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DIAMETER-IN Variant GRAPHS

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Abstract. The diameter of a graph $G$ is the maximal distance between two vertices of $G$. A graph $G$ is said to be diameter-edge-invariant, if $d(G - e) = d(G)$ for all its edges, diameter-vertex-invariant, if $d(G - v) = d(G)$ for all its vertices and diameter-adding-invariant if $d(G + e) = d(e)$ for all edges of the complement of the edge set of $G$. This paper describes some properties of such graphs and gives several existence results and bounds for parameters of diameter-invariant graphs.

Keywords: extremal graphs, diameter of graph

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1. Introduction

Let $G$ be an undirected, finite graph without loops or multiple edges. Then we denote by: $V(G)$ the vertex set of $G$; $E(G)$ the edge set of $G$; $\overline{G}$ the complement of $G$ with the edge set $E(\overline{G})$; $d_G(u, v)$ (or simply $d(u, v)$) the distance between two vertices $u, v$ in $G$; $e(u)$ the eccentricity of $u$. The radius $r(G)$ is the minimum of the vertex eccentricities, the diameter $d(G)$ is the maximum of the vertex eccentricities; $\deg_G(v)$ is the degree of vertex $v$ in $G$ and $\Delta(G)$ is the maximum degree of $G$. The notions and notations not defined here are used accordingly to the book [2].

Harary [9] introduced the concept of changing and unchanging of a graphical invariant $i$, asking for characterization of graphs $G$ for which $i(G - v), i(G - e)$ or $i(G + e)$ either differ from $i(G)$ or are equal to $i(G)$ for all $v \in V(G), e \in E(G)$ or $e \in E(\overline{G})$ respectively. Some of the most important invariants (for example in communications) are the radius and the diameter of a graph.

Even earlier, in late sixties A.Kotzig initiated the study of graphs for which $d(G - e) > d(G)$ for all $e \in E(G)$. These graphs are called diameter-minimal, for

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example see the papers of Glivjak, Kyš and Plesník [6], [7], [12]. Later on S. M. Lee [10], [11] initiated the study of graphs for which \( d(G - e) = d(G) \) for all \( e \in E(G) \) and he called them diameter-edge-invariant.

From the practical point of view we need to study the stability of the radius and the diameter of a graph \( G \), especially when an arbitrary edge or vertex is removed from \( G \). This operation can represent a single failure of communication line or any communication center (processor, etc.). The papers [1], [3], [5], [13] examine several properties of graphs in which radii do not change under these two conditions, and moreover, when an arbitrary edge is added to the graph \( G \). These graphs are defined as follows:

**Definition 1.1.** A graph \( G \) is:

1. radius-edge-invariant (r.e.i.) if \( r(G - e) = r(G) \) for every \( e \in E(G) \);
2. radius-vertex-invariant (r.v.i.) if \( r(G - v) = r(G) \) for every \( v \in V(G) \);
3. radius-adding-invariant (r.a.i.) if \( r(G + e) = r(G) \) for every \( e \in E(G) \).

According to this definition and to the previous study of diameter-edge-invariant graphs [10], [11], [13] we can define the following classes of graphs:

**Definition 1.2.** A graph \( G \) is:

1. diameter-edge-invariant (d.e.i.) if \( d(G - e) = d(G) \) for every \( e \in E(G) \);
2. diameter-vertex-invariant (d.v.i) if \( d(G - v) = d(G) \) for every \( v \in V(G) \);
3. diameter-adding-invariant (d.a.i.) if \( d(G + e) = d(G) \) for every \( e \in E(G) \).

Following this definition, in the beginning of Section 2 we will prepare some auxiliary results concerning operations on diameter-invariant graphs. Then, using them we will construct several d.e.i., d.v.i. and d.a.i. graphs. We will also characterize the d.v.i. and d.a.i. graphs of diameter 2. In Section 3 we will try to find some bounds for diameter-invariant-graphs.

2. Existence results

We first give some preliminary results about operations on graphs.

Recall that the join of graphs \( G \) and \( H \) is denoted \( G + H \) and consists of \( G \cup H \) and all edges of the form \( u_i v_j \) where \( u_i \in G, v_j \in H \). It is obvious that \( d(G + H) = 1 \) if \( G \) and \( H \) are complete graphs and \( d(G + H) = 2 \) otherwise. Also \( \deg_{G+H}(v) = \deg_G(v) + |V(H)| \) for all \( v \in V(G) \) and \( \deg_{G+H}(u) = \deg_H(u) + |V(G)| \) for all \( u \in V(H) \). Lee [10] gave several results for d.e.i. graphs.
Theorem 2.1. The join of graphs $G, H$ is diameter-vertex-invariant
(1) of diameter 1 if and only if $G = K_n, H = K_m, m \cdot n \neq 1$,
(2) of diameter 2 if and only if there are at least two edges in $E(G) \cup E(H)$ not
joined to the same vertex and
a) $G = K_1$ (or $H = K_1$) and $d(H) = 2$ ($d(G) = 2$), or
b) $|V(G)| > 1$ and $|V(H)| > 1$.

