cap A_k = (0)$. Consequently $A \cap (A_1 \lor \ldots \lor A_k) = A \cap (B \lor A_k) = (0)$. Thus the result follows by induction.

Obviously 2 $\Rightarrow$ 3 and 8 $\Rightarrow$ 9.
3 ⇒ 1: Suppose 3 holds. Let $A, B, C \in I(S)$ be such that $A \cap B = (0] = A \cap C$. Then $(a] \cap (b] = (0] = (a] \cap (c]$ for all $a \in A, b \in B$ and $c \in C$. Let $x \in A \cap (B \lor C)$. Then $x \in B \lor C$. Hence $(x] \subseteq (b_1] \lor \ldots \lor (b_m] \lor (c_1] \lor \ldots \lor (c_n]$ for some $b_1, \ldots, b_m \in B$ and $c_1, \ldots, c_n \in C$. Also $x \in A$. Consequently $(x] \cap (b_i] = (0]$ for $i = 1, \ldots, m$ and $(x] \cap (c_j] = (0]$ for $j = 1, \ldots, n$. By 3, $(x] \cap ((b_1] \lor \ldots \lor (b_m] \lor (c_1] \lor \ldots \lor (c_n]) = (0]$. It follows that $x = 0$. Thus $A \cap (B \lor C) = (0]$.

3 ⇒ 4: Suppose 3 holds. Let $M$ be any maximal filter of $S$. By Lemma 1.8, $S - M$ is a minimal prime semialideal. Let $x_1, \ldots, x_n \in S - M$ be such that $x_1 \lor \ldots \lor x_n$ exists. By Lemma 1.11, $a_1 \land x_1 = \ldots = a_n \land x_n = 0$ for some $a_1, \ldots, a_n \in M$. Let $a = a_1 \land \ldots \land a_n$. Then $a \in M$ and $a \land x_i = 0$ for $i = 1, \ldots, n$. By Lemma 1.2, $(a] \cap (x_i] = (0]$ for $i = 1, \ldots, n$. By Lemma 1.3, $(a] \cap (x_1 \lor \ldots \lor x_n] = (a] \cap ((x_1] \lor \ldots \lor (x_n]) = (0]$. It follows that $a \land (x_1 \lor \ldots \lor x_n) = 0$. Hence $x_1 \lor \ldots \lor x_n \in S - M$. Thus $S - M$ is an ideal.

4 ⇒ 5: Suppose 4 holds. Let $N$ be any minimal prime semialideal of $S$. By Lemma 1.8, $S - N$ is a maximal filter. By 4, $N = S - (S - N)$ is a minimal prime ideal.

5 ⇒ 6: Suppose 5 holds and let $A$ be any prime semialideal of $S$. By Lemma 1.9, $A \supseteq N$ for some minimal prime semialideal $N$. By 5, $N$ is a minimal prime ideal.

6 ⇒ 7: Suppose 6 holds and let $A$ be any proper filter of $S$. By Lemma 1.7, $S - A$ is a prime semialideal. By 6, $S - A$ contains a minimal prime ideal $N$. Clearly $A \cap N = \emptyset$.

7 ⇒ 8: Suppose 7 holds and let $a$ be any nonzero element of $S$. By 7, $(a]$ is disjoint from a minimal prime ideal $N$. Clearly $a \notin N$.

9 ⇒ 3: Suppose 9 holds. Let $a, a_1, \ldots, a_n \in S$ such that $(a] \cap (a_1] = \ldots = (a] \cap (a_n] = (0]$ and $(a] \cap ((a_1] \lor \ldots \lor (a_n]) \neq (0]$. Then there exists $x \in (a] \cap ((a_1] \lor \ldots \lor (a_n])$ such that $x \neq 0$. By 9 there is a prime ideal $A$ such that $x \notin A$. By Lemma 1.7, $S - A$ is a proper filter and clearly $a \in S - A$. Consequently $a_1, \ldots, a_n \in A$. It follows that $(a_1] \lor \ldots \lor (a_n] \subseteq A$ and so $x \in A$. Thus we get a contradiction. Hence $(a] \cap (a_1] = \ldots = (a] \cap (a_n] = (0] \Rightarrow (a] \cap ((a_1] \lor \ldots \lor (a_n]) = (0]$. \hfill \qedsymbol

**Theorem 2.4.** Let $S$ be a semilattice with 0. Then the following statements are equivalent:

1. $S$ is 0-distributive.
2. If $A$ is a nonempty subset of $S$ and $B$ is a proper filter intersecting $A$, there is a minimal prime ideal containing $A^*$ and disjoint from $B$.
3. If $A$ is a nonempty subset of $S$ and $B$ is a proper filter intersecting $A$, there is a prime ideal containing $A^*$ and disjoint from $B$.
4. If $A$ is a nonempty subset of $S$ and $B$ is a prime semiideal not containing $A$, there is a minimal prime ideal containing $A^*$ and contained in $B$. 

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5. If $A$ is a nonempty subset of $S$ and $B$ is a prime semiideal not containing $A$, there is a prime ideal containing $A^*$ and contained in $B$.
6. For each nonzero element $a$ of $S$ and each proper filter $B$ containing $a$, there is a prime ideal containing $(a)^*$ and disjoint from $B$.
7. For each nonzero element $a$ of $S$ and each prime semiideal $B$ not containing $a$, there is a prime ideal containing $(a)^*$ and contained in $B$.
8. If $A$ and $B$ are filters of $S$ such that $A$ and $B^0$ are disjoint, there is a minimal prime ideal containing $B^0$ and disjoint from $A$.
9. If $A$ and $B$ are filters of $S$ such that $A$ and $B^0$ are disjoint, there is a prime ideal containing $B^0$ and disjoint from $A$.
10. If $A$ is a filter of $S$ and $B$ is a prime semiideal containing $A^0$, there is a minimal prime ideal containing $A^0$ and contained in $B$.
11. If $A$ is a filter of $S$ and $B$ is a prime semiideal containing $A^0$, there is a prime ideal containing $A^0$ and contained in $B$.
12. For each nonzero element $a$ in $S$ and each filter $A$ disjoint from $(a)^*$, there is a prime ideal containing $(a)^*$ and disjoint from $A$.
13. For each nonzero element $a$ in $S$ and each prime semiideal $B$ containing $(a)^*$, there is a prime ideal containing $(a)^*$ and contained in $B$.

