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DIAMETER-IN Variant GRAPHS

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Abstract. The diameter of a graph $G$ is the maximal distance between two vertices of $G$. A graph $G$ is said to be diameter-edge-invariant, if $d(G-e) = d(G)$ for all its edges, diameter-vertex-invariant, if $d(G-v) = d(G)$ for all its vertices and diameter-adding-invariant if $d(G+e) = d(e)$ for all edges of the complement of the edge set of $G$. This paper describes some properties of such graphs and gives several existence results and bounds for parameters of diameter-invariant graphs.

Keywords: extremal graphs, diameter of graph

MSC 2000: 05C12, 05C35

1. Introduction

Let $G$ be an undirected, finite graph without loops or multiple edges. Then we denote by: $V(G)$ the vertex set of $G$; $E(G)$ the edge set of $G$; $\overline{G}$ the complement of $G$ with the edge set $E(\overline{G})$; $d_G(u,v)$ (or simply $d(u,v)$) the distance between two vertices $u, v$ in $G$; $e(u)$ the eccentricity of $u$. The radius $r(G)$ is the minimum of the vertex eccentricities, the diameter $d(G)$ is the maximum of the vertex eccentricities; $\text{deg}_G(v)$ is the degree of vertex $v$ in $G$ and $\Delta(G)$ is the maximum degree of $G$. The notions and notations not defined here are used accordingly to the book [2].

Harary [9] introduced the concept of changing and unchanging of a graphical invariant $i$, asking for characterization of graphs $G$ for which $i(G-v), i(G-e)$ or $i(G+e)$ either differ from $i(G)$ or are equal to $i(G)$ for all $v \in V(G), e \in E(G)$ or $e \in E(\overline{G})$ respectively. Some of the most important invariants (for example in communications) are the radius and the diameter of a graph.

Even earlier, in late sixties A.Kotzig initiated the study of graphs for which $d(G-e) > d(G)$ for all $e \in E(G)$. These graphs are called diameter-minimal, for

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example see the papers of Glivjak, Kyš and Plesník [6], [7], [12]. Later on S. M. Lee [10], [11] initiated the study of graphs for which \( d(G - e) = d(G) \) for all \( e \in E(G) \) and he called them diameter-edge-invariant.

From the practical point of view we need to study the stability of the radius and the diameter of a graph \( G \), especially when an arbitrary edge or vertex is removed from \( G \). This operation can represent a single failure of communication line or any communication center (processor, etc.). The papers [1], [3], [5], [13] examine several properties of graphs in which radii do not change under these two conditions, and moreover, when an arbitrary edge is added to the graph \( G \). These graphs are defined as follows:

**Definition 1.1.** A graph \( G \) is:

1. **radius-edge-invariant (r.e.i.)** if \( r(G - e) = r(G) \) for every \( e \in E(G) \);
2. **radius-vertex-invariant (r.v.i.)** if \( r(G - v) = r(G) \) for every \( v \in V(G) \);
3. **radius-adding-invariant (r.a.i.)** if \( r(G + e) = r(G) \) for every \( e \in E(G) \).

According to this definition and to the previous study of diameter-edge-invariant graphs [10], [11], [13] we can define the following classes of graphs:

**Definition 1.2.** A graph \( G \) is:

1. **diameter-edge-invariant (d.e.i.)** if \( d(G - e) = d(G) \) for every \( e \in E(G) \);
2. **diameter-vertex-invariant (d.v.i.)** if \( d(G - v) = d(G) \) for every \( v \in V(G) \);
3. **diameter-adding-invariant (d.a.i.)** if \( d(G + e) = d(G) \) for every \( e \in E(G) \).

Following this definition, in the beginning of Section 2 we will prepare some auxiliary results concerning operations on diameter-invariant graphs. Then, using them we will construct several d.e.i., d.v.i. and d.a.i. graphs. We will also characterize the d.v.i. and d.a.i. graphs of diameter 2. In Section 3 we will try to find some bounds for diameter-invariant-graphs.

### 2. Existence results

We first give some preliminary results about operations on graphs.

Recall that the join of graphs \( G \) and \( H \) is denoted \( G + H \) and consists of \( G \cup H \) and all edges of the form \( u_i v_j \) where \( u_i \in G, v_j \in H \). It is obvious that \( d(G + H) = 1 \) if \( G \) and \( H \) are complete graphs and \( d(G + H) = 2 \) otherwise. Also \( \deg_{G+H}(v) = \deg_G(v) + |V(H)| \) for all \( v \in V(G) \) and \( \deg_{G+H}(u) = \deg_H(u) + |V(G)| \) for all \( u \in V(H) \). Lee [10] gave several results for d.e.i. graphs.
**Theorem 2.1.** The join of graphs $G, H$ is diameter-vertex-invariant
(1) of diameter 1 if and only if $G = K_n, H = K_m, m \cdot n \neq 1$,
(2) of diameter 2 if and only if there are at least two edges in $E(G) \cup E(H)$ not
joined to the same vertex and
$$a) \ G = K_1 \ (or \ H = K_1) \ and \ d(H) = 2 \ (d(G) = 2), \ or$$
$$b) \ |V(G)| > 1 \ and \ |V(H)| > 1.$$  

**Proof.** (1) The first case is obvious, as every complete graph is d.v.i., except
$K_1$ and $K_2$. $G + H$ is a complete graph if and only if $G$ is a complete graph and $H$
is a complete graph.
(2) If $d(G + H) = 2$ and all edges in $E(G) \cup E(H)$ are connected to a single vertex
$v$ then $d(G + H - v) = 1$, a contradiction.
   a) Now let $G = K_1 = \{v\}$. Then $d(G + H - v) = d(G + H)$ if and only if $d(H) = 2$.
For all vertices $u \in V(H)$ we have $d(G + H - u) = 2$, as there exists at least one
edge $ab \in E(H - u)$ and $d(G + H - u) \leq 2r(G + H - u) \leq 2e(v) = 2$.
   b) Let $G$ and $H$ have both at least 2 vertices. Consider $v \in V(G + H)$ and a graph
$G + H - v$. For all $u, w \in V(G + H - v)$ we have $d(u, w) = 1$ if $u \in G, w \in H$ and
d($u, v) \leq 2$ if $u, w \in H$ (or $u, w \in G$). The fact that $E(G + H - v) \geq 1$ implies that
$d(G + H - v) = 2$. \hfill \Box$

The next observation is obvious.