Proof. (1) The first case is obvious, as every complete graph is d.v.i., except
$K_1$ and $K_2$. $G + H$ is a complete graph if and only if $G$ is a complete graph and $H$
is a complete graph.

(2) If $d(G + H) = 2$ and all edges in $E(G) \cup E(H)$ are connected to a single vertex
$v$ then $d(G + H - v) = 1$, a contradiction.

a) Now let $G = K_1 = \{v\}$. Then $d(G + H - v) = d(G + H)$ if and only if $d(H) = 2$.
For all vertices $u \in V(H)$ we have $d(G + H - u) = 2$, as there exists at least one
edge $ab \in E(H - u)$ and $d(G + H - u) \leq 2r(G + H - u) \leq 2e(v) = 2$.

b) Let $G$ and $H$ have both at least 2 vertices. Consider $v \in V(G + H)$ and a graph
$G + H - v$. For all $u, w \in V(G + H - v)$ we have $d(u, w) = 1$ if $u \in G$, $w \in H$ and
d($u, v$) $\leq 2$ if $u, w \in H$ (or $u, w \in G$). The fact that $E(G + H - v) \geq 1$ implies that
d($G + H - v) = 2$. □

The next observation is obvious.

Theorem 2.2. The join of graphs $G, H$ is diameter-adding-invariant of radius 2
if and only if $|E(G)| + |E(H)| \geq 2$.

Consider a finite connected graph $I$. Let $\{G_i: i \in V(I)\}$ be a class of graphs
indexed by a finite set $V(I)$.

The Sabidussi sum $S^+(\{G_i: i \in V(I)\})$ (or shortly $S^+$) of $\{G_i: i \in V(I)\}$ is a
graph defined as follows:

\[ V(S^+(\{G_i: i \in V(I)\})) = \bigcup \{V(G_i): i \in V(I)\}, \ E(S^+(\{G_i: i \in V(I)\})) = \bigcup \{E(G_i): i \in V(I)\} \cup \{xy: x \in V(G_i), y \in V(G_j), ij \in E(I)\}. \]

Sabidussi sum is sometimes called X-join. One can show that $d(S^+(\bigcup \{G_i: i \in V(I)\})) = d(I)$.

Theorem 2.3. Let $I$ be a graph of diameter $d \geq 2$. For any class of connected graphs $\{G_i : i \in V(I)\}$ with $|V(G_i)| \geq 2$ for all $i$, the Sabidussi sum $S^+\{(G_i : i \in V(I))\}$ is diameter-edge-invariant with diameter $d$. Moreover, if $I$ is diameter-edge-invariant then $S^+\{(G_i : i \in V(I))\}$ is diameter-edge-invariant without the restriction of $|V(G_i)| \geq 2$.

However, the assumption that $G_i$ be connected is unnecessary for $d \geq 3$.

Theorem 2.4. Let $I$ be a graph of diameter $d \geq 3$. For any class of graphs $\{G_i : i \in V(I)\}$ with $|V(G_i)| \geq 2$ for all $i$, the Sabidussi sum $S^+\{(G_i : i \in V(I))\}$ is diameter-edge-invariant with diameter $d$.

Proof. It is sufficient to show that in any $S^+ - e$ there are no vertices $u, v$ at distance greater than $d \geq 3$. If $u, v$ are from the same graph $G_i$ or if $u \in V(G_i)$, $v \in V(G_j)$, $d(i, j) > 1$, then there are at least 2 edge-disjoint paths of length at most $d$ joining $u$ and $v$. Therefore $d_{S^+-e}(u, v) \leq d$ for all $e \in E(S^+)$. Let $u \in V(G_i), v \in V(G_j)$ be two vertices such that $ij \in E(I)$ and suppose that there is no other path of length at most $d$ joining $u, v$. Since $d(I) > 2$, we have at least one vertex $w \in I$ adjacent to $i$ (or $j$), some other vertex $a \in V(G_i)$ (or $a \in V(G_j)$) and some vertex $b \in V(G_w)$. But then we have at least two edge-disjoint paths of length at most three joining $u$ and $v$—the edge $uv$ and the path $u-a-b-v$. Therefore $d_{S^+-e}(u, v) \leq 3 \leq d$ for all $e \in E(S^+)$.

We can prove similar result for d.v.i. graphs:

Theorem 2.5. Let $I$ be a graph of diameter $d \geq 2$. For any class of graphs $\{G_i : i \in V(I)\}$ with $|V(G_i)| \geq 2$ for all $i$, the Sabidussi sum $S^+\{(G_i : i \in V(I))\}$ is diameter-vertex-invariant with diameter $d$. Moreover, if $I$ is diameter-vertex-invariant then $S^+\{(G_i : i \in V(I))\}$ is diameter-vertex-invariant without the restriction of $|V(G_i)| \geq 2$.