**Proof.** $1 \Rightarrow 2$: Suppose 1 holds. Let $A$ be a nonempty subset of $S$ and $B$ any proper filter such that $B \cap A \neq \emptyset$. By Lemma 1.7, $S - B$ is a prime semiideal and by Lemma 1.9, $S - B \supset N$ for some minimal prime semiideal $N$. Clearly $N \cap B = \emptyset$. Also $S - B \supseteq A$ and so $N \supseteq A$. By Lemma 1.10, $N \supseteq A^*$. Since $S$ is 0-distributive, $N$ is a minimal prime ideal [see Theorem 2.3, 5].

By Lemma 1.7, it follows that $2 \Rightarrow 4$, $3 \Rightarrow 5$, $8 \Rightarrow 10$, $9 \Rightarrow 11$ and $12 \Rightarrow 13$.

Obviously $2 \Rightarrow 3$, $2 \Rightarrow 6$, $4 \Rightarrow 5$, $4 \Rightarrow 7$, $8 \Rightarrow 9$, $10 \Rightarrow 11 \Rightarrow 13$ and $5 \Rightarrow 7$.

$1 \Rightarrow 8$: Suppose 1 holds. Let $A$ and $B$ be filters of $S$ such that $A \cap B^0 \neq \emptyset$. By Lemma 1.12, there is a minimal prime semiideal $N$ such that $N \supseteq B^0$ and $N \cap A = \emptyset$. Since $S$ is 0-distributive it follows that $N$ is a minimal prime ideal [see Theorem 2.3, 5].

$8 \Rightarrow 12$: By Lemma 1.6, $(x)^* = [x]^0$ for all $x \in S$. Hence the result.

$6 \Rightarrow 1$: Suppose 6 holds. Let $a$ be any nonzero element of $S$. Now $[a]$ is a proper filter containing $a$. By 6, there is a prime ideal $N$ containing $(a)^*$ and disjoint from $[a]$. Clearly $a \notin N$. Thus $S$ is 0-distributive [see Theorem 2.3, 9].

$7 \Rightarrow 1$: Suppose 7 holds. Let $a$ be any nonzero element of $S$. Now $S - [a]$ is a prime semiideal not containing $a$. By 7 there is a prime ideal $N$ containing $(a)^*$ and contained in $S - [a]$. Clearly $a \notin N$. Thus $S$ is 0-distributive [See Theorem 2.3, 9].

$13 \Rightarrow 1$: Suppose 13 holds and let $a$ be any nonzero element of $S$. By Lemma 1.7, $S - [a]$ is a prime semiideal not containing $a$. Since $(a) \cap (a)^* = (0) \subseteq S - [a]$ it
follows that \( S - [a) \) contains \((a)^*\). By 13, there is a prime ideal \( N \) containing \((a)^*\) and contained in \( S - [a) \). Clearly \( a \in N \). Thus \( S \) is 0-distributive [see Theorem 2.3, 9].

\[ \square \]

**Theorem 2.5.** Let \( S \) be a semilattice with 0. Then the following statements are equivalent:

1. \( S \) is 0-distributive.
2. For any nonempty subset \( A \) of \( S \), \( A^* \) is the intersection of all minimal prime ideals not containing \( A \).
3. For any filter \( A \) of \( S \), \( A^0 \) is the intersection of all minimal prime ideals disjoint from \( A \).
4. For each \( a \) in \( S \), \((a)^* \) is an ideal.
5. Every normal semiideal of \( S \) is an intersection of minimal prime ideals.
6. For any finite number of ideals \( A, A_1, \ldots, A_n \) of \( S \),

\[ (A \cap (A_1 \lor \ldots \lor A_n))^* = (A \cap A_1)^* \cap \ldots \cap (A \cap A_n)^*. \]

7. For any three ideals \( A, B, C \) of \( S \),

\[ (A \cap (B \lor C))^* = (A \cap B)^* \cap (A \cap C)^*. \]

8. For any finite number of ideals \( A, A_1, \ldots, A_n \) of \( S \),

\[ ((A \lor A_1) \cap \ldots \cap (A \lor A_n))^* = A^* \cap (A_1 \cap \ldots A_n)^*. \]

9. For any three ideals \( A, B, C \) of \( S \),

\[ ((A \lor B) \cap (A \lor C))^* = A^* \cap (B \lor C)^*. \]

10. For any finite number of elements \( a, a_1, \ldots, a_n \) of \( S \),

\[ ([a] \cap ((a_1) \lor \ldots \lor (a_n)))^* = ([a] \cap (a_1))^* \cap \ldots \cap ([a] \cap (a_n))^*. \]

11. For any finite number of elements \( a_1, \ldots, a_n \) of \( S \),

\[ ((a_1) \lor \ldots \lor (a_n))^* = (a_1)^* \cap \ldots \cap (a_n)^*. \]

12. \( I(S) \) is pseudocomplemented.

**Proof.** 1 \( \Rightarrow \) 2: Follows by Lemma 1.10 and Theorem 2.3, 5.

1 \( \Rightarrow \) 3: Follows by Lemma 1.13 and Theorem 2.3, 5.

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By Lemma 1.6, \((a)^* = [a]^0\). Hence the result.

Suppose 4 holds. Let \(a, a_1, \ldots, a_n \in S\) be such that \((a) \cap (a_1) = \ldots = (a) \cap (a_n) = (0)\). Then \(a_1, \ldots, a_n \in (a)^*\). By 4 it follows that \((a_1) \lor \ldots \lor (a_n) \subseteq (a)^*\). Hence \((a) \cap ((a_1) \lor \ldots \lor (a_n)) = (0)\). Thus \(S\) is 0-distributive [see Theorem 2.3, 3].

Obviously 6 \(\Rightarrow\) 7, 8 \(\Rightarrow\) 9 and 6 \(\Rightarrow\) 10.

Suppose 2 holds. Let \(A\) be any normal semiideal of \(S\). Then \(A = B^*\) for some semiideal \(B\). By 2, \(B^*\) is the intersection of all minimal prime ideals not containing \(B\). Hence the result.

