Ladislav Nebeský
Hamiltonian colorings of graphs with long cycles


Persistent URL: http://dml.cz/dmlcz/134180

Terms of use:

© Institute of Mathematics AS CR, 2003

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.
HAMiltonian colorings of graphs with long cycles

Ladislav Nebeský, Praha

(Received April 11, 2002)

Abstract. By a hamiltonian coloring of a connected graph $G$ of order $n \geq 1$ we mean a mapping $c$ of $V(G)$ into the set of all positive integers such that $|c(x) - c(y)| \geq n - 1 - D_G(x, y)$ (where $D_G(x, y)$ denotes the length of a longest $x - y$ path in $G$) for all distinct $x, y \in G$. In this paper we study hamiltonian colorings of non-hamiltonian connected graphs with long cycles, mainly of connected graphs of order $n \geq 5$ with circumference $n - 2$.

Keywords: connected graphs, hamiltonian colorings, circumference

MSC 2000: 05C15, 05C38, 05C45, 05C78

Research supported by Grant Agency of the Czech Republic, grant No. 401/01/0218.
The *hamiltonian chromatic number* $hc(G)$ of $G$ is defined by

$$hc(G) = \min\{hc(c); c \text{ is a hamiltonian coloring of } G\}.$$  

Fig. 1 shows four connected graphs of order six, each of them with a hamiltonian coloring.

The notions of a hamiltonian coloring and the hamiltonian chromatic number of a connected graph were introduced by G. Chartrand, L. Nebeský and P. Zhang in [2]. These concepts have a transparent motivation: a connected graph $G$ is hamiltonian-connected if and only if $hc(G) = 1$.

The following useful result on the hamiltonian chromatic number was proved in [2]; its proof is easy.

**Proposition 1.** Let $G_1$ and $G_2$ be connected graphs. If $G_1$ is spanned by $G_2$, then $hc(G_1) \leq hc(G_2)$.

It was proved in [2] that

$$hc(G) \leq (n - 2)^2 + 1$$

for every connected graph $G$ of order $n \geq 2$ and that $hc(S) = (n - 2)^2 + 1$ for every star $S$ of order $n \geq 2$. These results were extended in [3]: there exists no connected graph of order $n \geq 5$ with $hc(G) = (n - 2)^2$, and if $T$ is a tree of order $n \geq 5$ obtained from a star of order $n - 1$ by inserting a new vertex into an edge, then $hc(T) = (n - 2)^2 - 1$.

The following definition will be used in the next sections. Let $G$ be a connected graph containing a cycle; by the circumference of $G$ we mean the length of a longest cycle in $G$; similarly as in [2] and [3], the circumference of $G$ will be denoted by $cir(G)$. If $G$ is a tree, then we put $cir(G) = 0$.

1. It was proved in [2] that if $G$ is a cycle of order $n \geq 3$, then $hc(G) = n - 2$. Proposition 1 implies that if $G$ is a hamiltonian graph of order $n \geq 3$, then $hc(G) \leq n - 2$.  

264
As was proved in [2], if $G$ is a connected graph of order $n \geq 4$ such that $\text{cir}(G) = n - 1$ and $G$ contains a vertex of degree 1, then $\text{hc}(G) = n - 1$. Thus, by Proposition 1, if $G$ is a connected graph of order $n \geq 4$ such that $\text{cir}(G) = n - 1$, then $\text{hc}(G) \leq n - 1$.

Consider arbitrary $j$ and $n$ such that $j \geq 0$ and $n - j \geq 3$. We denote by $\text{hc}_{\text{max}}(n, j)$ the maximum integer $i \geq 1$ with the property that there exists a connected graph $G$ of order $n$ such that $\text{cir}(G) = n - j$ and $\text{hc}(G) = i$.