**Theorem 2.2.** The join of graphs $G, H$ is diameter-adding-invariant of radius 2
if and only if $|E(G)| + |E(H)| \geq 2$.

Consider a finite connected graph $I$. Let $\{G_i: i \in V(I)\}$ be a class of graphs
indexed by a finite set $V(I)$.

The Sabidussi sum $S^+({\{G_i: i \in V(I)\}})$ (or shortly $S^+$) of $\{G_i: i \in V(I)\}$ is a
graph defined as follows:
$$V(S^+({\{G_i: i \in V(I)\}})) = \bigcup\{V(G_i): i \in V(I)\}, \ E(S^+({\{G_i: i \in V(I)\}}))$$
$$= \bigcup\{E(G_i): i \in V(I)\} \cup \{xy: x \in V(G_i), y \in V(G_j), ij \in E(I)\}.$$  

Sabidussi sum is sometimes called $X$-join. One can show that $d(S^+(\bigcup\{G_i: i \in V(I)\})) = d(I)$.


357
**Theorem 2.3.** Let \( I \) be a graph of diameter \( d \geq 2 \). For any class of connected graphs \( \{G_i: i \in V(I)\} \) with \( |V(G_i)| \geq 2 \) for all \( i \), the Sabidussi sum \( S^+({\{G_i: i \in V(I)\}}) \) is diameter-edge-invariant with diameter \( d \). Moreover, if \( I \) is diameter-edge-invariant then \( S^+({\{G_i: i \in V(I)\}}) \) is diameter-edge-invariant without the restriction of \( |V(G_i)| \geq 2 \).

However, the assumption that \( G_i \) be connected is unnecessary for \( d \geq 3 \).

**Theorem 2.4.** Let \( I \) be a graph of diameter \( d \geq 3 \). For any class of graphs \( \{G_i: i \in V(I)\} \) with \( |V(G_i)| \geq 2 \) for all \( i \), the Sabidussi sum \( S^+({\{G_i: i \in V(I)\}}) \) is diameter-edge-invariant with diameter \( d \).

**Proof.** It is sufficient to show that in any \( S^+ - e \) there are no vertices \( u, v \) at distance greater than \( d \geq 3 \). If \( u, v \) are from the same graph \( G_i \) or if \( u \in V(G_i), v \in V(G_j), d(i, j) > 1 \), then there are at least 2 edge-disjoint paths of length at most \( d \) joining \( u \) and \( v \). Therefore \( d_{S^+ - e}(u, v) \leq d \) for all \( e \in E(S^+) \).

Let \( u \in V(G_i), v \in V(G_j) \) be two vertices such that \( ij \in E(I) \) and suppose that there is no other path of length at most \( d \) joining \( u, v \). Since \( d(I) > 2 \), we have at least one vertex \( w \in I \) adjacent to \( i \) (or \( j \)), some other vertex \( a \in V(G_i) \) (or \( a \in V(G_j) \)) and some vertex \( b \in V(G_w) \). But then we have at least two edge-disjoint paths of length at most three joining \( u \) and \( v \)—the edge uv and the path \( u-a-b-v \). Therefore \( d_{S^+ - e}(u, v) \leq 3 \leq d \) for all \( e \in E(S^+) \). \( \square \)

We can prove similar result for d.v.i. graphs:

**Theorem 2.5.** Let \( I \) be a graph of diameter \( d \geq 2 \). For any class of graphs \( \{G_i: i \in V(I)\} \) with \( |V(G_i)| \geq 2 \) for all \( i \), the Sabidussi sum \( S^+({\{G_i: i \in V(I)\}}) \) is diameter-vertex-invariant with diameter \( d \). Moreover, if \( I \) is diameter-vertex-invariant then \( S^+({\{G_i: i \in V(I)\}}) \) is diameter-vertex-invariant without the restriction of \( |V(G_i)| \geq 2 \).

**Proof.** If \( |V(G_i)| \geq 2 \) then for any two vertices \( u, v \) at distance \( d(u, v) \geq 2 \), there are at least two vertex-disjoint paths of length \( d(u, v) \). Therefore \( d_{S^+ - w}(u, v) \leq d \) for all \( w \neq u, v \). Let \( i, j \) be two vertices of graph \( I \) such that \( d(i, j) = d(I) \). As \( V(G_i) \geq 2 \) and \( V(G_j) \geq 2 \), for all \( w \in V(S^+) \) there are at least two vertices at distance \( d \) in \( S^+ - w \). Finally, \( d(S^+ - w) = d(S^+) \) and \( S^+ \) is d.v.i. The second part of the result is obvious. \( \square \)
Theorem 2.6. Let $I$ be a diameter-adding-invariant graph of diameter $d \geq 2$. For any class of graphs $\{G_i : i \in V(I)\}$, the Sabidussi sum $S^+(\{G_i : i \in V(I)\})$ is diameter-adding-invariant with diameter $d$.