Suppose 4 holds. Let \(A, A_1, \ldots, A_n \in I(S)\). If \(Q\) is any minimal prime ideal of \(S\) such that \(Q \not\supseteq A \cap (A_1 \lor A_2 \lor \ldots \lor A_n)\), then \(Q \not\supseteq A \cap A_j\) for some \(j \in \{1, \ldots, n\}\).

By 2 it follows that \((A \cap (A_1 \lor \ldots \lor A_n))^* \supseteq (A \cap A_1)^* \cap \ldots \cap (A \cap A_n)^*\). The reverse inclusion is obvious.

Suppose 7 holds. Then for \(A, B, C \in I(S)\) we have \((A \cap (B \lor C))^* = (A \cap B)^* \cap (A \cap C)^*\). By replacing \(A\) by \(B \lor C\) it follows that \((B \lor C)^* = B^* \lor C^*\).

Suppose \(A \lor B = (0) = A \lor C\). Then \((a) \cap (b) = (0) = (a) \cap (c)\) for all \(a \in A, b \in B\) and \(c \in C\). Hence \(a \in B^* \lor C^*\) for all \(a \in A\). Hence \(a \in (B \lor C)^*\). Consequently \(A \subseteq (B \lor C)^*\). It follows that \(A \lor B = (0)\).

Suppose 2 holds, let \(A, A_1, \ldots, A_n\) be ideals of \(S\) and let \(Q\) be any minimal prime ideal such that \(Q \not\supseteq (A \lor A_1) \lor \ldots \lor (A \lor A_n)\). Then \(Q \not\supseteq A \lor A_1, \ldots, A \lor A_n\) and so \(Q \not\supseteq A\) or \(Q \not\supseteq A_j\) for \(j \in \{1, \ldots, n\}\). By 2 it follows that \(((A \lor A_1) \lor \ldots \lor (A \lor A_n))^* \supseteq A^* \lor (A_1^* \lor \ldots \lor A_n^*)\). The reverse inclusion is obvious.

Suppose 9 holds. Then for any three ideals \(A, B, C, S\), \(((A \lor B) \lor (A \lor C))^* = A^* \lor (B \lor C)^*\). By replacing \(C\) by \(B\) and \(A\) by \(C\) it follows that \((B \lor C)^* = B^* \lor C^*\).

Suppose \(A \lor B = (0) = A \lor C\). Then \((a) \cap (b) = (0) = (a) \cap (c)\) for all \(a \in A, b \in B\) and \(c \in C\). Hence \(a \in B^* \lor C^*\) for all \(a \in A\). Hence \(a \in (B \lor C)^*\) for all \(a \in A\). Consequently \(A \subseteq (B \lor C)^*\). It follows that \(A \lor B = (0)\). Thus \(S\) is 0-distributive.

Suppose 10 holds. Let \(a, a_1, \ldots, a_n \in S\) such that \((a) \cap (a_1) = \ldots = (a) \cap (a_n) = (0)\). Then \(((a) \cap (a_1))^* \lor \ldots \lor ((a) \cap (a_n))^* = S\). Hence \(((a) \cap (a_1))^* \lor \ldots \lor ((a) \cap (a_n))^* = S\). By 10, \(((a) \lor ((a_1) \lor \ldots \lor (a_n))^* = S\). Consequently \((a) \lor ((a_1) \lor \ldots \lor (a_n))^* = (0)\). It follows that \(S\) is 0-distributive [see Theorem 2.3, 3].

Suppose 6 holds. Then for any finite number of ideals \(A, A_1, \ldots, A_n\) of \(S\), \((A \cap (A_1 \lor \ldots \lor A_n))^* = (A \cap A_1)^* \lor \ldots \lor (A \cap A_n)^*\). By taking \(A = A_1 \lor \ldots \lor A_n\) it follows that \((A_1 \lor \ldots \lor A_n)^* = A_1^* \lor \ldots \lor A_n^*\). Hence the result.

Suppose 11 holds. Let \(a, a_1, \ldots, a_n \in S\) be such that \((a) \cap (a_1) = \ldots = (a) \cap (a_n) = (0)\). Then \(a \in (a_1)^* \lor \ldots \lor (a_n)^*\). By 11 it follows that \(a \in ((a_1) \lor \ldots \lor (a_n))^*\). Hence \((a) \lor ((a_1) \lor \ldots \lor (a_n))^* = (0)\). Thus \(S\) is 0-distributive [see Theorem 2.3, 3].
2 \Rightarrow 12: Suppose 2 holds. Let $A \in I(S)$. Then by 2 it follows that $A^*$ is an ideal. If $B \in I(S)$ is such that $A \cap B = (0)$ and $x \in B$, then $a \land x = 0$ for all $a \in A$ and so $x \in A^*$. Thus $B \subseteq A^*$. It follows that $A^*$ is the pseudocomplement of $A$.

12 \Rightarrow 1: Suppose 12 holds. Then every principal ideal of $S$ has a pseudocomplement in $I(S)$. Let $a, a_1, \ldots, a_n \in S$ be such that $(a) \cap (a_1) = \ldots = (a) \cap (a_n) = (0)$. Then $(a_i) \subseteq (a)^*$ for $i = 1, \ldots, n$ and so $((a_1) \lor \ldots \lor (a_n)) \subseteq (a)^*$. Consequently $(a) \cap ((a_1) \lor \ldots \lor (a_n)) = (0)$. Thus $S$ is 0-distributive [see Theorem 2.3, 3].

Remark 2.6. According to Varlet [8], an ideal of a semilattice $S$ is a nonempty subset $I$ of $S$ such that (i) $y \leq x$ and $x \in I$ imply $y \in I$; (ii) for any $x, y \in I$ there exists a $z \in I$ such that $z \geq x$ and $z \geq y$. According to him a semilattice $S$ with 0 is said to be 0-distributive if for any $a \in S$, the subset $(a)^* = \{ x \in S; \ x \land a = 0 \}$ is an ideal.