As follows from the results of [2] mentioned above,

$$\text{hc}_{\text{max}}(n, 0) = n - 2 \text{ for every } n \geq 3$$

and

$$\text{hc}_{\text{max}}(n, 1) = n - 1 \text{ for every } n \geq 4.$$ 

Using Proposition 1, it is not difficult to show that $\text{hc}_{\text{max}}(5, 2) = 6$. Combining Proposition 1 with Fig. 1 we easily get $\text{hc}_{\text{max}}(6, 2) \leq 10$. In this section, we will find an upper bound of $\text{hc}_{\text{max}}(n, 2)$ for $n \geq 7$.

Let $n \geq 7$, let $0 \leq i \leq \lfloor \frac{1}{3}(n - 2) \rfloor$, and let $V$ be a set of $n$ elements, say elements $u_0, u_1, \ldots, u_{n-4}, u_{n-3}, v, w$. We denote by $F(n, i)$ the graph defined as follows: $V(F(n, i)) = V$ and

$$E(F(n, i)) = \{u_0u_1, u_1u_2, \ldots, u_{n-4}u_{n-3}, u_{n-3}u_0\} \cup \{u_0v, u_iv\}.$$

**Lemma 1.** Let $n \geq 7$. Then there exists a hamiltonian coloring $c_i$ of $F(n, i)$ with

$$\text{hc}(c_i) = 3n - \left\lfloor \frac{1}{3}(n - 2) \right\rfloor - 6 - i$$

for each $i$, $0 \leq i \leq \left\lfloor \frac{1}{3}(n - 2) \right\rfloor$.

**Proof.** Put $j = \left\lfloor \frac{1}{3}(n - 2) \right\rfloor$. Let $0 \leq i \leq j$. Consider a mapping $c_i$ of $V(F(n, i))$ into $\mathbb{N}$ defined as follows:

$$c_i(u_0) = n - 1, \quad c_i(u_1) = n - 3, \quad \ldots, \quad c_i(u_{j-1}) = n - 2(j - 1) - 1,$$

$$c_i(u_j) = n - 2j - 1, \quad c_i(u_{j+1}) = 3n - 2j - 7, \quad c_i(u_{j+2}) = 3n - 2j - 9, \quad \ldots,$$

$$c_i(u_{n-4}) = n + 3, \quad c_i(u_{n-3}) = n + 1, \quad c_i(v) = 1 \quad \text{and} \quad c_i(w) = 3n - j - 6 - i.$$ 

(A diagram of $F(21, 0)$ with $c_0$ can be found in Fig. 2.)

Consider arbitrary distinct vertices $r$ and $s$ of $F(n, i)$ such that $c_i(r) \geq c_i(s)$. Put $D'_i(r, s) = D'_{F(n, i)}(r, s)$. Obviously, $c_i(r) > c_i(s)$. If $(r, s) = (u_{j+1}, u_j)$ or $(u_{n-3}, u_0)$ or $(u_{f+1}, u_f)$, where $0 \leq f \leq n - 4$, then $c_i(r) - c_i(s) = D'_i(r, s)$. If $(r, s) = (u_j, v)$, then $D'_i(r, s) + 2 \geq c_i(r) - c_i(s) \geq D'_i(r, s)$. Otherwise, $c_i(r) - c_i(s) > D'_i(r, s)$. Thus $c_i$ is a hamiltonian coloring of $F(n, i)$. We see that $\text{hc}(c_i) = c_i(w)$. \hfill $\square$

Let $n \geq 7$. We define $F'(n) = F(n, 0) - u_0w + vw$. 

265
Corollary 1. Let \( n \geq 7 \). Then there exists a hamiltonian coloring \( c'_0 \) of \( F'(n) \) with \( \text{hc}(c'_0) = 3n - \lfloor \frac{1}{3}(n - 2) \rfloor - 7 \).