Proof. If $|V(G_i)| \geq 2$ then for any two vertices $u, v$ at distance $d(u, v) \geq 2$, there are at least two vertex-disjoint paths of length $d(u, v)$. Therefore $d_{S^+ - w}(u, v) \leq d$ for all $w \neq u, v$. Let $i, j$ be two vertices of graph $I$ such that $d(i, j) = d(I)$. As $V(G_i) \geq 2$ and $V(G_j) \geq 2$, for all $w \in V(S^+)$ there are at least two vertices at distance $d$ in $S^+ - w$. Finally, $d(S^+ - w) = d(S^+)$ and $S^+$ is d.v.i. The second part of the result is obvious.
Theorem 2.6. Let \( I \) be a diameter-adding-invariant graph of diameter \( d \geq 2 \). For any class of graphs \( \{G_i: i \in V(I)\} \), the Sabidussi sum \( S^+(\{G_i: i \in V(I)\}) \) is diameter-adding-invariant with diameter \( d \).

Proof. We will prove this theorem by contradiction. Let \( S^+ \) be not a d.a.i. graph. It is clear that for all vertices \( a, b \in G_k \) there is \( d(S^+ + ab) = d(S^+) = d(I) \). Thus we have two vertices \( v \in G_i, u \in G_j \) such that \( d(S^+ + uv) < d(S^+) = d(I) \). But then \( d(I + ij) \leq d(S^+ + uv) < d(S^+) = d(I) \), a contradiction. \( \square \)

The corona \( G \odot H \) of graphs \( G \) and \( H \) was defined by Frucht and Harary ([4], see also [2]) as the graph obtained by taking one copy of \( G \) of order \( p_G \) and \( p_G \) copies of \( H \), and then joining the \( i \)'th vertex of \( G \) to every vertex in the \( i \)'th copy of \( H \). If the \( i \)'th vertex is named \( v \), then the copy belonging to \( v \) will be named \( H_v \).

It is clear that if \( p_G > 1 \), \( r(G) = r_G \), \( d(G) = d_G \), then \( r(G \odot H) = r_G + 1 \), \( d(G \odot H) = d_G + 2 \) and \( v \) is a central vertex of \( G \odot H \) if and only if \( v \) is a central vertex of \( G \). Moreover, \( h \in H_v \) is a peripheral vertex of \( G \odot H \) if and only if \( v \) is a peripheral vertex in \( G \). Since \( d(G \odot H - v) = \infty \) for \( v \in G \) and \( e_{G \odot H - h_v}(h) > d(G \odot H) \) for the peripheral vertex \( v \) of the graph \( G \) and \( h \in H_v \), the corona of two graphs will never be d.e.i. or d.v.i.

The paper [1] gives the following theorem:

Theorem 2.7. For any graphs \( G, H \), such that \( |V(G)| \geq 3 \), the corona \( G \odot H \) is radius-adding-invariant if and only if \( G \) is radius-adding-invariant.

For the diameter of \( G \odot H \) the following theorem holds:

Theorem 2.8. For any graphs \( G, H \), such that \( |V(G)| \geq 3 \), \( H \neq K_1 \) the corona \( G \odot H \) is diameter-adding-invariant if and only if \( G \) is diameter-adding-invariant.

Proof. (\( \Rightarrow \)) Suppose that \( G \odot H \) is d.a.i., but \( G \) is not d.a.i. Let \( e \in E(\overline{G}) \) be an edge such that \( d(G + e) < d(G) \). Therefore

\[
d(G \odot H + e) = d((G + e) \odot H) = d(G + e) + 2 < d(G) + 2 = d(G \odot H),
\]

a contradiction.

(\( \Leftarrow \)) We consider various possibilities for an edge \( e \in E(\overline{G \odot H}) \).

1. If \( e \in E(\overline{G}) \), then

\[
d(G \odot H + e) = d(G + e) + 2 = d(G) + 2 = d(G \odot H).
\]

2. If \( e \in E(\overline{H_v}) \) for any \( v \in V(G) \), then for all \( w \in V(G \odot H) \) we have \( e_{G \odot H}(w) = e_{G \odot H + e}(w) \) and thus \( d(G \odot H) = d(G \odot H + e) \).

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(3) Suppose $e = uh_v$ where $u \in V(G), h_v \in H_v, v \neq u$. Let $d(G \circ H + e) < d(G \circ H)$. If $x$ and $y$ are two peripheral vertices of $G \circ H$ such that $d(x, y) = d(G \circ H)$, then the $x$-$y$ geodesic in $G \circ H + e$ must contain $e$. Moreover, if $x \notin H_v$ and $y \notin H_v$ then $u-h_v-v$ is a part of the $x$-$y$ geodesic in $G \circ H + e$. But then for all such pairs $d_{G \circ H + uv}(x, y) < d(G \circ H)$. On the other hand let, for example, $x \in H_v$. It is clear that for all $z \in H_v, z \neq x$ we have $d_{G \circ H + e}(y, z) \geq d_{G \circ H + e}(y, x) + 1$. But then again $d_{G \circ H + uv}(x, y) < d(G \circ H)$ and $d_{G \circ H + uv}(x, h_v) < d(G \circ H)$. This leads to the case (1) which was discussed above.