Let $S$ be a 0-distributive semilattice in Varlet’s sense. Then for each $a \in S$, $(a)^*$ is a Varlet ideal and therefore an ideal in our sense. Thus $S$ is 0-distributive in our sense. The converse is not true. Consider the semilattice $S = \{ 0, a, b, c \}$ in which the ordering is defined by $0 < a, b, c; a \parallel b; a \parallel c$; and $b \parallel c$. Clearly $S$ is 0-distributive in our sense but not in Varlet’s sense.

We give below some additional characterizations when the semilattice is finite.

**Theorem 2.7.** Let $S$ be a finite semilattice. Then the following statements are equivalent:

1. $S$ is 0-distributive.
2. If $a, b, c$ are elements of $S$ such that $(a) \cap (b) = (0) = (a) \cap (c)$ then $(a) \cap ((b) \lor (c)) = (0)$.
3. Every maximal filter of $S$ is prime.
4. Each nonzero element of $S$ is contained in a prime filter.
5. If $A$ is a nonempty subset of $S$ and $B$ is a proper filter intersecting $A$, there is a prime filter containing $B$ and disjoint from $A^*$.
6. If $A$ is a nonempty subset of $S$ and $B$ is a prime semiideal not containing $A$, there is a prime filter containing $S - B$ and disjoint from $A^*$.
7. For each nonzero element $a$ of $S$ and each proper filter $B$ containing $a$, there is a prime filter containing $B$ and disjoint from $(a)^*$.
8. For each nonzero element $a$ of $S$ and each prime semiideal $B$ not containing $a$, there is a prime filter containing $S - B$ and disjoint from $(a)^*$.
9. If $A$ and $B$ are filters of $S$ such that $A$ and $B^0$ are disjoint, there is a prime filter containing $A$ and disjoint from $B^0$.
10. If $A$ is a filter of $S$ and $B$ is a prime semiideal containing $A^0$, there is a prime filter containing $S - B$ and disjoint from $A^0$. 

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11. For each nonzero element $a$ in $S$ and each filter $A$ disjoint from $(a)^*$, there is a prime filter containing $A$ and disjoint from $(a)^*$.

12. For each nonzero element $a$ in $S$ and each prime semiideal $B$ containing $(a)^*$, there is a prime filter containing $S - B$ and disjoint from $(a)^*$.

Proof. Obviously $1 \Rightarrow 2, 6 \Rightarrow 8, 10 \Rightarrow 12, 5 \Rightarrow 7$ and $9 \Rightarrow 11$.

$2 \Rightarrow 1$: Suppose 2 holds and let $a, a_1, \ldots, a_n \in S$ be such that $(a) \cap (a_1) = \ldots = (a) \cap (a_n) = (0)$. Let $A = (a_1) \cup \ldots \cup (a_n)$, let $B = \{b_1, \ldots, b_m\}$ be the set of existing supremas of nonempty subsets of $A$ and $b \in B$. Then $(a_1) \cup \ldots \cup (a_n) = (b_1) \cup \ldots \cup (b_m)$ and $b = c_1 \lor \ldots \lor c_k$ for some $c_1, \ldots, c_k \in A$. If $p, q \in \{1, \ldots, k\}$, clearly $b$ is an upperbound of $\{c_p, c_q\}$. Thus the set $C$ of upperbounds of $\{c_p, c_q\}$ is nonempty and $\inf C = c_p \lor c_q$. Also $(a) \cap (c_p) = (0) = (a) \cap (c_q)$, so that $(a) \cap ((c_p) \lor (c_q)) = (0)$ by 2. It is easily seen that every nonempty subset of $\{c_1, \ldots, c_k\}$ has a supremum and by induction it follows that $(a) \cap (b) = (a) \cap ((c_1) \lor \ldots \lor (c_k)) = (0)$. Hence $(a) \cap ((a_1) \lor \ldots \lor (a_n)) = (a) \cap ((b_1) \cup \ldots \cup (b_m)) = (a) \cap (b_1) \cup \ldots \cup ((a) \cap (b_m)) = (0)$. Consequently $S$ is 0-distributive [see Theorem 2.3, 3].

$1 \Rightarrow 3$: Suppose 1 holds. Let $M$ be any maximal filter of $S$. Since $S$ is finite, every filter of $S$ is principal. Let $a, b \in S - M$ be such that $(a) \cap (b) \neq 0$. Let $(a) \cap (b) = \{c_1, \ldots, c_n\}$ and $c = c_1 \land \ldots \land c_n$. Then $c \geq a, b$ as $c_i \geq a, b$ for all $i$. If $d \in S$ and $d \geq a, b$, then $d = c_j$ for some $j$, so that $d \geq c$. Thus $c = a \lor b$. Also $S - M$ is an ideal [see Theorem 2.3, 4]. Hence $a \lor b \in S - M$. It follows that $(a) \cap (b) = (a \lor b) \not\in M$, proving $M$ is prime.

$3 \Rightarrow 4$: Suppose 3 holds. Let $a$ be any nonzero element of $S$. By Lemma 1.5, $(a)$ is contained in a maximal filter $M$. By 3, $M$ is prime. Clearly $a \in M$.

$4 \Rightarrow 1$: Suppose 4 holds. Let $a$ be any nonzero element of $S$. By 4, $a \in B$ for some prime filter $B$. By Lemma 1.14, $S - B$ is a prime ideal and clearly $a \not\in S - B$. It follows that $S$ is 0-distributive [see Theorem 2.3, 9].

$3 \Rightarrow 5$: Suppose 3 holds. Let $A$ be a nonempty subset of $S$ and $B$ a proper filter such that $B \cap A \neq \emptyset$. By Lemma 1.5, $B \subseteq M$ for some maximal filter $M$. By 3, $M$ is prime. By Lemma 1.8, $S - M$ is a minimal prime semiideal and clearly $S - M \not\subseteq A$. Hence $S - M \supseteq A^*$ and so $M \cap A^* = \emptyset$.

$5 \Rightarrow 6$: Suppose 5 holds. Let $A$ be a nonempty subset of $S$ and $B$ a prime semiideal such that $B \not\subseteq A$. By Lemma 7, $S - B$ is a proper filter and clearly $(S - B) \cap A \neq \emptyset$. By 5 there is a prime filter containing $S - B$ and disjoint from $A^*$.

$7 \Rightarrow 8$: Similar to $5 \Rightarrow 6$.