Proof. Put \( c'_0 = c_1 \), where \( c_1 \) is defined in the proof of Lemma 1. It is clear that \( c'_0 \) is a hamiltonian coloring of \( F'(n) \). Applying Lemma 1, we get the desired result. \( \square \)

Lemma 2. Let \( n \geq 7 \). Then there exists a hamiltonian coloring \( c^+_i \) of \( F(n, i) \) with
\[
\text{hc}(c^+_i) = 2n - 4 + 2\lfloor \frac{1}{2}(n - 2) \rfloor - i
\]
for each \( i, \frac{1}{3}(n - 2) + 1 \leq i \leq \frac{1}{2}(n - 2) \).

Proof. Put \( j = \frac{1}{2}(n - 2) \) and \( k = \frac{1}{2}(n - 2) \). Let \( j + 1 \leq i \leq k \). Consider a mapping \( c^+_i \) of \( V(F(n, i)) \) into \( \mathbb{N} \) defined as follows:
\[
\begin{align*}
c^+_i(u_0) &= 3k + 1, \quad c^+_i(u_1) = 3k - 1, \quad \ldots, \quad c^+_i(u_{k-1}) = k + 3, \quad c^+_i(u_k) = k + 1, \\
c^+_i(u_{k+1}) &= 2(n - 3) + k + 1, \quad c^+_i(u_{k+2}) = 2(n - 3) + k - 1, \quad \ldots, \\
c^+_i(u_{n-4}) &= 3k + 5, \quad c^+_i(u_{n-3}) = 3k + 3, \quad c^+_i(v) = 1 \quad \text{and} \quad c^+_i(w) = 2n - 4 + 2k - i.
\end{align*}
\]
(A diagram of \( F(21, 7) \) with \( c^+_7 \) can be found in Fig. 3.)

Put \( D'_i = D'_{F(n, i)} \). We see that \( c^+_i(u_k) - c^+_i(v) = D'_i(u_k, v) \) and \( c^+_i(w) - c^+_i(u_{k+1}) = D'_i(w, u_{k+1}) \). It is easy to show that \( c^+_i \) is a hamiltonian coloring of \( F(n, i) \). We have \( \text{hc}(c^+_i) = c^+_i(w) \). \( \square \)

Theorem 1. Let \( n \geq 7 \). Then
\[
\text{hc}_{\text{max}}(n, 2) \leq 3n - \lfloor \frac{1}{3}(n - 2) \rfloor - 6.
\]

Proof. Consider an arbitrary connected graph \( G \) of order \( n \) with \( \text{cir}(G) = n - 2 \). Obviously, \( G \) is spanned by a connected graph \( F \) such that \( \text{cir}(F) = n - 2 \) and \( F \)
has exactly one cycle. By Proposition 1, $hc(G) \leq hc(F)$. Thus we need to show that $hc(F) \leq 3n - \lfloor \frac{1}{3}(n - 2) \rfloor - 6$.

If $F$ is isomorphic to $F'(n)$, then the result follows from Corollary 1. Let $F$ be not isomorphic to $F'(n)$. Then there exists $i, 0 \leq i \leq \lfloor \frac{1}{3}(n - 2) \rfloor$, such that $F$ is isomorphic to $F(n, i)$. If $0 \leq i \leq \lfloor \frac{1}{3}(n - 2) \rfloor$, then the result follows from Lemma 1. Let $\lfloor \frac{1}{3}(n - 2) \rfloor \leq i \leq \lfloor \frac{1}{2}(n - 2) \rfloor$. By Lemma 2, $hc(F) \leq 2n - 4 + 2\lfloor \frac{1}{3}(n - 2) \rfloor - i \leq 2n - 4 + 2\lfloor \frac{1}{2}(n - 2) \rfloor - \lfloor \frac{1}{3}(n - 2) \rfloor - 1 \leq 3n - \lfloor \frac{1}{3}(n - 2) \rfloor - 7$, which completes the proof. \hfill \Box

**Corollary 2.** Let $n \geq 7$. Then

$$hc_{\text{max}}(n, 2) \leq \frac{1}{3}(8n - 14).$$

2. Consider arbitrary $j$ and $n$ such that $j \geq 0$ and $n - j \geq 3$. We denote by $hc_{\text{min}}(n, j)$ the minimum integer $i \geq 1$ with the property that there exists a connected graph $G$ of order $n$ such that $\text{cir}(G) = n - j$ and $hc(G) = i$. Since every hamiltonian-connected graph of order $\geq 3$ is hamiltonian, we get $hc_{\text{min}}(n, 0) = 1$ for every $n \geq 3$. In this section we will find an upper bound of $hc_{\text{min}}(n, j)$ for $j \geq 1$ and $n \geq j(j+3)+1$.

We start with two auxiliary definitions. If $U$ is a set, then we denote

$$E_{\text{com}}(U) = \{A \subseteq U; |A| = 2\}.$$

If $W_1$ and $W_2$ are disjoint sets, then we denote

$$E_{\text{combi}}(W_1, W_2) = \{A \in E_{\text{com}}(W_1 \cup W_2); |A \cap W_1| = 1 = |A \cap W_2|\}.$$
Lemma 3. Consider arbitrary $j, k$ and $n$ such that $j \geq 1, k \geq j + 1$, and

$$k + j(k + 1) \leq n \leq k + (k - 1)^2 + 2j.$$

Then there exists a $k$-connected graph $G$ of order $n$ such that $\text{cir}(G) = n - j$ and $\text{hc}(G) \leq 2j(k - 1) + 1$.

Proof. Clearly, there exist $f_1, \ldots, f_{k - 1}$ such that

$$j \leq f_g \leq k - 1 \quad \text{for all } g, \ 0 \leq g \leq k - 1$$

and

$$f_1 + \ldots + f_{k - 1} = n - 2j - k.$$

Consider pairwise disjoint finite sets $U, W_1, \ldots, W_k$ and $W_{k+1}$ such that $|U| = k,$

$$|W_g| = f_g \quad \text{for each } g, 0 \leq g \leq k - 1$$

and $|W_k| = |W_{k+1}| = j.$ We denote by $G$ the graph with

$$V(G) = U \cup W_1 \cup \ldots W_k \cup W_{k+1}$$

and

$$E(G) = E_{\text{com}}(V_1) \cup \ldots \cup E_{\text{com}}(V_{k+1}) \cup E_{\text{combi}}(U, V_1 \cup \ldots \cup V_{k+1}).$$

It is easy to see that $G$ is a $k$-connected graph of order $n$ and $\text{cir}(G) = n - j$.

Put $D'(x, y) = D'_G(x, y)$ for $x, y \in U$. It is clear that

$$D'(u, u^*) = 2j$$

for all distinct $u, u^* \in U,$

$$D'(u, w) = j$$

for all $u \in U$ and $w \in W_1 \cup \ldots \cup W_{k+1},$

$$D'(w, w^*) = 0$$

for all $w$ and $w^*$ such that there exist distinct $g, g^* \in \{1, \ldots, k+1\}$ such that $w \in W_g$ and $w^* \in W_{g^*},$ and

$$D'(w, w^*) = j$$

for all distinct $w$ and $w^*$ such that there exists $h \in \{1, \ldots, k+1\}$ such that $w, w^* \in W_h.$

268
Put $f_k = f_{k+1} = j$. Consider a mapping $c$ of $V(G)$ into $\mathbb{N}$ with the properties that

$$c(U) = \{1, 2j + 1, 4j + 1, \ldots, 2j(k - 1) + 1\}$$

and

$$c(W_g) = \{ j + 1, 3j + 1, \ldots, 2j(f_g - 1) + j + 1 \}$$

for each $g, 1 \leq g \leq k + 1$. It is easy to see that $c$ is a hamiltonian coloring of $G$. Hence $hc(G) \leq hc(c) = 2j(k - 1) + 1$. □

**Theorem 2.** Let $n$ and $j$ be integers such that $j \geq 1$ and $n \geq j(j + 3) + 1$, and let $k$ be the smallest integer such that

$$k \geq j + 1 \; \text{and} \; (k - 1)^2 + k \geq n - 2j.$$  

Then

$$hc_{\text{min}}(n, j) \leq 2j(k - 1) + 1.$$  

**Proof.** The theorem immediately follows from Lemma 3. □

**Corollary 3.** Let $n \geq 5$ and let $k$ be the smallest integer such that

$$k \geq 2 \; \text{and} \; n \leq (k - 1)^2 + k + 2.$$  

Then

$$hc_{\text{min}}(n, 1) \leq 2k - 1.$$  

**Corollary 4.** Let $n \geq 11$ and let $k$ be the smallest integer such that

$$k \geq 3 \; \text{and} \; n \leq (k - 1)^2 + k + 4.$$  

Then

$$hc_{\text{min}}(n, 2) \leq 4k - 3.$$  

3. As follows from results obtained in [2], if (a) $n \geq 3$, then for every $k \in \{1, 2, \ldots, n-1\}$ there exists a connected graph $G$ of order $n \geq 4$ such that $hc(G) = k$, and if (b) $G$ is a graph of order $n$ such that $hc(G) \geq n$, then $\text{cir}(G) \neq n, n - 1$.

For $n = 4$ or $5$, it is easy to find a connected graph of order $n$ with $hc(G) = n$: $hc(P_4) = 4$ and $hc(2K_2 + K_1) = 5$. On the other hand, there exists no connected graph of order 6 with $hc(G) = 6$. We can state the following question: Given $n \geq 7$,
does there exist a connected graph $G$ of order $n$ with $hc(G) = n$? Answering this question for $n \geq 8$ is the subject of the present section.

Let $1 \leq j \leq i$. Consider mutually distinct elements $r, s, u, v, w$ and finite sets $X$ and $Y$ such that $|X| = i$, $|Y| = j$ and the sets $X$, $Y$ and $\{r, s, u, v, w\}$ are pairwise disjoint. We define a graph $G(i, j)$ as follows:

$$V(G(i, j)) = X \cup Y \cup \{r, s, u, v, w\} \text{ and } E(G(i, j))$$

$$= \{uv\} \cup E_{com}(X) \cup E_{com}(Y) \cup E_{combi}(\{u, w\}, X \cup \{r\})$$

$$\cup E_{combi}(\{v, w\}, Y \cup \{s\}).$$

Obviously, $\operatorname{cir}(G(i, j)) = i + j + 3 = |V(G(i, j)| - 2$.

**Proposition 2.** Let $1 \leq j \leq i$. Put $D'(t_1, t_2) = D'_{G(i, j)}(t_1, t_2)$ for all $t_1, t_2 \in V(G(i, j))$. Then

1. $D'(x, y) = 0$ for all $x \in X$ and all $y \in Y$,
2. $D'(x, s) = 0, D'(x, r) = D'(x, v) = 1$ and $D'(x, u) = D'(x, w) = 2$
   for all $x \in X$,
3. $D'(y, r) = 0, D'(y, s) = D'(y, u) = 1$ and $D'(y, v) = D'(y, w) = 2$
   for all $y \in Y$,
4. $D'(x_1, x_2) = 2$ for all distinct $x_1, x_2 \in X$,
5. $D'(y_1, y_2) = 2$ for all distinct $y_1, y_2 \in Y$,
6. $D'(r, s) = 0$,
7. $D'(r, v) = D'(s, u) = 1$,
8. $D'(u, v) = 2$,
9. $D'(s, v) = D'(s, w) = j + 1$,
10. $D'(v, w) = j + 2$,
11. $D'(r, u) = D'(r, w) = \min(i + 1, j + 2)$,

and

12. $D'(u, w) = \min(i + 2, j + 3)$.

**Proof** is easy.

270
Lemma 4. Let $1 \leq j \leq i$. Then $hc(G(i, j)) \geq i + j + 5$.

Proof. Suppose, to the contrary, that there exists a hamiltonian coloring $c$ of $G(i, j)$ such that $hc(c) \leq i + j + 4$. Thus $hc(c) \leq 2i + 4$. We may assume that there exists $t \in V(G(i, j))$ such that $c(t) = 1$.

Put $X^+ = X \cup \{u, w\}$. By virtue of (2), (4) and (12),

(13) $|c(x_1^+) - c(x_2^+)| \geq 2$ for all distinct $x_1^+, x_2^+ \in X^+$.

By virtue of (2), (7) and (12),

(14) $c(r) \neq c(x^+) \neq c(v)$ for all $x^+ \in X^+$,

(15) $c(r) \neq c(v), \ c(s) \neq c(u)$

and

$|c(u) - c(v)| \geq 2$.

Obviously, $|X^+| = i + 2$. As follows from (13),

(16) $\max c(X^+) \geq 2i + 2 + \min c(X^+)$.

Thus $hc(c) \geq 2i + 3$. Since $hc(c) \leq i + j + 4$, we get

(17) $i - 1 \leq j \leq i$.

If $\{c(r), c(v)\} = \{1, 2\}$, then (14) implies that $\max c(X^+) \geq 2i + 5$; a contradiction. If $\{c(r), c(v)\} = \{hc(c), hc(c) - 1\}$, then $\max c(X^+) \leq 2i + 2$; a contradiction. Thus

(18) $\{1, 2\} \neq \{c(r), c(v)\} \neq \{hc(c), hc(c) - 1\}$.

Moreover, if

$c(u) = \min c(X^+) \text{ and } c(v) = c(u) + 2$

or

$c(u) = \max c(X^+) \text{ and } c(v) = c(u) - 2$,

then $\max c(X^+) \geq 2i + 3 + \min c(X^+)$.

Combining (11) and (12) with (17), we have

(19) $|c(r) - c(u)| \geq i + 1, |c(r) - c(w)| \geq i + 1 \text{ and } |c(u) - c(w)| \geq i + 2$.

We denote by $c'$ a mapping of $V(G(i, j))$ into $\mathbb{N}$ defined as follows:

$c'(t) = hc(c) + 1 - c(t)$ for each $t \in V(G(i, j))$. 

271
We see that $c'$ is a hamiltonian coloring of $G(i, j)$ and that $hc(c') = hc(c)$. Obviously, $c(u) \leq c(v)$ or $c'(u) \leq c'(v)$. Without loss of generality we assume that $c(u) \leq c(v)$. Thus

$$c(v) \geq c(u) + 2$$

and if $c(u) = 1$ and $hc(c) = 2i + 3$, then $c(v) \geq 4$.

We distinguish two cases.

Case 1. Assume that $j = i - 1$. Then $hc(c) = 2i + 3$. By virtue of (9) and (10),

$$|c(s) - c(v)| \geq i, |c(s) - c(w)| \geq i \quad \text{and} \quad |c(v) - c(w)| \geq i + 1.$$  

If $c(r) < c(u) < c(w)$ or $c(r) < c(w) < c(u)$ or $c(u) < c(w) < c(r)$ or $c(w) < c(u) < c(r)$, then (19) implies that $hc(c) \geq 2i + 4$, which is a contradiction.

Let $c(w) < c(r) < c(u)$. As follows from (19), $c(u) = 2i + 3$ and therefore $c(v) \geq 2i + 5$; a contradiction.

Finally, let $c(u) < c(r) < c(w)$. Thus $c(w) = 2i + 3$ and therefore $c(u) = 1$ and $c(r) = i + 2$. Since $c(u) = 1$ and $hc(c) = 2i + 3$, we get $c(v) \geq 4$. If $c(v) < c(s)$, then $c(s) \geq i + 4$ and therefore $|c(s) - c(w)| \leq i - 1$; a contradiction. Let $c(s) < c(v)$. Since $c(s) \neq c(u)$, we have $c(s) \geq 2$. This implies that $c(v) \geq i + 2$. Since $c(w) = 2i + 3$, we get $c(v) = i + 2$. Thus $c(v) = c(r)$, which contradicts (15).

Case 2. Assume that $i = j$. Recall that $hc(c) \leq 2i + 4$. By virtue of (9) and (10),

$$|c(s) - c(v)| \geq i + 1, |c(s) - c(w)| \geq i + 1 \quad \text{and} \quad |c(v) - c(w)| \geq i + 2.$$  

If $c(r) < c(w) < c(u)$ or $c(w) < c(r) < c(u)$, then (19) implies that $c(u) \geq 2i + 3$ and therefore $c(v) \geq 2i + 5$, which is a contradiction.

Let $c(r) < c(u) < c(w)$. Then $c(w) = 2i + 4$ and therefore $c(r) = 1$ and $c(u) = i + 2$. This implies that $c(v) \geq i + 4$ and therefore $|c(v) - c(w)| \leq i$; a contradiction.

Let $c(u) < c(w) < c(r)$. Then $c(u) = 1$, $c(w) = i + 3$ and $c(r) = 2i + 4$. Since $3 \leq c(v) \neq c(r)$, we get $|c(v) - c(w)| \leq i$; a contradiction.

Let $c(w) < c(u) < c(r)$. Then $c(w) = 1$, $c(u) = i + 3$ and $c(r) = 2i + 4$. Assume that $c(s) < c(v)$; since $c(w) = 1$, we get $c(s) \geq i + 2$ and therefore $c(v) \geq 2i + 3$; since $c(r) = 2i + 4$ and $c(v) \neq c(r)$, we get $c(v) = 2i + 3$, which contradicts (18). Assume that $c(v) < c(s)$; since $c(u) = i + 3$, we get $c(v) \geq i + 5$ and therefore $c(s) \geq 2i + 6$; a contradiction.

Finally, let $c(u) < c(r) < c(w)$. Then $c(w) \geq 2i + 3$. If $c(v) < c(s)$, then $c(v) \geq 3$ and $c(s) \geq i + 4$ and therefore $c(w) \geq 2i + 5$; a contradiction. Assume that $c(s) < c(v)$.
If $c(s) \geq 2$, then $c(v) \geq i + 3$ and therefore $c(w) \geq 2i + 5$, which is a contradiction.

Let $c(s) = 1$. Then $c(u) = 2$, $c(r) = i + 3$ and $c(w) = 2i + 4$. This implies that $c(v) = i + 2$. Obviously, $\min c(X^+) = 2$. Since $c(v) = i + 2$ and $c(r) = i + 3$, we see that $c(x^+) \not\in \{i + 2, i + 3\}$ for each $x^+ \in X^+$. Therefore $\max c(X^+) \geq 2i + 3 + \min c(X^+) = 2i + 5$, which is a contradiction.

Thus the proof of the lemma is complete. □

**Theorem 3.** For every $n \geq 8$, there exists a connected graph $G$ of order $n$ with $\text{cir}(G) = n - 2$ and $\text{hc}(G) = n$.

**Proof.** For every $f$ and $h$ such that $f \leq h$ we define

$$\text{EVEN}[f, h] = \{g; f \leq g \leq h, g \text{ is even}\}$$

and

$$\text{ODD}[f, h] = \{g; f \leq g \leq h, g \text{ is odd}\}.$$

We will use graphs $G(i, j)$ in the proof.

Consider an arbitrary $n \geq 8$. We distinguish four cases.

**Case 1.** Let $n = 4f + 8$, where $f \geq 0$. Put

$$G_1 = G(2f + 2, 2f + 1).$$

Then the order of $G_1$ is $n$. Let $c_1$ be an injective mapping of $V(G_1)$ into $\mathbb{N}$ such that

$$c_1(r) = c_1(s) = 2f + 5, \quad c_1(u) = 1, \quad c_1(v) = 3, \quad c_1(w) = 4f + 8,$$

$$c_1(X) = \text{EVEN}[4, 4f + 6] \quad \text{and} \quad c_1(Y) = \text{EVEN}[6, 4f + 6].$$

(For $f = 0$, $G_1$ and $c_1$ are presented in Fig. 4.) Combining (1)–(12) with the definition of a hamiltonian coloring, we see that $c_1$ is a hamiltonian coloring of $G_1$. Clearly, $\text{hc}(c_1) = 4f + 8 = n$. Lemma 4 implies that $\text{hc}(c_1) = \text{hc}(G_1)$. Thus $\text{hc}(G_1) = n$. 

\[\text{Fig. 4}\]
Case 2. Let $n = 4f + 9$, where $f \geq 0$. Put

$$G_2 = G(2f + 2, 2f + 2).$$

Then the order of $G_2$ is $n$. Let $c_2$ be an injective mapping of $V(G_2)$ into $\mathbb{N}$ such that

$$c_2(r) = c_2(s) = 2f + 6, \quad c_2(u) = 1,$$

$$c_2(v) = 3, \quad c_2(w) = 4f + 9 \quad \text{and} \quad c_2(X) = c_2(Y) = \text{ODD}[5, 4f + 7].$$

(For $f = 0$, $G_2$ and $c_2$ are presented in Fig. 5.) By virtue of (1)–(12), $c_2$ is a hamiltonian coloring of $G_2$. Obviously, $hc(c_2) = n$. As follows from Lemma 4, $hc(G_2) = n$.

Case 3. Let $n = 4f + 10$, where $f \geq 0$. Put

$$G_3 = G(2f + 3, 2f + 2).$$

The order of $G_3$ is $n$. Let $c_3$ be an injective mapping of $V(G_3)$ into $\mathbb{N}$ such that

$$c_3(r) = 2f + 5, \quad c_3(s) = 2f + 6, \quad c_3(u) = 1, \quad c_3(v) = 3, \quad c_3(w) = 4f + 10,$$

$$c_3(X) = \text{EVEN}[4, 4f + 8] \quad \text{and} \quad c_3(Y) = \text{ODD}[5, 4f + 7].$$

(See Fig. 6 for $f = 0$.) By (1)–(12), $c_3$ is a hamiltonian coloring of $G_3$. By Lemma 4, $hc(G_3) = hc(c_3) = n$. 

Fig. 5

Fig. 6
Case 4. Let \( n = 4f + 11 \), where \( n \geq 0 \). Put

\[ G_4 = G(2f + 4, 2f + 2). \]

The order of \( G_4 \) is \( n \) again. Let \( c_4 \) be an injective mapping of \( V(G_4) \) into \( \mathbb{N} \) such that

\[
c_4(r) = 2f + 6, \quad c_4(s) = 2f + 7, \quad c_4(u) = 1, \quad c_4(v) = 4, \quad c_4(w) = 4f + 11, \\
c_4(X) = \text{ODD}[3, 4f + 9] \text{ and } c_4(Y) = \text{EVEN}[6, 4f + 8].
\]

(See Fig. 7 for \( f = 0 \).) Combining (1)–(12) with Lemma 4, we see that \( \text{hc}(G_4) = \text{hc}(c_4) = n \).

Thus the proof is complete. \( \square \)

The author conjectures that there exists no connected graph \( G \) of order 7 such that \( \text{hc}(G) = 7 \).

The author sincerely thanks the referee for helpful comments and suggestions.

References


Author’s address: Ladislav Nebeský, Univerzita Karlova v Praze, Filozofická fakulta, nám. J. Palacha 2, 116 38 Praha 1, e-mail: Ladislav.Nebesky@ff.cuni.cz.