Proof. We will prove this theorem by contradiction. Let $S^+$ be not a d.a.i. graph. It is clear that for all vertices $a, b \in G_k$ there is $d(S^+ + ab) = d(S^+) = d(I)$. Thus we have two vertices $v \in G_i, u \in G_j$ such that $d(S^+ + uv) < d(S^+) = d(I)$. But then $d(I + ij) \leq d(S^+ + uv) < d(S^+) = d(I)$, a contradiction. \qed

The corona $G \circ H$ of graphs $G$ and $H$ was defined by Frucht and Harary ([4], see also [2]) as the graph obtained by taking one copy of $G$ of order $p_G$ and $p_G$ copies of $H$, and then joining the $i$'th vertex of $G$ to every vertex in the $i$'th copy of $H$. If the $i$'th vertex is named $v$, then the copy belonging to $v$ will be named $H_v$.

It is clear that if $p_G > 1$, $r(G) = r_G$, $d(G) = d_G$, then $r(G \circ H) = r_G + 1$, $d(G \circ H) = d_G + 2$ and $v$ is a central vertex of $G \circ H$ if and only if $v$ is a central vertex of $G$. Moreover, $h \in H_v$ is a peripheral vertex of $G \circ H$ if and only if $v$ is a peripheral vertex in $G$. Since $d(G \circ H - v) = \infty$ for $v \in G$ and $e_{G \circ H - h_v}(h) > d(G \circ H)$ for the peripheral vertex $v$ of the graph $G$ and $h \in H_v$, the corona of two graphs will never be d.e.i. or d.v.i.

The paper [1] gives the following theorem:

Theorem 2.7. For any graphs $G, H$, such that $|V(G)| \geq 3$, the corona $G \circ H$ is radius-adding-invariant if and only if $G$ is radius-adding-invariant.

For the diameter of $G \circ H$ the following theorem holds:

Theorem 2.8. For any graphs $G, H$, such that $|V(G)| \geq 3, H \neq K_1$ the corona $G \circ H$ is diameter-adding-invariant if and only if $G$ is diameter-adding-invariant.

Proof. ($\Rightarrow$) Suppose that $G \circ H$ is d.a.i., but $G$ is not d.a.i. Let $e \in E(\overline{G})$ be an edge such that $d(G + e) < d(G)$. Therefore

$$d(G \circ H + e) = d((G + e) \circ H) = d(G + e) + 2 < d(G) + 2 = d(G \circ H),$$

a contradiction.

($\Leftarrow$) We consider various possibilities for an edge $e \in E(\overline{G \circ H})$.

1. If $e \in E(\overline{G})$, then

$$d(G \circ H + e) = d(G + e) + 2 = d(G) + 2 = d(G \circ H).$$

2. If $e \in E(\overline{H_v})$ for any $v \in V(G)$, then for all $w \in V(G \circ H)$ we have $e_{G \circ H}(w) = e_{G \circ H + e}(w)$ and thus $d(G \circ H) = d(G \circ H + e)$. 

359
(3) Suppose $e = uh_v$ where $u \in V(G), h_v \in H_v, v \neq u$. Let $d(G \circ H + e) < d(G \circ H)$. If $x$ and $y$ are two peripheral vertices of $G \circ H$ such that $d(x, y) = d(G \circ H)$, then the $x$-$y$ geodesic in $G \circ H + e$ must contain $e$. Moreover, if $x \notin H_v$ and $y \notin H_v$ then $u-h_v-v$ is a part of the $x$-$y$ geodesic in $G \circ H + e$. But then for all such pairs $d_{G \circ H + uv}(x, y) < d(G \circ H)$.

On the other hand let, for example, $x \in H_v$. It is clear that for all $z \in H_v, z \neq x$ we have $d_{G \circ H + e}(y, z) \geq d_{G \circ H + e}(y, x) + 1$. But then again $d_{G \circ H + uv}(x, y) < d(G \circ H)$ and $d_{G \circ H + uv}(x, h_v) < d(G \circ H)$. This leads to the case (1) which was discussed above.

(4) Finally, suppose $e = h_u h_v$ where $u, v \in V(G), h_u \in H_u, h_v \in H_v, v \neq u$. Let $d(G \circ H + e) < d(G \circ H)$. It is obvious that for all $h' \in H_u, h' \in H_v, h'_v \neq u, h'_v \neq v$ we have $e_{G \circ H + e}(h'_v) \geq e_{G \circ H + e}(h_u)$ and $e_{G \circ H + e}(h'_v) \geq e_{G \circ H + e}(h_v)$. Thus if $x$ and $y$ are two peripheral vertices of $G \circ H$ different from $h_u, h_v$ such that $d(x, y) = d(G \circ H)$, then the $x$-$y$ geodesic in $G \circ H + e$ must contain $e$. Moreover, the $x$-$y$ geodesic must contain a subpath of length three of the form $u-h_u-h_v-v, h''_v-h_u-h_v-v$ or $h''_u-h_u-h_v-v$.

Consider the graph $G \circ H + uv$. To obtain an $x$-$y$ path of length less than $d(G \circ H)$ it is sufficient to take $u$-v instead of $u-h_u-h_v-v, h''_v-u-u-v$ or $h''_u-u-u-v$ instead of $h''_u-h_u-h_v-v$ in the $x$-$y$ geodesic formed in $G \circ H + u+h_v$. Thus $d_{G \circ H + u+h_v}(x, y) \geq d_{G \circ H + uv}(x, y)$ and since $d_{G \circ H + uv}(h'_u, h'_v) = d_{G \circ H + uv}(h_u, h_v) = d_{G \circ H + uv}(h'_u, h'_v) = d_{G \circ H + uv}(h_u, h_v) = d_{G \circ H + uv}(a, b) < d(G \circ H)$ for all $a, b \in V(G \circ H)$. Therefore $d(G \circ H + uv) < d(G \circ H)$. But this is the case (1) which was discussed above.