(4) Finally, suppose $e = h_u h_v$ where $u, v \in V(G), h_u \in H_u, h_v \in H_v, v \neq u$. Let $d(G \circ H + e) < d(G \circ H)$. It is obvious that for all $h'_u \in H_u, h'_v \in H_v, h'_u \neq h_u, h'_v \neq h_v$ we have $e_{G \circ H + e}(h'_u) \geq e_{G \circ H + e}(h_u)$ and $e_{G \circ H + e}(h'_v) \geq e_{G \circ H + e}(h_v)$. Thus if $x$ and $y$ are two peripheral vertices of $G \circ H$ different from $h_u, h_v$ such that $d(x, y) = d(G \circ H)$, then the $x$-$y$ geodesic in $G \circ H + e$ must contain $e$. Moreover, the $x$-$y$ geodesic must contain a subpath of length three of the form $u-h_u-h_v-v$, $h''_u-h_u-h_v-v$ or $h''_u-h_u-h_v-h''_v$.

Consider the graph $G \circ H + uv$. To obtain an $x$-$y$ path of length less than $d(G \circ H)$ it is sufficient to take $u$-$v$ instead of $u-h_u-h_v-v$, $h''_u-u-v$ instead of $h''_u-h_u-h_v-v$ or $h''_u-h_u-h_v-h''_v$ in the $x$-$y$ geodesic formed in $G \circ H + h_u h_v$. Thus $d_{G \circ H + h_u h_v}(x, y) \geq d_{G \circ H + uv}(x, y)$ and since $d_{G \circ H + uv}(h'_u, h'_v) = d_{G \circ H + uv}(h_u, h_v) = d_{G \circ H + uv}(h_u, h'_v) = d_{G \circ H + uv}(h'_u, h_v)$ we have $d_{G \circ H + uv}(a, b) < d(G \circ H)$ for all $a, b \in V(G \circ H)$. Therefore $d(G \circ H + uv) < d(G \circ H)$. But this is the case (1) which was discussed above.

If $H = K_1$ and $G$ is d.a.i. having $|V(G)| \geq 3$ then $G \circ H$ is not necessarily d.a.i.

Consider the group $\mathbb{Z}_{2r+1}$ and define a graph $G_{\mathbb{Z}_{2r+1}}$ in the following way:

$$V(G) = \{(i, j); \ i, j \in \mathbb{Z}_{2r+1}\},$$

$$(i_1, j_1)(i_2, j_2) \in E(G) \iff |i_1 - i_2| \leq 1 \land |j_1 - j_2| \leq 1.$$

If $(i_1, j_1)$ and $(i_2, j_2)$ are two vertices of $G_{\mathbb{Z}_{2r+1}}$, then $d((i_1, j_1), (i_2, j_2)) = \max\{|i_1 - i_2|, 2r+1 - |i_1 - i_2|, 2r+1 - |j_1 - j_2|\}$. Since for each vertex $u = (i, j)$, there are $8r$ vertices $u_k = (i_k, j_k), i_k = i + k \mod(2r+1) \lor j_k = i + r + 1 \mod(2r+1) \lor j_k = j + r \mod(2r+1) \lor j_k = j + r + 1 \mod(2r+1)$ such that $d(u, u_k) = r$, the graph $G_{\mathbb{Z}_{2r+1}}$ is self-centered of radius $r$.

Now, consider a graph $G'$ obtained in the following way: Suppose $V(G') = V(G_{\mathbb{Z}_{2r+1}}) + v, E(G') = E(G_{\mathbb{Z}_{2r+1}}) + uv$ where $u = (i, j) \in V(G_{\mathbb{Z}_{2r+1}})$. We have $e_{G'}(v) = d(G_{\mathbb{Z}_{2r+1}}) + 1 = d(G')$. Let $f \in E(G')$ be an arbitrary edge. If $f \in E(G_{\mathbb{Z}_{2r+1}})$, then $e_{G_{\mathbb{Z}_{2r+1}}}(w) = e_{G_{\mathbb{Z}_{2r+1}}}(f(w)) = e_{G_{\mathbb{Z}_{2r+1}}}(v) = d(G' + f)$. If $f \notin E(G_{\mathbb{Z}_{2r+1}})$, then $f$ is of type $v(i', j')$ where $i \neq i'$ or $j \neq j'$. It is sufficient to take the vertex $a = (i + r \mod(2r+1), j +
Consider the two following graphs $I_1$, $I_2$:

$$
\begin{array}{c}
1 & 2 & r-1 & r \\
\cdot & \cdot & \cdot & a \\
\cdot & \cdot & \cdot & b \\
\cdot & \cdot & \cdot & c \\
\end{array}
\quad
\begin{array}{c}
1 & 2 & r-2 & r-1 \\
\cdot & \cdot & \cdot & a \\
\cdot & \cdot & \cdot & b \\
\cdot & \cdot & \cdot & c \\
\end{array}
$$

Figure 1

In the first case $d = 2r$, in the second $d = 2r - 1$. Since in both graphs there are three pairs of vertices $\{a, b\}$, $\{b, c\}$, $\{c, a\}$ at distance $d$, and adding a single edge may change at most two of these distances, both graphs are d.a.i. of diameter $d$ for all $r \geq 1$.