$8 \Rightarrow 1$: Suppose 8 holds and let $a$ be any nonzero element of $S$. Now $S - (a)$ is a prime semiideal not containing $a$. By 8 there is a prime filter $N$ containing $(S - (a)) = (a)$ and disjoint from $(a)^*$. By Lemma 1.14, $S - N$ is a prime ideal and clearly $a \not\in S - N$. Thus $S$ is 0-distributive [see Theorem 2.3, 9].
3 ⇒ 9: Suppose 3 holds. Let $A$ and $B$ be filters of $S$ such that $A$ and $B^0$ are disjoint. By Lemma 1.12, there is a minimal prime semiideal $N$ such that $N \supseteq B^0$ and $N \cap A = \emptyset$. By Lemma 1.8, $S - N$ is a maximal filter. Clearly $S - N \supseteq A$ and $(S - N) \cap B^0 = \emptyset$. By 3, $S - N$ is prime.

9 ⇒ 10: Suppose 9 holds. Let $A$ be a filter of $S$ and $B$ a prime semiideal such that $B \supseteq A^0$. By Lemma 1.7, $S - B$ is a proper filter and clearly $(S - B) \cap A^0 = \emptyset$. By 9, there is a prime filter containing $S - B$ and disjoint from $A^0$.

11 ⇒ 12: Similar to 5 ⇒ 6.

12 ⇒ 4: Suppose 12 holds. Let $a$ be any nonzero element of $S$. Now $S - [a]$ is a prime semiideal not containing $(a)$. Since $(a) \cap (a)^* = (0) \subseteq S - [a]$ it follows that $(a)^* \subseteq S - [a]$. By 12 there is a prime filter $N$ containing $S - (S - [a]) = [a]$ and disjoint from $(a)^*$. Clearly $a \in N$. □

**Theorem 2.8.** Let $S$ be a finite semilattice. Then the following statements are equivalent:

1. $S$ is 0-distributive.
2. For any finite number of filters $A, A_1, \ldots, A_n$ of $S$ such that $A \cap A_i \neq \emptyset$ for all $i \in \{1, \ldots, n\}$,

   \[ ((A \cap A_1) \lor \ldots \lor (A \cap A_n))^0 = A^0 \cap (A_1 \lor \ldots \lor A_n)^0. \]

3. For any three filters $A, B, C$ of $S$ such that $A \cap B \neq \emptyset$ and $A \cap C \neq \emptyset$,

   \[ ((A \cap B) \lor (A \cap C))^0 = A^0 \cap (B \lor C)^0. \]

4. For all $a, b, c$ in $S$ such that $[a] \cap [b] \neq \emptyset$ and $[a] \cap [c] \neq \emptyset$,

   \[ (((a) \cap [b]) \lor ([a] \cap [c]))^0 = [a]^0 \cap ([b] \lor [c])^0. \]

5. For any finite number of filters $A, A_1, \ldots, A_n$ of $S$ such that $A_1 \cap \ldots \cap A_n \neq \emptyset$,

   \[ (A \lor (A_1 \cap \ldots \cap A_n))^0 = (A \lor A_1)^0 \cap \ldots \cap (A \lor A_n)^0. \]

6. For any three filters $A, B, C$ of $S$ such that $B \cap C \neq \emptyset$,

   \[ (A \lor (B \cap C))^0 = (A \lor B)^0 \cap (A \lor C)^0. \]

7. For any finite number of elements $a, a_1, \ldots, a_n$ of $S$ such that $[a_1] \cap \ldots \cap [a_n] \neq \emptyset$,

   \[ ([a] \lor ([a_1] \cap \ldots \cap [a_n]))^0 = ([a] \lor [a_1])^0 \cap \ldots \cap ([a] \lor [a_n])^0. \]

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8. For all \( a, b, c \) in \( S \), with \( [b] \cap [c] \neq \emptyset \),
\[
([a] \lor ([b] \cap [c]))^0 = ([a] \lor [b])^0 \cap ([a] \lor [c])^0.
\]

9. For any finite number of elements \( a_1, \ldots, a_n \) of \( S \) such that \( [a_1] \cap \ldots \cap [a_n] \neq \emptyset \),
\[
([a_1] \cap \ldots \cap [a_n])^0 = [a_1]^0 \cap \ldots \cap [a_n]^0.
\]

10. For all \( a, b \) in \( S \) with \( [a] \cap [b] \neq \emptyset \),
\[
([a] \cap [b])^0 = [a]^0 \cap [b]^0.
\]

11. For all \( a, b, c \) in \( S \),
\[
((a) \cap ((b) \lor (c)))^* = ((a) \cap (b))^* \cap ((a) \cap (c))^*.
\]

12. For all \( a, b, c \) in \( S \),
\[
(((a) \lor (b)) \cap ((a) \lor (c)))^* = (a)^* \cap ((b) \lor (c))^*.
\]

13. For all \( a, b \) in \( S \),
\[
((a) \lor (b))^* = (a)^* \cap (b)^*.
\]

**Proof.** 1 \( \Rightarrow \) 2: Suppose 1 holds and let \( A, A_1, \ldots, A_n \) be filters of \( S \) such that \( A \cap A_i \neq \emptyset \) for all \( i \in \{1, \ldots, n\} \). If \( Q \) is any minimal prime ideal of \( S \) such that \( Q \cap (A \cap A_1) \lor \ldots \lor (A \cap A_n) = \emptyset \), then \( Q \cap (A \cap A_1) = \ldots = Q \cap (A \cap A_n) = \emptyset \). By Lemma 1.14, \( S - Q \) is a prime filter and \( S - Q \supseteq (A \cap A_1), \ldots, (A \cap A_n) \). Hence \( S - Q \supseteq A \) or \( S - Q \supseteq A_1 \lor \ldots \lor A_n \) and so \( Q \cap A = \emptyset \) or \( Q \cap (A_1 \lor \ldots \lor A_n) = \emptyset \). It follows that \( (A \cap A_1) \lor \ldots \lor (A \cap A_n))^0 \supseteq A^0 \cap (A_1 \lor \ldots \lor A_n)^0 \) [see Theorem 2.5, 3]. The reverse inclusion is obvious.