If $H = K_1$ and $G$ is d.a.i. having $|V(G)| \geq 3$ then $G \circ H$ is not necessarily d.a.i.: Consider the group $\mathbb{Z}_{2r+1}$ and define a graph $G_{\mathbb{Z}_{2r+1}}$ in the following way:

$$V(G) = \{(i, j); i, j \in \mathbb{Z}_{2r+1}\},$$

$$(i_1, j_1)(i_2, j_2) \in E(G) \iff |i_1 - i_2| \leq 1 \land |j_1 - j_2| \leq 1.$$ 

If $(i_1, j_1)$ and $(i_2, j_2)$ are two vertices of $G_{\mathbb{Z}_{2r+1}}$, then $d((i_1, j_1), (i_2, j_2)) = \max\{|i_1 - i_2|, 2r + 1 - |i_1 - i_2|, \min\{|j_1 - j_2|, 2r + 1 - |j_1 - j_2|\}\} \leq r$. Since for each vertex $u = (i, j)$, there are $8r$ vertices $u_k = (i_k, j_k), i_k = i + r \text{ mod}(2r+1) \lor j_k = i + r \text{ mod}(2r+1) \lor j_k = j + r \text{ mod}(2r+1) \lor j_k = j + r \text{ mod}(2r+1)$ such that $d(u, u_k) = r$, the graph $G_{\mathbb{Z}_{2r+1}}$ is self-centered of radius $r$.

Now, consider a graph $G'$ obtained in the following way: Suppose $V(G') = V(G_{\mathbb{Z}_{2r+1}}) + v, E(G') = E(G_{\mathbb{Z}_{2r+1}}) + uv$ where $u = (i, j) \in V(G_{\mathbb{Z}_{2r+1}})$. We have $e_{G'}(v) = d(G_{\mathbb{Z}_{2r+1}}) + 1 = d(G')$. Let $f \in E(G')$ be an arbitrary edge. If $f \in E(G_{\mathbb{Z}_{2r+1}})$, then $e_{G_{\mathbb{Z}_{2r+1}}}(w) = e_{G_{\mathbb{Z}_{2r+1}}} + f(w)$ for all $w \in V(G_{\mathbb{Z}_{2r+1}})$ and thus $d(G') = e_{G'}(v) = e_{G'} + f(v) = d(G' + f)$. If $f \notin E(G_{\mathbb{Z}_{2r+1}})$, then $f$ is of type $(i', j')$ where $i \neq i'$, or $j \neq j'$. It is sufficient to take the vertex $a = (i + r \text{ mod}(2r+1), j' +$
r \mod(2r + 1)) to obtain a vertex such that \( d(a,v) = d(G_{2r+1}) + 1 = d(G') \). Thus \( G' \) is d.a.i.

Now consider a graph \( G' \circ K_1 \). Let \( H_v = \{b\} \) be a copy of \( K_1 \) belonging to \( v \in G' \). One can show that \( d(G' \circ K_1 + bu) = d(G') + 1 < d(G' \circ K_1) \). Thus \( G' \circ K_1 \) is not d.a.i.

Consider the two following graphs \( I_1, I_2 \):

\[
\begin{array}{cccccc}
1 & 2 & r-1 & r & \cdots & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \cdots & \cdots \\
&&&&a&& \\
\downarrow & \downarrow & \downarrow & \downarrow & \cdots & \cdots \\
&&&&b&& \\
\downarrow & \downarrow & \downarrow & \downarrow & \cdots & \cdots \\
&&&&c&& \\
\end{array}
\quad
\begin{array}{cccccc}
1 & 2 & r-2 & r-1 & \cdots & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \cdots & \cdots \\
&&&&a&& \\
\downarrow & \downarrow & \downarrow & \downarrow & \cdots & \cdots \\
&&&&b&& \\
\downarrow & \downarrow & \downarrow & \downarrow & \cdots & \cdots \\
&&&&c&& \\
\end{array}
\]

Figure 1

In the first case \( d = 2r \), in the second \( d = 2r - 1 \). Since in both graphs there are three pairs of vertices \( \{a,b\} \), \( \{b,c\} \), \( \{c,a\} \) at distance \( d \), and adding a single edge may change at most two of these distances, both graphs are d.a.i. of diameter \( d \) for all \( r \geq 1 \).

Lee [10] showed, as a consequence of Theorem 2.3, that any connected graph is an induced subgraph of a d.e.i. graph of diameter \( d \geq 2 \). Walikar et. al. [3] showed that for every graph \( G \), the graph \( H \) formed as \( K_2 + G + K_2 \) is d.e.i. As a consequence they got that every graph could be embedded in a d.e.i. graph. Later in this section we will show that for each graph \( G \), there is an d.e.i., d.v.i. and d.a.i. graph \( H \) of diameter \( d \) having \( G \) as an induced subgraph.

**Lemma 2.9.** Let \( G \) be a graph with at least two vertices. Then the graph \( H = K_2 + G + K_2 \) is diameter-vertex-invariant and diameter-adding-invariant of diameter 2.

**Proof.** One can show that \( d(H) = 2 \). As \( |E(H)| > 1 \), it is clear that \( H \) is d.a.i. We can write \( H = (K_2 + G) + K_2 \). Thus by Theorem 2.1 \( H \) is d.v.i. \( \square \)

**Theorem 2.10.** Every graph \( G \) can be embedded as an induced subgraph in a diameter-edge-invariant, diameter-vertex-invariant and diameter-adding-invariant graph \( H \) of diameter \( d \geq 2 \).