Lee [10] showed, as a consequence of Theorem 2.3, that any connected graph is an induced subgraph of a d.e.i. graph of diameter $d \geq 2$. Walikar et. al. [3] showed that for every graph $G$, the graph $H$ formed as $K_2 + G + K_2$ is d.e.i. As a consequence they got that every graph could be embedded in a d.e.i. graph. Later in this section we will show that for each graph $G$, there is an d.e.i., d.v.i. and d.a.i. graph $H$ of diameter $d$ having $G$ as an induced subgraph.

**Lemma 2.9.** Let $G$ be a graph with at least two vertices. Then the graph $H = K_2 + G + K_2$ is diameter-vertex-invariant and diameter-adding-invariant of diameter 2.

**Proof.** One can show that $d(H) = 2$. As $|E(\overline{H})| > 1$, it is clear that $H$ is d.a.i. We can write $H = (K_2 + G) + K_2$. Thus by Theorem 2.1 $H$ is d.v.i. \hfill \Box

**Theorem 2.10.** Every graph $G$ can be embedded as an induced subgraph in a diameter-edge-invariant, diameter-vertex-invariant and diameter-adding-invariant graph $H$ of diameter $d \geq 2$.

**Proof.** Suppose $G$ has at least two vertices. It is sufficient to take the graph $K_2 + G + K_2$ for $d = 2$ and the Sabidussi sum $S^+(\{G_i \equiv G: i \in V(I)\})$ where $I$ is a graph $I_1$ if $d = 2k$ or $I_2$ if $d = 2k + 1$. It follows from the results of the previous section that $S^+$ is d.e.i., d.v.i. and d.a.i.
If $G = K_1$ then it is a subgraph of any graph, and as for each $d$ there exists d.e.i.,
d.v.i. and d.a.i. graph $H$, the theorem holds.

Because of the previous theorem, we cannot obtain a forbidden subgraph character-
ization for all d.e.i., d.v.i., and d.a.i. graphs.

Bálint and Vacek in [1] constructed several r.e.i., r.v.i. and r.a.i. graphs. We will
now show that there are graphs which radius and diameter are both invariant.

**Theorem 2.11.** Let $r$, $d$ be natural numbers such that $2 \leq r < d \leq 2r$. Let $G$ be a graph with at least two vertices. Then there exists a radius-edge-
invariant, diameter-edge-invariant, radius-vertex-invariant and diameter-vertex-
invariant graph $H$ such that $r(H) = r$, $d(H) = d$, $C(H) = V(G)$ and $G$ is an
induced subgraph of $H$.

[1] gives a somewhat weaker result with similar graph construction for radius-
invariant graphs only.

**Proof.** For $d \neq 2r - 1$ consider the following graph $Q$:

![Figure 2](image)

$Q$ is formed by 2 central vertices $c_1$, $c_2$; by $2(d - 1) + 1$ rows of vertices in $2(r - 1)$
columns and by 4 additional vertices $v_1$, $v_2$, $u_1$, $u_2$. Every column except 1 and
$2(r - 1)$ (counted from the left side) has $2(d - r) + 1$ vertices. Columns 1 and $2(r - 1)$
have $2(2(d - r) + 1)$ vertices. Vertices $c_1$, $c_2$ are adjacent to all vertices in columns
$r - 1$ and $r$. Vertices $v_1$, $v_2$ ($u_1$, $u_2$) are adjacent and joined to all vertices in row 1
$(2(d - r) + 1)$ and columns 1 and $2(r - 1)$. Vertex in row $k$ and column $l$ is adjacent
to all vertices in row $k$ and columns $l - 1, l + 1$ and to all vertices in column $l$ and rows $k - 1, k + 1$ except the case when $l = r - 1$ or $l = r$.

It is clear that $e(c_1) = e(c_2) = r$, $e(v) > r$ otherwise, and $d(u_i, v_j) = \min\{d(v_i, c_1) + d(c_1, u_j), 2(d - r) + 2\}$. Since $d \neq 2r - 1$ we have $2(d - r) + 2 \leq d$ or $2r \leq d$ and thus $d(u_i, v_j) \leq d$. For any other vertex $x$, $x \neq c_i$, $x \neq u_i$ (or $v_i$) we have $d(x, u_i) \leq \min\{2(d - r) + 1, 2r - 2\} \leq d$. Now, let $y, z$ be arbitrary vertices except $u_i, v_j, c_i$. When $y, z$ belong to the same row and the same half (right or left) of $Q$ we obviously have $d(y, z) < r < d$. Consider a shortest cycle $F$ such that $y, z \in F$. The length of the cycle $F$ can be at most $2 + 2(d - r) + 2(r - 1) = 2d$ if it is made as a sequence of $y - c_1, c_1 - z, z - u_i$ (or $z - v_i$), $u_i - y$ (or $v_i - y$) geodesics or less otherwise. This implies $d(x, y) \leq d$. Thus for all $w \in V(Q)$ we have $e(w) \leq d$.