Obviously 2 \( \Rightarrow \) 3 \( \Rightarrow \) 4, 5 \( \Rightarrow \) 6 \( \Rightarrow \) 8 and 5 \( \Rightarrow \) 7 \( \Rightarrow \) 8.

4 \( \Rightarrow \) 10: Follows by taking \( c = b \) in 4.

1 \( \Rightarrow \) 5: Suppose 1 holds. Let \( A, A_1, \ldots, A_n \) be filters of \( S \) such that \( A \cap \ldots \cap A_n \neq \emptyset \). If \( Q \) is any minimal prime ideal of \( S \) such that \( Q \cap (A \lor (A_1 \cap \ldots \cap A_n)) = \emptyset \), then \( Q \cap A = \emptyset = Q \cap (A_1 \cap \ldots \cap A_n) \). By Lemma 1.14, \( S - Q \) is a prime filter and clearly \( S - Q \supseteq A \lor A_j \) and so \( Q \cap (A \lor A_j) = \emptyset \) for some \( j \in \{1, \ldots, n\} \). It follows that \( (A \lor (A_1 \cap \ldots \cap A_n))^0 \supseteq (A \lor A_1)^0 \cap \ldots \cap (A \lor A_n)^0 \) [see Theorem 2.5, 3]. The reverse inclusion is obvious.

10 \( \Rightarrow \) 9: Suppose 10 holds and let \( a_1, \ldots, a_n \in S \) be such that \( [a_1] \cap \ldots \cap [a_n] \neq \emptyset \). Then \( ([a_1] \cap [a_2])^0 = [a_1]^0 \cap [a_2]^0 \). Assume \( ([a_1] \cap \ldots \cap [a_{k-1}]) = [a_1]^0 \cap \ldots \cap [a_{k-1}]^0 \) for \( 2 < k \leq n \). Let \( x \in [a_1]^0 \cap \ldots \cap [a_k]^0 \). Then \( x \in [a_1]^0 \cap \ldots \cap [a_{k-1}]^0 = ([a_1] \cap \ldots \cap [a_{k-1}])^0 \) by our induction hypothesis. Hence \( x \lor y = 0 \) for some \( y \in ([a_1] \cap \ldots \cap [a_{k-1}]) \). Thus \( x \in [y]^0 \cap [a_k]^0 \). Assume \( ([a_1] \cap \ldots \cap [a_k])^0 \subseteq ([a_1] \cap \ldots \cap [a_k])^0 \) so that \( ([a_1]^0 \cap \ldots \cap [a_k]^0) \subseteq ([a_1] \cap \ldots \cap [a_k])^0 \). The reverse inclusion is obvious. By induction it follows that \( ([a_1] \cap \ldots \cap [a_n])^0 = [a_1]^0 \cap \ldots \cap [a_n]^0 \).

9 \( \Rightarrow \) 1: Suppose 9 holds. Let \( a \in S \) and let \( a_1, \ldots, a_n \in (a)^* \) be such that \( a_1 \lor \ldots \lor a_n \) exists. Then \( a \lor a_1 = \ldots = a \lor a_n = 0 \) and so \( a \in [a_1]^0 \cap \ldots \cap [a_n]^0 = 0 \).
\[(a_1 \cap \ldots \cap a_n)^0 \] by 9. That is \(a \in [a_1 \lor \ldots \lor a_n]^0\). Hence \(a \land (a_1 \lor \ldots \lor a_n) = 0\), so that \(a_1 \lor \ldots \lor a_n \in (a)^*\). Thus \((a)^*\) is an ideal. It follows that \(S\) is 0-distributive [see Theorem 2.5, 4].

8 \(\Rightarrow\) 1: Suppose 8 holds and let \(a, b, c \in S\) such that \((a) \cap (b) = (0) = (a) \cap (c)\). Let \(X = \{x_1, \ldots, x_n\}\) be the set of existing suprema of nonempty subsets of \((b) \cup (c)\) and \(x \in X\). Then \((b) \lor (c) = (x_1) \cup \ldots \cup (x_n)\) and \(x = y_1 \lor \ldots \lor y_m\) for some \(y_1, \ldots, y_m \in (b) \cup (c)\). If \(p, q \in \{1, \ldots, m\}\), clearly \(x\) is an upperbound of \(\{y_p, y_q\}\). Thus the set \(Y\) of upperbounds of \(\{y_p, y_q\}\) is nonempty and inf \(Y = y_p \lor y_q\). Also \(a \land y_p = 0 = a \land y_q\). Hence \(\big((a) \lor [y_p]\big)^0 = \big((a) \lor [y_q]\big)^0\). Let \(z \in (a) \cap ((y_p) \lor (y_q))\). Then \(z \leq a\) and \(z \leq y_p \lor y_q\). Now \(z \in S = \big((a) \lor [y_p]\big)^0 \cap \big((a) \lor [y_q]\big)^0 = \big((a) \lor ([y_p] \cap [y_q])\big)^0 = \big((a) \lor [y_p] \lor [y_q]\big)^0\) by 8, so that \(z \land t = 0\) for some \(t \in [a] \lor [y_p] \lor [y_q]\). Thus \(z = z \land a \lor (y_p) \lor (y_q) \leq z \land t = 0\) and consequently \((a) \cap ((y_p) \lor (y_q)) = (0)\). It is easily seen that every nonempty subset of \(\{y_1, \ldots, y_m\}\) has a supremum and by induction it follows that \((a) \land (x) = (a) \land ((y_1) \lor \ldots \lor (y_m)) = (0)\). Hence \((a) \land ((b) \lor (c)) = (a) \land ((x_1) \lor \ldots \lor (x_n)) = ((a) \land (x_1)) \lor \ldots \lor ((a) \land (x_n)) = (0)\). Thus \(S\) is 0-distributive [see Theorem 2.7, 2].

1 \(\Rightarrow\) 11: Suppose 1 holds. Then for all \(A, B, C \in I(S)\) we have \((A \cap (B \lor C))^* = (A \land B)^* \cap (A \lor C)^*\) [see Theorem 2.5, 7]. Hence 11 follows.

1 \(\Rightarrow\) 12: Suppose 1 holds. Then for all \(A, B, C \in I(S)\) we have \((A \lor B) \cap (A \land C)^* = A^* \cap (B \land C)^*\) [see Theorem 2.5, 9]. Hence 12 follows.

12 \(\Rightarrow\) 13: Follows by taking \(c = b\) in 12.

13 \(\Rightarrow\) 1: Suppose 13 holds. Let \(a, b, c \in S\) be such that \((a) \cap (b) = (0) = (a) \cap (c)\). Then \((a) \in (b)^* \land (c)^* = ((b) \lor (c))^*\) by 13. Hence \((a) \land ((b) \lor (c)) = (0)\). Thus \(S\) is 0-distributive [see Theorem 2.7, 2].

11 \(\Rightarrow\) 1: Suppose 11 holds. Let \(a, b, c \in S\) be such that \((a) \cap (b) = (0) = (a) \land (c)\). Then \(((a) \cap (b))^* \land ((a) \land (c))^* = S\). Hence By 11, \(((a) \land ((b) \lor (c)))^* = S\). It follows that \((a) \land ((b) \lor (c)) = (0)\). Thus \(S\) is 0-distributive [see Theorem 2.7, 2]. \(\square\)

**Theorem 2.9.** Any one of the conditions 3 to 12 of Theorem 2.7 is sufficient for a semilattice \(S\) with 0 (not necessarily finite) to be 0-distributive. These conditions are also necessary in the case of a lattice.

**Proof.** Suppose 3 of Theorem 2.7 holds and let \(M\) be any maximal filter of \(S\). By Lemma 1.8, \(S - M\) is a minimal prime semiideal. Let \(x_1, \ldots, x_n \in S - M\) and suppose \(x_1 \lor \ldots \lor x_n\) exists. By 3, \(M\) is prime and clearly \([x_i] \notin M\) for \(i = 1, \ldots, n\). Hence by Lemma 1.2, \([x_1 \lor \ldots \lor x_n] = [x_1] \cap \ldots \cap [x_n] \notin M\). Consequently \(x_1 \lor \ldots \lor x_n \in S - M\) and so \(S - M\) is an ideal. It follows that \(S\) is 0-distributive [see Theorem 2.3, 4]. \(\square\)
The sufficiency of the condition 4 of Theorem 2.7 follows by Lemma 1.14 and Theorem 2.3 [see Theorem 2.3, 9]. The sufficiency of the conditions 5 to 12 of Theorem 2.7 follows by Lemma 1.14 and Theorem 2.4 [see Theorem 2.4, 3, 5, 6, 7, 9, 11, 12, 13].

**Theorem 2.10.** Any one of the conditions 2 to 10 of Theorem 2.8 is sufficient for a semilattice $S$ with 0 (not necessarily finite) to be 0-distributive. These conditions are also necessary in the case of a lattice.

**Proof.** Obviously $2 \Rightarrow 3 \Rightarrow 4$ and $5 \Rightarrow 6 \Rightarrow 8$.

$4 \Rightarrow 10$: Follows by taking $c = b$ in 4.

$10 \Rightarrow 9$: Same proof as in Theorem 2.8.

Suppose 9 holds. Let $a \in S$ and let $a_1, \ldots, a_n \in (a)^*$ be such that $a_1 \vee \ldots \vee a_n$ exists.

Then $a \wedge a_1 = \ldots = a \wedge a_n = 0$ and so $a \in [a_1]^0 \cap \ldots \cap [a_n]^0 = ([a_1] \cap \ldots \cap [a_n])^0$ by 9. That is $a \in [a_1 \vee \ldots \vee a_n]^0$. It follows that $a \wedge (a_1 \vee \ldots \vee a_n) = 0$. Hence $a_1 \vee \ldots \vee a_n \in (a)^*$. Thus $(a)^*$ is an ideal and so $S$ is 0-distributive [see Theorem 2.5, 4].

$8 \Rightarrow 7$: Suppose 8 holds and let $a, a_1, \ldots, a_n \in S$ be such that $[a_1] \cap \ldots \cap [a_n] \neq \emptyset$.

Then $(a) \vee ([a_1] \cap [a_2]))^0 = ([a] \vee ([a_1])^0 \cap ([a] \vee [a_2])^0$. Assume $([a] \vee ([a_1] \cap \ldots \cap [a_{k-1}]))^0 = ([a] \vee [a_1])^0 \cap \ldots \cap ([a] \vee [a_{k-1}])^0$ for $2 < k \leq n$. Let $x \in ([a] \vee [a_1])^0 \cap \ldots \cap ([a] \vee [a_{k-1}])^0$. Then $x \in ([a] \vee [a_1])^0 \cap \ldots \cap ([a] \vee [a_{k-1}])^0 = ([a] \vee (([a_1] \cap \ldots \cap [a_{k-1}])))^0$ by our induction hypothesis and $x \in ([a] \vee [a_k])^0$. Hence $x \wedge y = a$ for some $y \in [a] \vee ([a_1] \cap \ldots \cap [a_{k-1}])$ and $x \wedge z = 0$ for some $z \in [a] \vee [a_k]$. Thus $x \wedge a \wedge t = 0$ for some $t \in [a_1] \cap \ldots \cap [a_{k-1}]$ and $x \wedge a \wedge a_k = 0$ so that $x \in [a \wedge t]^0 \cap [a \wedge a_k]^0 = ([a] \vee [t])^0 \cap ([a] \vee [a_k])^0 = ([a] \vee ([t] \cap [a_k]))^0$ by 8. Consequently $x \wedge a \wedge u = 0$ for some $u \in [t] \cap [a_k] \subseteq [a_1] \cap \ldots \cap [a_k]$ and so $x \in ([a] \vee ([a_1] \cap \ldots \cap [a_k]))^0$. Thus $([a] \vee [a_1])^0 \cap \ldots \cap ([a] \vee [a_k])^0 \subseteq ([a] \vee ([a_1] \cap \ldots \cap [a_k]))^0$. The reverse inclusion is obvious. By induction it follows that $([a] \vee [a_1])^0 \cap \ldots \cap ([a] \vee [a_n])^0 = ([a] \vee ([a_1] \cap \ldots \cap [a_n]))^0$.

Suppose 7 holds. Let $a \in S$ and let $a_1, \ldots, a_n \in (a)^*$ be such that $a_1 \vee \ldots \vee a_n$ exisits. Then $a \wedge a_1 = \ldots = a \wedge a_n = 0$ and so $a \in [a_1]^0 \cap \ldots \cap [a_n]^0$. Replacing $a$ by $a_1 \vee \ldots \vee a_n$ in 7, we have $([a_1] \cap \ldots \cap [a_n])^0 = [a_1]^0 \cap \ldots \cap [a_n]^0$. Thus $a \in ([a_1] \cap \ldots \cap [a_n])^0 = [a_1 \vee \ldots \vee a_n]^0$. Hence $a \wedge (a_1 \vee \ldots \vee a_n) = 0$ and consequently $a_1 \vee \ldots \vee a_n \in (a)^*$. Thus $(a)^*$ is an ideal. It follows that $S$ is 0-distributive [see Theorem 2.5, 4].

**Remark 2.11.** The conditions 3 to 12 of Theorem 2.7 are not necessary for an infinite semilattice to be 0-distributive. These conditions are both necessary and sufficient in the case of a lattice.

Clearly each of the conditions 3 to 12 implies the condition 4. Hence it is enough to prove that 4 is not necessary.
Let $C$ be an infinite chain without the least element and $S = C \cup \{0, a, b, d\}$. Define an ordering on $S$ as follows: $0 < a, b, d$; $a \parallel b$; $a \parallel d$; $b \parallel d$ and $a, b, d < c$ for all $c \in C$. Clearly $S$ is a 0-distributive semilattice with respect to this ordering. But no prime filter of $S$ contains the nonzero element $a$. Thus 4 is not necessary.

**Remark 2.12.** The conditions 2 to 10 of Theorem 2.8 are not necessary for an infinite semilattice to be 0-distributive. These conditions are both necessary and sufficient in the case of a lattice.

Clearly $2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 10$, $5 \Rightarrow 6 \Rightarrow 8$, $7 \Rightarrow 8$, and $9 \Rightarrow 10$. Hence it is enough to prove that 8 and 10 are not necessary.

Let $C$ be an infinite chain without the least element and $S = C \cup \{0, a, b, d\}$. Define an ordering on $S$ as follows: $0 < a, b, d, e$; $a < e$; $a \parallel b$; $a \parallel d$; $b \parallel d$; $b \parallel e$; $d \parallel e$; $a, b, d, e < c$ for all $c \in C$; $e \parallel c$ for all $c \in C$. It is easily seen that $S$ is a 0-distributive semilattice with respect to this ordering. Now $[e] \lor [b] = S = [e] \lor [d]$, so that $((e) \lor (b))^0 \land ((e) \lor (d))^0 = S$. Also $[e] \lor ([b] \land [d]) = [a]$ and hence $((e) \lor ([b] \land [d]))^0 = \{0, b, d\}$. Thus $((e) \lor ([b] \land [d]))^0 \neq ((e) \lor [b])^0 \land ((e) \lor [d])^0$, proving 8 is not necessary.

Consider the 0-distributive semilattice $S$ from Remark 2.11. Now $([a] \land [b])^0 = \{0\}$ and $[a]^0 \land [b]^0 = \{0, d\}$. Thus $([a] \land [b])^0 \neq [a]^0 \land [b]^0$, proving 10 is not necessary.

**Remark 2.13.** The condition 2 of Theorem 2.7 and the conditions 11, 12, 13 of Theorem 2.8 are necessary for a semilattice (not necessarily finite) to be 0-distributive.

**Proof.** The necessity of the condition 2 of Theorem 2.7 is obvious. The necessity of the conditions 11, 12, 13 of Theorem 2.8 follows by Theorem 2.5 [see Theorem 2.5, 10, 8, 11].

**Remark 2.14.** The condition 2 of Theorem 2.7 and the conditions 11, 12, 13 of Theorem 2.8 are not necessary for an infinite semilattice with 0 to be 0-distributive.

Clearly the condition 12 of Theorem 2.8 implies the condition 13 of Theorem 2.8 and the condition 13 of Theorem 2.8 implies the condition 2 of Theorem 2.7. Hence it is enough to show that the conditions 11 and 12 of Theorem 2.8 are not sufficient.

Let $C_1$, $C_2$, $C_3$ be infinite chains without greatest and least elements and let $S = C_1 \cup C_2 \cup C_3 \cup \{0, a, b, c, d, e, f, g, 1\}$. Define an ordering on $S$ as follows. $0 < a$, $b, c, d$; $a < e$; $b < f$; $c < g$; $d < e$; $d < f$; $d < g$; $e < c_1 < 1$ for all $c_1 \in C_1$; $e < c_2 < 1$ for all $c_2 \in C_2$; $f < c_1$ for all $c_1 \in C_1$; $f < c_3 < 1$ for all $c_3 \in C_3$; $g < c_2$ for all $c_2 \in C_2$; $g < c_3$ for all $c_3 \in C_3$; $a \parallel b$; $a \parallel c$; $a \parallel d$; $a \parallel f$; $a \parallel g$; $a \parallel c_3$ for all $c_3 \in C_3$; $b \parallel c$; $b \parallel d$; $b \parallel e$; $b \parallel g$; $b \parallel c_2$ for all $c_2 \in C_2$; $c \parallel d$; $c \parallel e$; $c \parallel f$; $c \parallel c_1$ for all $c_1 \in C_1$; $c_1 \parallel c_2$ for all $c_1 \in C_1$ and $c_2 \in C_2$; $c_1 \parallel c_3$ for all $c_1 \in C_1$ and $c_3 \in C_3$; $c_2 \parallel c_3$ for all $c_2 \in C_2$ and $c_3 \in C_3$. Clearly $S$ is a semilattice with respect to this ordering.
Also for all $x, y, z \in S$, we have $((x \cap ((y] \lor (z]))^* = ((x] \cap (y])^* \cap ((x] \cap (z])^*$ and $((x] \lor (y]) \cap ((x] \lor (z])^* = (x]^* \cap ((y] \lor (z])^*$. Now $(d] \cap (a] = (0] = (d] \cap B$ where $B = (b] \lor (c]$. But $(d] \cap ((a] \lor B) \neq (0]$. Thus $S$ is not 0-distributive.

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References


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