**Proof.** Suppose \( G \) has at least two vertices. It is sufficient to take the graph \( K_2 + G + K_2 \) for \( d = 2 \) and the Sabidussi sum \( S^+(\{G_i \equiv G : i \in V(I)\}) \) where \( I \) is a graph \( I_1 \) if \( d = 2k \) or \( I_2 \) if \( d = 2k + 1 \). It follows from the results of the previous section that \( S^+ \) is d.e.i., d.v.i. and d.a.i.
If \( G = K_1 \) then it is a subgraph of any graph, and as for each \( d \) there exists d.e.i., d.v.i. and d.a.i. graph \( H \), the theorem holds.

Because of the previous theorem, we cannot obtain a forbidden subgraph characterization for all d.e.i., d.v.i., and d.a.i. graphs.

Bálint and Vacek in [1] constructed several r.e.i., r.v.i. and r.a.i. graphs. We will now show that there are graphs which radius and diameter are both invariant.

**Theorem 2.11.** Let \( r, d \) be natural numbers such that \( 2 \leq r < d \leq 2r \). Let \( G \) be a graph with at least two vertices. Then there exists a radius-edge-invariant, diameter-edge-invariant, radius-vertex-invariant and diameter-vertex-invariant graph \( H \) such that \( r(H) = r, d(H) = d, C(H) = V(G) \) and \( G \) is an induced subgraph of \( H \).

[1] gives a somewhat weaker result with similar graph construction for radius-invariant graphs only.

**Proof.** For \( d \neq 2r - 1 \) consider the following graph \( Q \):

![Graph Q](image-url)

\( Q \) is formed by 2 central vertices \( c_1, c_2 \); by \( 2(d - 1) + 1 \) rows of vertices in \( 2(r - 1) \) columns and by 4 additional vertices \( v_1, v_2, u_1, u_2 \). Every column except 1 and \( 2(r - 1) \) (counted from the left side) has \( 2(d - r) + 1 \) vertices. Columns 1 and \( 2(r - 1) \) have \( 2(2(d - r) + 1) \) vertices. Vertices \( c_1, c_2 \) are adjacent to all vertices in columns \( r - 1 \) and \( r \). Vertices \( v_1, v_2, (u_1, u_2) \) are adjacent and joined to all vertices in row 1 \( (2(d - r) + 1) \) and columns 1 and \( 2(r - 1) \). Vertex in row \( k \) and column \( l \) is adjacent
to all vertices in row \( k \) and columns \( l - 1, l + 1 \) and to all vertices in column \( l \) and rows \( k - 1, k + 1 \) except the case when \( l = r - 1 \) or \( l = r \).

It is clear that \( e(c_1) = e(c_2) = r \), \( e(v) > r \) otherwise, and \( d(u_i, v_j) = \min\{d(v_i, c_1) + d(c_1, u_j), 2(d - r) + 2\} \). Since \( d \neq 2r - 1 \) we have \( 2(d - r) + 2 \leq d \) or \( 2r \leq d \) and thus \( d(u_i, v_j) \leq d \). For any other vertex \( x, x \neq c_i, x \neq u_i \) (or \( v_i \)) we have \( d(x, v_i) \leq \min\{2(d - r) + 1, 2r - 2\} \leq d \). Now, let \( y, z \) be arbitrary vertices except \( u_i, v_i, c_i \). When \( y, z \) belong to the same row and the same half (right or left) of \( Q \) we obviously have \( d(y, z) < r < d \). Consider a shortest cycle \( F \) such that \( y, z \in F \). The length of the cycle \( F \) can be at most \( 2 + 2(d - r) + 2(r - 1) = 2d \) if it is made as a sequence of \( y - c_1, c_1 - z, z - u_i \) (or \( z - v_i \)), \( u_i - y \) (or \( v_i - y \)) geodesics or less otherwise. This implies \( d(x, y) \leq d \). Thus for all \( w \in V(Q) \) we have \( e(w) \leq d \).

To obtain vertices \( o, p \) such that \( d(o, p) = d \) it is sufficient to take the vertex \( o \) in row \( 1 \) and column \( 1 \) and the vertex \( p \) in row \( 2(d - r) + 1 \) and column \( d + 1 \). This implies that \( r(Q) = r \) and \( d(Q) = d \). Note: There are more than one pair of such vertices.

Since for every vertex \( a, a \neq c_i \) there are at least two edge and vertex-disjoint \( c_1 - a \) (or \( c_2 - a \)) paths, and, in addition there are four vertices in the graph \( Q \) at distance \( r \) from \( c_1, c_2 \), we have \( r(Q - e) = r(Q - b) = r \) for all \( e \in E(Q) \), \( b \in V(Q) \), \( Q \) is r.e.i. and r.v.i.

Next, we will show that \( Q \) is also d.e.i. and d.v.i. We have already proved that \( e_{Q-e}(c_i) = e_{Q-b}(c_i) = r \). Consider the eccentricity of the vertices \( v_i \) (\( u_j \)). Let \( s \) be any vertex except \( v_i (u_j) \) and suppose \( s \) does not belong to row \( 1 \) (or \( 2(d - r) + 1 \)). Thus there are at least two edge and vertex-disjoint \( u_i \)-s geodesics. It is clear that \( d_{Q-u_1u_2}(u_1, u_2) = 2 \) and for all vertices \( t \) in row \( 1 \) we have \( d(u_i, t) \leq (r - 1) + 2 \leq d \).

Thus for all \( e \in E(Q) \), \( b \in V(Q) \) we have \( e_{Q-e}(u_i) \leq d \) and \( e_{Q-b}(u_i) \leq d \).