To obtain vertices $o, p$ such that $d(o, p) = d$ it is sufficient to take the vertex $o$ in row $1$ and column $1$ and the vertex $p$ in row $2(d - r) + 1$ and column $d + 1$. This implies that $r(Q) = r$ and $d(Q) = d$. Note: There are more than one pair of such vertices.

Since for every vertex $a, a \neq c_i$ there are at least two edge and vertex-disjoint $c_1 - a$ (or $c_2 - a$) paths, and, in addition, there are four vertices in the graph $Q$ at distance $r$ from $c_1, c_2$, we have $r(Q - e) = r(Q - b) = r$ for all $e \in E(Q), b \in V(Q)$, $Q$ is r.e.i. and r.v.i.

Next, we will show that $Q$ is also d.e.i. and d.v.i. We have already proved that $e_{Q-e}(c_i) = e_{Q-b}(c_i) = r$. Consider the eccentricity of the vertices $v_i (u_j)$. Let $s$ be any vertex except $v_i (u_j)$ and suppose $s$ does not belong to row $1$ (or $2(d - r) + 1$). Thus there are at least two edge and vertex-disjoint $u_i$-s geodesics. It is clear that $d_{Q-u_1u_2}(u_1, u_2) = 2$ and for all vertices $t$ in row $1$ we have $d(u_i, t) \leq (r - 1) + 2 \leq d$. Thus for all $e \in E(Q), b \in V(Q)$ we have $e_{Q-e}(u_i) \leq d$ and $e_{Q-b}(u_i) \leq d$.

Now let $y, z$ be arbitrary vertices except $u_i, v_i, c_i$. One can show that if vertices $y, z$ do not lie in the same row and the same half of the graph $Q$, then the length of at most one of the $y-c_1, c_1-z, z-u_i$ (or $z-v_i$), $u_i-y$ (or $v_i-y$) geodesics is different in $Q$ and in $Q - e (Q - b)$. It follows directly from the construction of $Q$ that the difference in lengths of these paths can be at most $1$. Consider a shortest cycle $F'$ such that $y, z \in F'$. The length of the cycle $F'$ can be at most $2 + 2(d - r) + 2(r - 1) + 1 = 2d + 1$ if it is made as a sequence of $y-c_1, c_1-z, z-u_i$ (or $z-v_i$), $u_i-y$ (or $v_i-y$) geodesics in $Q - e (Q - b)$. Thus $d_{Q-e}(y, z) \leq d$ and $d_{Q-b}(y, z) \leq d$.

We can obtain vertices $o, p \in V(Q - b)$ such that $d(o, p) = d$ in the same way as in $Q$. Finally, for $d \neq 2r - 1$ the graph $Q$ is r.e.i., r.v.i., d.e.i. and d.v.i. of radius $r$ and diameter $d$.

For $d = 2r - 1$ it is sufficient to take only $d - 1$ rows of vertices. It is clear that $d(u_i, v_j) = d$. All other facts could be proved similarly as above and we leave the details to the reader.
The desired graph \( H \) is obtained from the graph \( Q \) by substituting the graph \( G \) instead of the vertices \( c_1, c_2 \). □

**Theorem 2.12.** Let \( r, d \) be natural numbers such that \( r \leq d \leq 2r \). Then there exists a radius-adding-invariant and diameter-adding-invariant graph \( G \) such that \( r(G) = r \) and \( d(G) = d \).

**Proof.** It is sufficient to take the tree \( I_1 \) if \( d = 2r \) and the following tree for \( d = 2r - 1 \).

\[
\begin{array}{cccc}
1 & 2 & r-2 & r-1 \\
\end{array}
\]

Figure 3

Otherwise the desired graph can be constructed as follows: Denote \( G_0 = G_{2k+1} \) where \( k = 2r - d \geq 2 \). From [1] we have that \( G_0 \) is r.a.i. Since \( G_0 \) is self-centered and \( r(G_0 + e) \leq d(G_0 + e) \leq d(G_0) = r(G_0) \) it is also d.a.i.

We will construct a graph \( G_{i+1} \) from the graph \( G_i \) as \( G_{i+1} = G_i \circ H, H \neq K_1 \). From Theorem 2.7 and from Theorem 2.8 it follows directly that every graph \( G_i \) is r.a.i. and d.a.i. For \( i = d - r \) we have an r.a.i. and d.a.i. graph \( G_{d-r} \) such that \( r(G_{d-r}) = i \cdot 1 + r(G_0) = (d - r) + (2r - d) = r \) and \( d(G_i) = i \cdot 2 + d(G_0) = 2(d - r) + (2r - d) = d \). □

Walikar, Buckley and Itagi [13] showed that any graph \( G \) of diameter 2 is d.e.i. if and only if every edge of \( G \) is contained in a triangle and if there are at least two geodesics for all vertices \( v, w \) at distance 2. As we have already stated, a graph \( G \) of diameter \( d = 2 \) is d.a.i. if and only if \( E(\overline{G}) \geq 2 \). For d.v.i. graphs we have the following result.

**Theorem 2.13.** Suppose that a graph \( G \) has diameter 2. Then \( G \) is diameter-vertex-invariant if and only if

1. for all \( u, v \in V(G) \) such that \( d(u, v) = 2 \) there are at least two \( u-v \) geodesics,
2. there are at least two edges \( a_1a_2, b_1b_2 \in E(\overline{G}) \) not incident with the same vertex.

**Proof.** (\( \Longrightarrow \))

(1) Suppose there is only one such geodesic \( u-x-v \). Then \( d_{G-x}(u,v) \geq 3 \), a contradiction.
(2) Let all edges in $E(G)$ have one joint incident vertex $v$. Then $G - v$ is a complete graph. Therefore $d(G - v) = 1$ which is again a contradiction.

($\Leftarrow$) Consider an arbitrary vertex $w \in V(G)$ and the graph $G - w$. From (2) it follows that we have $E(G - w) \geq 1$, and thus $d(G - w) > 1$. For any two vertices $u, v \in V(G - w)$ there is $d_G(u, v) \leq 2$. If $d_G(u, v) = 2$, then from (1) it follows that there must be some path $u - a - v$ in $G - w$. Therefore $d(u, v) = 2$. 

3. Some bounds

A $k$-depth spanning tree ($k$-DST) of a graph $G$ is a spanning tree of $G$ of height $k$. It must be true that $k \leq d$, and if $k = d$, such trees must be rooted at a peripheral vertex. A breadth first search algorithm beginning with any vertex $v$ such that $e(v) = k$ will always produce a $k$-DST. Moreover, if $d(u, v) = i$ then the vertex $u$ belongs to level $i$. We will consider only breadth first search distance spanning trees later in this paper.

Theorem 3.1. Let $G$ be a diameter-edge-invariant graph with $n$ vertices and diameter $d$. Then for all $v \in V(G)$

(1) $2 \leq \deg(v) \leq n - \frac{1}{2}(3d - 6)$ (except $d = 2$ where it is $2 \leq \deg(v) \leq n - 1$) if $d$ is even and

(2) $2 \leq \deg(v) \leq n - \frac{1}{2}(3d - 5)$ if $d$ is odd.

Moreover, all these bounds are sharp.

Proof. The lower bound is obvious as $G$ has no bridges. Consider a $d$-DST rooted at a peripheral vertex $x$.

There must be at least one vertex $y$ on level $d$. As $G$ is d.e.i. there are at least two edge-disjoint $x$-$y$ paths of length $d$ in $G$. Thus there are no levels $i$, $i + 1$ both with only one vertex. Because of this we have at most $\frac{1}{2}d + 1$ levels with only one vertex if $d$ is even and at most $\frac{1}{2}(d + 1)$ levels with only one vertex if $d$ is odd.

Any vertex $v$ on level $i$ can be adjacent only to vertices on levels $i - 1$, $i$, $i + 1$. Thus there are at least $d - 2$ remaining levels with vertices which are not adjacent to $v$. At most $\frac{1}{2}d$ ($\frac{1}{2}(d - 1)$ if $d$ is odd) of these levels have only one vertex.

Therefore

$$\deg(v) \leq n - 1 - 2\left(\frac{d}{2} - 2\right) + \frac{d}{2} = n - \frac{3d - 6}{2}$$

if $d$ is even and

$$\deg(v) \leq n - 1 - 2(d - 2) + \frac{d - 1}{2} = n - \frac{3d - 5}{2}$$

if $d$ is odd.
There is one exception. For $d = 2$ it is $\frac{1}{2}(3d - 6) = 0$. But for any graph $G$ it must hold $\deg(v) \leq n - 1$.

To obtain a graph which reaches the bound it is sufficient to take $H_1 = K_{n - \frac{d}{2}d + 1}$ in the graph $G_1$ if $d$ is even and $H_2 = K_{n - (3d - 1)/2}$ in the graph $G_2$ if $d$ is odd. In both graphs $x$ has the minimal and $z$ the maximal possible degree.

Lee [11] gave the bound for the minimal number of vertices in d.e.i. graphs of diameter $d$ which is $\frac{3}{2}d + 1$ vertices if $d$ is even and $\frac{3}{2}(d + 1)$ vertices if $d$ is odd.

**Theorem 3.2.** Let $G$ be a diameter-vertex-invariant graph with $n$ vertices and diameter $d$. Then for all $v \in V(G)$

1. $\deg(v) = n - 1$, if $d = 1$,
2. $2 \leq \deg(v) \leq n - 1$ if $d = 2$,
3. $2 \leq \deg(v) \leq n - 3$ if $d = 3$,
4. $2 \leq \deg(v) \leq n - 4$ if $d = 4$ unless $n = 2d + 2 = 10$, for which it is $2 \leq \deg(v) \leq 5$,
5. $2 \leq \deg(v) \leq n - 2d + 3$ if $d \geq 5$.

These bounds are sharp.

**Proof.** The first two statements are obvious. If $d = 3$ then there is no vertex $v$ such that $e(v) = n - 2$. Otherwise there is a unique vertex $u$ such that $d(u, v) = 2$. Thus $d(G - u) \leq 2r(G - u) = 2e_{G - u}(v) = 2$, a contradiction.

Suppose that $d(G) \geq 4$. Consider two vertices $u, v$ such that $d(u, v) = d$ and two $d$-DST $T_1, T_2$ rooted at peripheral vertices $v$ and $u$. Since $G$ has no cut-vertices, each of these trees has at least 2 vertices on each of the levels $1, \ldots, d - 1$. We will prove the bound by a contradiction.

Let there be a vertex $w$ such that $\deg(w) > n - 2d + 3$. If it belongs to level $i$, then it could be adjacent only to vertices on levels $i - 1, i, i + 1$ (if such exist). Since $\deg(w) > n - 2d + 3$, for $d - 2$ levels there remain at most $2d - 5$ vertices. Thus

1. $w$ is adjacent to every vertex on level $i - 1, i, i + 1$, or
(2) for all trees $T_1, T_2$ there is exactly 1 vertex on each of the levels 0 and $d$ and 2 vertices on every other level except $i - 1, i, i + 1$.

Moreover, it is clear that there is a diametral path $P$ such that $w \in P$.

(1) At least one tree $T_i$ contains the vertex $w$ on level $i \geq \lceil \frac{1}{2}d \rceil$. Let it be the tree $T_1$ and let it contain only one vertex (for example $u$) on level $d$. Then we can prove that $d(G - u) = d - 1$: Let $a_1, a_2$ be two vertices on levels higher than $i$ and $b_1, b_2$ be two vertices on levels lower than $i$. Therefore $d(a_i, b_k) < d(u, b_k) \leq d$. As $d(a_i, w) < \frac{1}{2}d$ we have $d(a_1, a_2) < d$. Moreover, $G$ is d.v.i., and thus the vertices $b_1, b_2$ lie on a cycle. The vertex $w$ is adjacent to all vertices on level $i - 1$ and therefore the length of this cycle must be less than $2d$. Thus $d(b_1, b_2) < d$. Finally, $d(G - u) = d - 1$, a contradiction. As a result of this part we already get that $\Delta(G) \leq n - 2d + 4$.

Let the tree $T_1$ contain two vertices on level $d$ and let $\Delta(G) = n - 2d + 4$. Thus there are exactly 2 vertices on each level 1, ..., $i - 2$. Let us mark the vertices on level 2 as $c_1, c_2$. It must be $\deg(c_1) > 2$ and $\deg(c_2) > 2$. Otherwise, if $xc_2 \in E(G), x \neq v$ then

$$d(G - x) \geq e_{G-x}(c_j) \geq d(c_i, u) = d(c_i, v) + d(v, u) = d + 1 > d.$$ 

If $c_1c_2 \in E(G)$ or if $i - 1 > 2$ (and thus there are only 2 vertices on level 2), then in $G - v$ all vertices on levels lower than $i$ lie on a cycle of length less than $2d$. Similarly as in previous part $d(G - v) = d - 1$.

Now, consider the case in which $c_1c_2 \in E(G)$ and $i - 1 = 2$. Then $d_{G-v}(c_1, c_2) \leq 4$ and thus for any vertex $y \in V(G - v)$ we have $e_{G-v}(y) \leq \max\{4, d - 1\}$. Finally, it holds $\Delta(G) \leq n - 2d + 3$ with the exception of $d = 4$. In that case we cannot use the same arguments as those given in the previous paragraph. Therefore, we obtain only the inequality $\Delta(G) \leq n - 2d + 4 = n - 4$.

If $n = 2d + 2 = 10$, then there are at most 3 vertices on level 2. In that case $d_{G-v}(c_1, c_2) \leq 2$ and thus $e_{G-v}(y) \leq \max\{2, d - 1\} < d$ for all $y \in V(G - v)$. Therefore $\Delta(G) \leq n - 2d + 3 = 5$.

(2) Suppose $\Delta(G) \geq n - 2d + 4$. We can use the same arguments and notations as above. If, for example $d(u, w) < \frac{1}{2}d$ then $d(G - u) = d - 1$. If $d(u, w) = d(w, v) = \frac{1}{2}d$ then for a tree $T_1$ rooted at central vertex $v$ with the vertex $w$ on level $i$ either $w$ is adjacent to every vertex on level $i - 1$ or $w$ is adjacent to every vertex on level $i + 1$. Thus $d(G - v) = d - 1$ in the first case or $d(G - u) = d - 1$ in the second case.

Suppose $4 \neq d \geq 3$ or $2d + 2 = 10 = n$. The graph $G$ (where $H = K_{n-2d}$, see Figure 5) certifies that our bounds are sharp. The following graph (see Figure 6) is for $d = 4, n \neq 10$ ($H = K_{n-10}$).

For $d = 2$ it is sufficient to take $C_4$ and substitute any vertex of $C_4$ with $K_{n-3}$. □
**Theorem 3.3.** