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Mathematica Bohemica, Vol. 128 (2003), No. 3, 319--324

Persistent URL: http://dml.cz/dmlcz/134184

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GENERALIZED DEDUCTIVE SYSTEMS
IN SUBREGULAR VARIETIES

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(Received November 18, 2002)

Abstract. An algebra $\mathcal{A} = (A, F)$ is subregular alias regular with respect to a unary term function $g$ if for each $\Theta, \Phi \in \text{Con} \mathcal{A}$ we have $\Theta = \Phi$ whenever $[g(a)]_\Theta = [g(a)]_\Phi$ for each $a \in A$. We borrow the concept of a deductive system from logic to modify it for subregular algebras. Using it we show that a subset $C \subseteq A$ is a class of some congruence on $\Theta$ containing $g(a)$ if and only if $C$ is this generalized deductive system. This method is efficient (needs a finite number of steps).

Keywords: regular variety, subregular variety, deductive system, congruence class, difference system

MSC 2000: 08A30, 08B05, 03B22

Let $\mathcal{A} = (A, F)$ be an algebra and $\emptyset \neq C \subseteq A$ a subset. The problem to decide whether $C$ is a class of some congruence $\Theta \in \text{Con} \mathcal{A}$ has been a problem of long standing. In general, it was solved by A.I. Mal’cev in 1954. However, his method is far from being effective. Essential progress was done for certain subsets of $A$ for algebras having a constant 0. A. Ursini introduced a concept of an ideal in universal algebra [8] and it was shown by him and H.-P. Gumm [7] that in varieties permutable at 0 every 0-class of each congruence on $\mathcal{A}$ is just an ideal of $\mathcal{A}$ and vice versa. It turns out that for varieties which are permutable at 0 and weakly regular this method is effective, i.e. for a finite algebra of a finite type it can be decided by a finite number of steps of the corresponding algorithmical scheme. This method was extended for an arbitrary congruence class of algebra $\mathcal{A}$ of a regular and permutable variety and it was generalized by the author and R. Bělohlávek [2] to algebras in regular varieties. Recently, we have used another method, the so called deductive systems,

This work is supported by the Research Grant J98:153100011 of the Czech Republic Government.
to characterize 0-classes in weakly regular varieties (see [5]) or arbitrary congruence classes in algebras of regular varieties, see [3].

If the concept of regularity is weakened to the so called subregularity (see e.g. [1]), one can still use an effective method to characterize certain congruence classes. This is the aim of our paper.

Let us recall that an algebra $\mathcal{A} = (A, F)$ is regular if every $\Theta, \Phi \in \text{Con} \mathcal{A}$ coincide whenever they have a class in common. An algebra $\mathcal{A}$ with a constant 0 is weakly regular if every $\Theta, \Phi \in \text{Con} \mathcal{A}$ coincide whenever $[0]_{\Theta} = [0]_{\Phi}$.

These concepts have a common generalization.

**Definition 1.** Let $g$ be a unary term function of an algebra $\mathcal{A} = (A, F)$. $\mathcal{A}$ is regular with respect to $g$ if $\Theta = \Phi$ for $\Theta, \Phi \in \text{Con} \mathcal{A}$ whenever $[g(a)]_{\Theta} = [g(a)]_{\Phi}$ for each $a \in A$. Let $g$ be a unary term of variety $\mathcal{V}$. We say that $\mathcal{V}$ is regular with respect to $g$ if each $\mathcal{A} \in \mathcal{V}$ has this property (with respect to the corresponding term function $g^A$).

Regularity with respect to $g$ is known also under the name subregularity, see [1], provided the term $g$ is implicitly given.

Let us mention that if $g(x) = x$ (the identity term) then regularity with respect to $g$ is the regularity; if 0 is a constant of $\mathcal{A}$ and $g(x) = 0$ then regularity with respect to $g$ is just the weak regularity.

**Definition 2.** Let $g$ be a unary term of a variety $\mathcal{V}$. A finite set $\{p_1, \ldots, p_n\}$ of ternary terms $p_1, \ldots p_n$ of $\mathcal{V}$ is called a $g$-difference system for $\mathcal{V}$ if

$$[p_1(x, y, z) = g(z) \& \ldots \& p_n(x, y, z) = g(z)] \text{ if and only if } x = y.$$

**Example.** If $g(z) = 0$ where 0 is a constant of $\mathcal{V}$ then the $g$-difference system is just the Gödel equivalence system as introduced in [4] (of course, then every $p_i(x, y, z)$ is independent of the last variable thus it is properly binary). If $g(z) = z$ then we have the difference system as introduced in [3].

If $g(z) = z$ and $\mathcal{V}$ is a variety of groups then for $p(x, y, z) = x - y + z$ the singleton $\{p\}$ is a $g$-difference system; if $\mathcal{V}$ is the variety of Boolean algebras then $\{p\}$ is a $g$-difference system for $p(x, y, z) = x \oplus y \oplus z$, where $\oplus$ denotes the so called symmetrical difference.

Analogously, if $\mathcal{V}$ is the variety of pseudocomplemented semilattices and $g(x) = x^{**}$ then $\{p\}$ is a $g$-difference system for $p(x, y, z) = (x + y) + z$ where

$$x + y = (((x \land y^*)^* \land (x^* \land y^*))^*).$$
An example of a difference system having more than one term was found for MV-algebras in [3].

The following useful result was proved in [1]:

**Proposition 1.** Let \( g \) be a unary term of a variety \( \mathcal{V} \). Then \( \mathcal{V} \) is regular with respect to \( g \) if and only if there exist ternary terms \( p_1, \ldots, p_m \) such that

\[
\{p_1, \ldots, p_m\} \text{ is a } g \text{-difference system of } \mathcal{V}.
\]

Moreover, every variety \( \mathcal{V} \) which is regular with respect to \( g \) is \( n \)-permutable for some \( n \geq 2 \).

Let us note that \( m \) and \( n \) in Proposition 1 need not coincide. E.g. for groups we have \( n = 2 \) and \( m = 1 \).

In the sequel we will use the following result which is considered to be a folklore but its formal proof can be found in [6]:

**Proposition 2.** A variety \( \mathcal{V} \) is \( n \)-permutable for some \( n \geq 2 \) if and only if for each \( A \in \mathcal{V} \) and every binary relation \( R \) on \( A \) the following implication holds: if \( R \) is reflexive, transitive and compatible then \( R \in \text{Con} \mathcal{A} \).

Recall that a relation \( R \) on an algebra \( \mathcal{A} = (A, F) \) is compatible (with respect to \( F \)) if for each \( n \)-ary \( f \in F \) and \( a_1, \ldots, a_n, b_1, \ldots, b_n \in A \),

\[
\langle a_i, b_i \rangle \in R \ (i = 1, \ldots, n) \Rightarrow \langle f(a_1, \ldots, a_n), f(b_1, \ldots, b_n) \rangle \in R;
\]

in other words, \( R \) is compatible if it is a subalgebra of the square \( \mathcal{A} \times \mathcal{A} \).

The crucial concept of our paper is the following one:

**Definition 3.** Let \( g \) be a unary term function of an algebra \( \mathcal{A} = (A, F) \) and let \( t_1, \ldots, t_n \) be ternary term functions of \( \mathcal{A} \), \( z \in A \). A subset \( D \subseteq A \) is called a \((g, z)\)-deductive system of \( \mathcal{A} \) with respect to \( \{t_1, \ldots, t_n\} \) if

(i) \( g(z) \in D \),

(ii) \( a \in D \) and \( t_i(a, b, z) \in D \) for \( i = 1, \ldots, n \) imply \( b \in D \),

(iii) \( a \in D \) implies \( t_i(g(z), a, z) \in D \) for \( i = 1, \ldots, n \).

Let us note that (i) and (ii) imply the converse of (iii), thus

\[
a \in D \iff t_i(g(z), a, z) \in D \text{ for } i = 1, \ldots, n.
\]

**Example.** Let “\( \Rightarrow \)” be the connective implication of an arbitrary (e.g. classical, non-classical, intuitionistic, multiple-valued, etc.) logic and \( D \) the subset of “tautologies”. Then for \( g(z) = 1 \) (the tautology) and \( n = 1, t_1(x, y, z) := x \Rightarrow y \) we surely have
1 ∈ D,

a ∈ D and (a ⇒ b) ∈ D implies b ∈ D,

Let R be a binary relation on a set A and x ∈ A. Denote [x]_R = {a ∈ A; ⟨a, x⟩ ∈ R}.

**Definition 4.** Let t_1, ..., t_n be ternary term functions of an algebra \( \mathcal{A} = (A, F) \) and \( D ⊆ A \), z ∈ A. Define a binary relation \( \Theta_{D,z} \) on A induced by \{t_1, ..., t_n\} as follows:

\[(*) \quad ⟨a, b⟩ ∈ \Theta_{D,z} \text{ if and only if } t_i(b, a, z) ∈ D \text{ for } i = 1, ..., n.\]

We are ready to characterize the classes \([g(z)]_{\Theta_{D,z}}\) of \( \Theta_{D,z} \).

**Lemma 1.** Let \( t_1, ..., t_n \) be ternary term functions of an algebra \( \mathcal{A} = (A, F) \), let \( g \) be a unary term function of \( \mathcal{A} \) and \( z ∈ A \). If \( D \) is a \((g, z)\)-deductive system of \( \mathcal{A} \) with respect to \( \{t_1, ..., t_n\} \) and \( \Theta_{D,z} \) is induced by \( \{t_1, ..., t_n\} \) then \( D = [g(z)]_{\Theta_{D,z}} \).

**Proof.** Let \( a ∈ D \). By (iii) we have \( t_i(g(z), a, z) ∈ D \) for \( i = 1, ..., n \) and, by (*), \( ⟨a, g(z)⟩ ∈ \Theta_{D,z} \) which yields \( a ∈ [g(z)]_{\Theta_{D,z}} \). Conversely, if \( a ∈ [g(z)]_{\Theta_{D,z}} \) then \( ⟨a, g(z)⟩ ∈ \Theta_{D,z} \), thus \( t_i(g(z), a, z) ∈ D \) for \( i = 1, ..., n \). Applying (i) we infer \( g(z) ∈ D \) and, by virtue of (ii), also \( a ∈ D \). Together, \( D = [g(z)]_{\Theta_{D,z}} \). \( \square \)

**Lemma 2.** Let \( t_1, ..., t_n \) be ternary term functions of an algebra \( \mathcal{A} = (A, F) \), let \( g \) be a unary term function of \( \mathcal{A} \) and \( z ∈ A \), \( D ⊆ A \). Let \( \Theta_{D,z} \) be induced by \( \{t_1, ..., t_n\} \). If \( \Theta_{D,z} \) is reflexive and transitive and \( D = [g(z)]_{\Theta_{D,z}} \) then \( D \) is a \((g, z)\)-deductive system of \( \mathcal{A} \) with respect to \( \{t_1, ..., t_n\} \).

**Proof.** Suppose \( a ∈ D \) and \( t_i(a, b, z) ∈ D \) for \( i = 1, ..., n \). Then \( ⟨b, a⟩ ∈ \Theta_{D,z} \).

Since \( D = [g(z)]_{\Theta_{D,z}} \), also \( ⟨a, g(z)⟩ ∈ \Theta_{D,z} \). Due to transitivity of \( \Theta_{D,z} \), we have \( b ∈ [g(z)]_{\Theta_{D,z}} \), i.e. \( D \) satisfies (ii) of Definition 3. The condition (i) follows by reflexivity of \( \Theta_{D,z} \) (since \( g(z) ∈ [g(z)]_{\Theta_{D,z}} = D \)).

If \( a ∈ D \) then \( ⟨a, g(z)⟩ ∈ \Theta_{D,z} \), thus \( t_i(g(z), a, z) ∈ D \) for \( i = 1, ..., n \), i.e. \( D \) satisfies also (iii) and hence it is a \((g, z)\)-deductive system of \( \mathcal{A} \) w.r.t. \( \{t_1, ..., t_n\} \). \( \square \)

Since congruences are compatible relations on an algebra \( \mathcal{A} = (A, F) \), we must respect also the substitution property (with respect to F) to describe their classes. Hence, we define:

**Definition 5.** Let \( g \) be a unary and \( p_1, ..., p_n \) \( n \)-ary term functions of an algebra \( \mathcal{A} = (A, F) \). We say that \( D ⊆ A \) is a compatible \((g, z)\)-deductive system of \( \mathcal{A} \).
with respect to \( \{p_1, \ldots, p_n\} \) if \( D \) is a \((g, z)\)-deductive system of \( \mathcal{A} \) with respect to \( \{p_1, \ldots, p_n\} \) and for each \( k \)-ary operation \( f \in F \) and every \( a_1, \ldots, a_k, b_1, \ldots, b_k \in A \) the following implication holds:

\[
\begin{align*}
\text{if } p_i(a_1, b_1, z) \in D, \ldots, p_i(a_k, b_k, z) \in D & \text{ for } i = 1, \ldots, n \\
\text{then } p_i(f(a_1, \ldots, a_k), f(b_1, \ldots, b_k), z) \in D & \text{ for } i = 1, \ldots, n.
\end{align*}
\]

**Theorem 1.** Let \( g \) be a unary term of a variety \( \mathcal{V} \) and \( \{p_1, \ldots, p_n\} \) a \( g \)-difference system for \( \mathcal{V} \). Let \( \mathcal{A} = (A, F) \in \mathcal{V} \), \( \Theta \in \text{Con} \mathcal{A} \), \( z \in A \) and \( D = [g(z)]_\Theta \). Then

(a) \( \Theta_{D,z} = \Theta \);

(b) \( D \) is a compatible \((g, z)\)-deductive system of \( \mathcal{A} \) with respect to \( \{p_1, \ldots, p_n\} \).

**Proof.** If \( \langle a, b \rangle \in \Theta_{D,z} \) then \( p_i(b, a, z) \in D = [g(z)]_\Theta \) for \( i = 1, \ldots, n \) and hence \( \langle p_i(b, a, z), g(z) \rangle \in \Theta \). Applying Proposition 1, we infer \( \langle b, a \rangle \in \Theta \), thus also \( \langle a, b \rangle \in \Theta \) proving \( \Theta_{D,z} \subseteq \Theta \).

Conversely, if \( \langle a, b \rangle \in \Theta \) then \( \langle b, a \rangle \in \Theta \) and, by Proposition 1 again, \( \langle p_i(b, a, z), g(z) \rangle \in \Theta \) for \( i = 1, \ldots, n \), thus \( p_i(b, a, z) \in [g(z)]_\Theta = D \). By \((\ast)\) of Definition 4 we conclude \( \langle a, b \rangle \in \Theta_{D,z} \) giving \( \Theta \subseteq \Theta_{D,z} \). We have shown \( \Theta = \Theta_{D,z} \).

By Lemma 2, \( D \) is a \((g, z)\)-deductive system of \( \mathcal{A} \) with respect to \( \{p_1, \ldots, p_n\} \). Since \( \Theta \in \text{Con} \mathcal{A} \) is compatible, it is an easy exercise to show that also \( D \) is compatible.

**Theorem 2.** Let \( g \) be a unary term of a variety \( \mathcal{V} \) and \( \{p_1, \ldots, p_n\} \) a \( g \)-difference system for \( \mathcal{V} \). Let \( \mathcal{A} = (A, F) \in \mathcal{V} \), \( z \in A \) and let \( D \) be a compatible \((g, z)\)-deductive system of \( \mathcal{A} \) with respect to \( \{p_1, \ldots, p_n\} \). Then the relation \( \Theta_{D,z} \) induced by \( \{p_1, \ldots, p_n\} \) is a congruence on \( \mathcal{A} \) and \( D = [g(z)]_{\Theta_{D,z}} \).

**Proof.** By Proposition 1, \( \mathcal{V} \) satisfies \( p_i(x, x, z) = g(z) \) for \( i = 1, \ldots, n \) and hence the relation \( \Theta_{D,z} \) induced by \( \{p_1, \ldots, p_n\} \) is reflexive. Since the \((g, z)\)-deductive system \( D \) is compatible, also \( \Theta_{D,z} \) is compatible. Prove transitivity of \( \Theta_{D,z} \): let \( \langle a, b \rangle \in \Theta_{D,z} \) and \( \langle b, c \rangle \in \Theta_{D,z} \). Then \( p_i(b, c, z) \in D \) for \( i = 1, \ldots, n \) and, by virtue of compatibility of \( \Theta_{D,z} \),

\[
\langle a, b \rangle \in \Theta_{D,z} \Rightarrow \langle p_i(c, a, z), p_i(c, b, z) \rangle \in \Theta_{D,z}
\]

whence \( p_j(p_i(c, b, z), p_i(c, a, z), z) \in D \) for \( j = 1, \ldots, n \). However, \( D \) is a \((g, z)\)-deductive system of \( \mathcal{A} \) with respect to \( \{p_1, \ldots, p_n\} \), thus, by \((\ast)\) of Definition 3, we conclude \( p_i(c, a, z) \in D \) for \( i = 1, \ldots, n \). Hence \( \langle a, c \rangle \in \Theta_{D,z} \).

By Proposition 1, \( \mathcal{V} \) is \( m \)-permutable for some \( m \geq 2 \) and, by Proposition 2, \( \Theta_{D,z} \) is also symmetrical. Together, we have \( \Theta_{D,z} \in \text{Con} \mathcal{A} \). By Lemma 1 we conclude \( D = [g(z)]_{\Theta_{D,z}} \).

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**Corollary 1.** Let $\mathcal{V}$ be a variety which is regular with respect to $g$. Then $\mathcal{V}$ has a $g$-difference system $\{p_1, \ldots, p_n\}$ and for each $\mathcal{A} = (A, F) \in \mathcal{V}$, $z \in A$ and $D \subseteq A$, $D$ is a congruence class containing $g(z)$ if and only if $D$ is a $(g, z)$-deductive system of $\mathcal{A}$ with respect to $\{p_1, \ldots, p_n\}$.

Although the involved method of $(g, z)$-deductive systems enables us to characterize only the congruence classes containing $g(a)$ for some $a \in A$ and for $\mathcal{A} = (A, F)$ from a variety which is regular with respect to $g$, this method is effective in the following sense: if $\mathcal{A}$ is finite and of a finite type, we need to verify only a finite number of conditions of Definition 3 and Definition 5. Thus there exists an algorithmical scheme deciding whether a subset $C \subseteq A$ is a congruence class of $\mathcal{A}$ in a finite number of steps. This scheme depends on the computability of functions $p_1, \ldots, p_n$.

Applying the same reasoning and a computation as in [3], we obtain:

**Corollary 2.** Let $\mathcal{V}$ be a variety regular with respect to $g$ and of a finite type with $k$ fundamental operation symbols. Let $\sigma(f_i)$ be the arity of the $i$-th operation symbol $f_i$. Let $\{p_1, \ldots, p_n\}$ be its $g$-difference system. If $\mathcal{A} = (A, F) \in \mathcal{V}$ is finite and $C \subseteq A$, $a \in A$, $g(a) \in C$ and $|A| = m$, $|C| = r$ then there exists an algorithmical scheme for deciding whether $C$ is a congruence class and this scheme needs

$$n \sum_{i=1}^{k} m^{2\sigma_i(f_i)} + k \cdot m^2 \cdot n + r \cdot (m \cdot n + m + n)$$

steps.

**References**


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