Now let \( y, z \) be arbitrary vertices except \( u_i, v_i, c_i \). One can show that if vertices \( y, z \) do not lie in the same row and the same half of the graph \( Q \), then the length of at most one of the \( y-c_1, c_1-z, z-u_i \) (\( z-v_i \)), \( u_i-y \) (\( v_i-y \)) geodesics is different in \( Q \) and in \( Q - e \) (\( Q - b \)). It follows directly from the construction of \( Q \) that the difference in lengths of these paths can be at most \( 1 \). Consider a shortest cycle \( F' \) such that \( y, z \in F' \). The length of the cycle \( F' \) can be at most \( 2 + 2(d - r) + 2(r - 1) + 1 = 2d + 1 \) if it is made as a sequence of \( y-c_1, c_1-z, z-u_i \) (or \( z-v_i \)), \( u_i-y \) (or \( v_i-y \)) geodesics in \( Q - e \) (\( Q - b \)). Thus \( d_{Q-e}(y, z) \leq d \) and \( d_{Q-b}(y, z) \leq d \).

We can obtain vertices \( o, p \in V(Q - b) \) such that \( d(o, p) = d \) in the same way as in \( Q \). Finally, for \( d \neq 2r - 1 \) the graph \( Q \) is r.e.i., r.v.i., d.e.i. and d.v.i. of radius \( r \) and diameter \( d \).

For \( d = 2r - 1 \) it is sufficient to take only \( d - 1 \) rows of vertices. It is clear that \( d(u_i, v_j) = d \). All other facts could be proved similarly as above and we leave the details to the reader.
The desired graph $H$ is obtained from the graph $Q$ by substituting the graph $G$ instead of the vertices $c_1, c_2$.  

**Theorem 2.12.** Let $r, d$ be natural numbers such that $r \leq d \leq 2r$. Then there exists a radius-adding-invariant and diameter-adding-invariant graph $G$ such that $r(G) = r$ and $d(G) = d$.

**Proof.** It is sufficient to take the tree $I_1$ if $d = 2r$ and the following tree for $d = 2r - 1$.

![Tree Diagram](image)

Figure 3

Otherwise the desired graph can be constructed as follows: Denote $G_0 = G_{2k+1}$ where $k = 2r - d \geq 2$. From [1] we have that $G_0$ is r.a.i. Since $G_0$ is self-centered and $r(G_0 + e) \leq d(G_0 + e) \leq d(G_0) = r(G_0)$ it is also d.a.i.

We will construct a graph $G_{i+1}$ from the graph $G_i$ as $G_{i+1} = G_i \circ H$, $H \neq K_1$. From Theorem 2.7 and from Theorem 2.8 it follows directly that every graph $G_i$ is r.a.i. and d.a.i. For $i = d - r$ we have an r.a.i. and d.a.i. graph $G_{d-r}$ such that $r(G_{d-r}) = i \cdot 1 + r(G_0) = (d - r) + (2r - d) = r$ and $d(G_i) = i \cdot 2 + d(G_0) = 2(d - r) + (2r - d) = d$.

Walikar, Buckley and Itagi [13] showed that any graph $G$ of diameter 2 is d.e.i. if and only if every edge of $G$ is contained in a triangle and if there are at least two geodesics for all vertices $v, w$ at distance 2. As we have already stated, a graph $G$ of diameter $d = 2$ is d.a.i. if and only if $E(G) \geq 2$. For d.v.i. graphs we have the following result.

**Theorem 2.13.** Suppose that a graph $G$ has diameter 2. Then $G$ is diameter-vertex-invariant if and only if

1. for all $u, v \in V(G)$ such that $d(u, v) = 2$ there are at least two $u$-$v$ geodesics,
2. there are at least two edges $a_1a_2, b_1b_2 \in E(G)$ not incident with the same vertex.

**Proof.** ($\Rightarrow$)

(1) Suppose there is only one such geodesic $u$-$x$-$v$. Then $d_{G-x}(u, v) \geq 3$, a contradiction.

364
(2) Let all edges in $E(G)$ have one joint incident vertex $v$. Then $G - v$ is a complete graph. Therefore $d(G - v) = 1$ which is again a contradiction.

($\Leftarrow$) Consider an arbitrary vertex $w \in V(G)$ and the graph $G - w$. From (2) it follows that we have $E(G - w) \geq 1$, and thus $d(G - w) > 1$. For any two vertices $u, v \in V(G - w)$ there is $d_G(u, v) \leq 2$. If $d_G(u, v) = 2$, then from (1) it follows that there must be some path $u-a-v$ in $G - w$. Therefore $d(u, v) = 2$. \hfill $\Box$

3. Some bounds

A $k$-depth spanning tree ($k$-DST) of a graph $G$ is a spanning tree of $G$ of height $k$. It must be true that $k \leq d$, and if $k = d$, such trees must be rooted at a peripheral vertex. A breadth first search algorithm beginning with any vertex $v$ such that $e(v) = k$ will always produce a $k$-DST. Moreover, if $d(u, v) = i$ then the vertex $u$ belongs to level $i$. We will consider only breadth first search distance spanning trees later in this paper.

**Theorem 3.1.** Let $G$ be a diameter-edge-invariant graph with $n$ vertices and diameter $d$. Then for all $v \in V(G)$

1. $2 \leq \deg(v) \leq n - \frac{1}{2}(3d - 6)$ (except $d = 2$ where it is $2 \leq \deg(v) \leq n - 1$) if $d$ is even and $d$ is odd.
2. $2 \leq \deg(v) \leq n - \frac{1}{2}(3d - 5)$ if $d$ is odd.

Moreover, all these bounds are sharp.

**Proof.** The lower bound is obvious as $G$ has no bridges. Consider a $d$-DST rooted at a peripheral vertex $x$.

There must be at least one vertex $y$ on level $d$. As $G$ is d.e.i. there are at least two edge-disjoint $x-y$ paths of length $d$ in $G$. Thus there are no levels $i$, $i + 1$ both with only one vertex. Because of this we have at most $\frac{1}{2}d + 1$ levels with only one vertex if $d$ is even and at most $\frac{1}{2}(d + 1)$ levels with only one vertex if $d$ is odd.

Any vertex $v$ on level $i$ can be adjacent only to vertices on levels $i - 1$, $i$, $i + 1$. Thus there are at least $d - 2$ remaining levels with vertices which are not adjacent to $v$. At most $\frac{1}{2}d$ ($\frac{1}{2}(d - 1)$ if $d$ is odd) of these levels have only one vertex.

Therefore

\[
\deg(v) \leq n - 1 - 2\left(\frac{d}{2} - 2\right) + \frac{d}{2} = n - \frac{3d - 6}{2}
\]

if $d$ is even and

\[
\deg(v) \leq n - 1 - 2(d - 2) + \frac{d - 1}{2} = n - \frac{3d - 5}{2}
\]

if $d$ is odd.
There is one exception. For \( d = 2 \) it is \( \frac{1}{2} (3d - 6) = 0 \). But for any graph \( G \) it must hold \( \deg(v) \leq n - 1 \).

To obtain a graph which reaches the bound it is sufficient to take \( H_1 = K_{n - \frac{d}{2} + 1} \) in the graph \( G_1 \) if \( d \) is even and \( H_2 = K_{n - (3d - 1)/2} \) in the graph \( G_2 \) if \( d \) is odd. In both graphs \( x \) has the minimal and \( z \) the maximal possible degree. \( \square \)

Lee [11] gave the bound for the minimal number of vertices in d.e.i. graphs of diameter \( d \) which is \( \frac{3}{2}d + 1 \) vertices if \( d \) is even and \( \frac{3}{2}(d + 1) \) vertices if \( d \) is odd.

**Theorem 3.2.** Let \( G \) be a diameter-vertex-invariant graph with \( n \) vertices and diameter \( d \). Then for all \( v \in V(G) \)

1. \( \deg(v) = n - 1 \), if \( d = 1 \),
2. \( 2 \leq \deg(v) \leq n - 1 \) if \( d = 2 \),
3. \( 2 \leq \deg(v) \leq n - 3 \) if \( d = 3 \),
4. \( 2 \leq \deg(v) \leq n - 4 \) if \( d = 4 \) unless \( n = 2d + 2 = 10 \), for which it is \( 2 \leq \deg(v) \leq 5 \),
5. \( 2 \leq \deg(v) \leq n - 2d + 3 \) if \( d \geq 5 \).

These bounds are sharp.

**Proof.** The first two statements are obvious. If \( d = 3 \) then there is no vertex \( v \) such that \( e(v) = n - 2 \). Otherwise there is a unique vertex \( u \) such that \( d(u, v) = 2 \). Thus \( d(G - u) \leq 2r(G - u) = 2e_{G - u}(v) = 2 \), a contradiction.

Suppose that \( d(G) \geq 4 \). Consider two vertices \( u, v \) such that \( d(u, v) = d \) and two \( d \)-DST \( T_1, T_2 \) rooted at peripheral vertices \( v \) and \( u \). Since \( G \) has no cut-vertices, each of these trees has at least 2 vertices on each of the levels \( 1, \ldots, d - 1 \). We will prove the bound by a contradiction.

Let there be a vertex \( w \) such that \( \deg(w) > n - 2d + 3 \). If it belongs to level \( i \), then it could be adjacent only to vertices on levels \( i - 1, i, i + 1 \) (if such exist). Since \( \deg(w) > n - 2d + 3 \), for \( d - 2 \) levels there remain at most \( 2d - 5 \) vertices. Thus

1. \( w \) is adjacent to every vertex on level \( i - 1, i, i + 1 \), or
(2) for all trees $T_1, T_2$ there is exactly 1 vertex on each of the levels 0 and $d$ and 2 vertices on every other level except $i - 1, i, i + 1$.

Moreover, it is clear that there is a diametral path $P$ such that $w \in P$.

(1) At least one tree $T_i$ contains the vertex $w$ on level $i \geq \lceil \frac{1}{2}d \rceil$. Let it be the tree $T_1$ and let it contain only one vertex (for example $u$) on level $d$. Then we can prove that $d(G - u) = d - 1$: Let $a_1, a_2$ be two vertices on levels higher than $i$ and $b_1, b_2$ be two vertices on levels lower than $i$. Therefore $d(a_i, b_k) < d(u, b_k) \leq d$. As $d(a_i, w) < \frac{1}{2}d$ we have $d(a_1, a_2) < d$. Moreover, $G$ is d.v.i., and thus the vertices $b_1, b_2$ lie on a cycle. The vertex $w$ is adjacent to all vertices on level $i - 1$ and therefore the length of this cycle must be less than $2d$. Thus $d(b_1, b_2) < d$. Finally, $d(G - u) = d - 1$, a contradiction. As a result of this part we already get that $\Delta(G) \leq n - 2d + 4$.

Let the tree $T_1$ contain two vertices on level $d$ and let $\Delta(G) = n - 2d + 4$. Thus there are exactly 2 vertices on each level $1, \ldots, i - 2$. Let us mark the vertices on level 2 as $c_1, c_2$. It must be $\deg(c_1) > 2$ and $\deg(c_2) > 2$. Otherwise, if $xc_j \in E(G), x \neq v$ then

$$d(G - x) \geq e_{G-x}(c_j) \geq d(c_i, u) = d(c_i, v) + d(v, u) = d + 1 > d.$$  

If $c_1c_2 \in E(G)$ or if $i - 1 > 2$ (and thus there are only 2 vertices on level 2), then in $G - v$ all vertices on levels lower than $i$ lie on a cycle of length less than $2d$. Similarly as in previous part $d(G - v) = d - 1$.

Now, consider the case in which $c_1c_2 \in E(G)$ and $i - 1 = 2$. Then $d_{G-v}(c_1, c_2) \leq 4$ and thus for any vertex $y \in V(G - v)$ we have $e_{G-v}(y) \leq \max\{4, d - 1\}$. Finally, it holds $\Delta(G) \leq n - 2d + 3$ with the exception of $d = 4$. In that case we cannot use the same arguments as those given in the previous paragraph. Therefore, we obtain only the inequality $\Delta(G) \leq n - 2d + 4 = n - 4$.

If $n = 2d + 2 = 10$, then there are at most 3 vertices on level 2. In that case $d_{G-v}(c_1, c_2) \leq 2$ and thus $e_{G-v}(y) \leq \max\{2, d - 1\} < d$ for all $y \in V(G - v)$. Therefore $\Delta(G) \leq n - 2d + 3 = 5$.

(2) Suppose $\Delta(G) \geq n - 2d + 4$. We can use the same arguments and notations as above. If, for example $d(u, w) < \frac{1}{2}d$ then $d(G - w) = d - 1$. If $d(u, w) = d(w, v) \geq \frac{1}{2}d$ then for a tree $T_1$ rooted at central vertex $v$ with the vertex $w$ on level $i$ either $w$ is adjacent to every vertex on level $i - 1$ or $w$ is adjacent to every vertex on level $i + 1$. Thus $d(G - v) = d - 1$ in the first case or $d(G - u) = d - 1$ in the second case.

Suppose $4 \neq d \geq 3$ or $2d + 2 = 10 = n$. The graph $G$ (where $H = K_n - 2d$, see Figure 5) certifies that our bounds are sharp. The following graph (see Figure 6) is for $d = 4, n \neq 10 (H = K_{n-10})$.

For $d = 2$ it is sufficient to take $C_4$ and substitute any vertex of $C_4$ with $K_{n-3}$. □

Similarly as the previous theorem we can prove the following result:
Theorem 3.3. Diameter-vertex-invariant graph of diameter $d \geq 3$ has at least $2d + 2$ vertices.

To obtain a d.v.i. graph with $2d + 2$ vertices is sufficient to take $K_2$ instead of $H$ in Figure 5.

To prove this, consider a diametral $u,v$ path and the cycle $F$ of length $d + 1$ in the graph $G + uv$ formed by the $u,v$ path and the edge $uv$. The eccentricity of every vertex $w$ in the subgraph $F$ is $\lceil \frac{d}{2} \rceil$. Also $d_F(s,t) = d_{G+uv}(s,t)$ for all $s,t \in F$. Moreover, since $G$ is d.a.i., there are at least two vertices $x,y \in V(G+uv)$ such that $d_{G+uv}(x,y) = d$.

Case 1: $x \in F$

Let $z$ be the last joint vertex of the $x,y$ geodesic and of the cycle $F$. One can prove that $d_{G+uv}(z,y) \geq \lceil \frac{d}{2} \rceil$. For every $a \in V(G + uv)$ we have:

1. $a$ is adjacent to at most 3 successive vertices of $F$. Otherwise $d_G(u,v) < d(G)$.
2. $a$ is adjacent to at most 3 successive vertices of any $z,y$ geodesic. Otherwise $d_{G+uv}(x,y) < d(G)$.
3. $a$ is adjacent to at most 4 vertices of the cycle $F$ and of some $z,y$ geodesic together. (Only if $a$ is adjacent to $z$ and its neighbours.) Otherwise $d_{G+uv}(x,y) < d(G)$.

368
(4) if \( a = z \) then it is adjacent to at most 3 vertices of the cycle \( F \) and of some \( z-y \) geodesic together.

Case 2: \( x \notin F, y \notin F \)

It is clear that the \( x-y \) geodesic contains at most \( \lceil \frac{1}{2}d \rceil \) vertices of cycle \( F \). If two vertices \( b, c \) belong to \( F \) and to the \( x-y \) geodesic, then some \( b-c \) geodesic belongs to \( F \). For every \( a \in V(G + uv) \) we have:

(1) \( a \) is adjacent to at most 3 successive vertices of \( F \). Otherwise \( d(u, v)_G < d(G) \).

(2) \( a \) is adjacent to at most 3 successive vertices of any \( x-y \) geodesic. Otherwise \( d_{G+uv}(x, y) < d(G) \).

(3) If the cycle \( F \) and the \( x-y \) geodesic have \( \lceil \frac{1}{2}d \rceil \) vertices in common, then \( a \) is adjacent to at most 4 vertices of the cycle \( F \) and the \( x-y \) geodesic together. If the cycle \( F \) and the \( x-y \) geodesic have \( \lceil \frac{1}{2}d \rceil - i \) vertices in common, then \( a \) is adjacent to at most \( 4 + i \) vertices of the cycle \( F \) and the \( x-y \) geodesic together. Otherwise \( d_{G+uv}(x, y) < d(G) \).

(4) If \( a \) belongs both to \( x-y \) geodesic and to the cycle \( F \) then it is adjacent to at most 3 vertices of the cycle \( F \) and the \( x-y \) geodesic together.

Thus \( a \) is adjacent to at most \( n - 1 - (d + 1 + \lceil \frac{1}{2}d \rceil - 4) \) vertices which is the same as the bounds.

To obtain a graph which certifies that the bounds are the best possible it is sufficient to take the graphs \( I_1 \) (\( I_2 \)) and substitute some central vertex with the graph \( K_{n-3d/2} \) (or \( K_{n-(3d+1)/2} \)). □

The next bound follows immediately from the proof of the previous theorem.

**Theorem 3.5.** Diameter-adding-invariant graph of diameter \( d \) has at least

(1) \( \frac{3}{2}d + 1 \) vertices if \( d \) is even,

(2) \( \frac{1}{2}(3d + 1) \) vertices if \( d \) is odd.

**References**


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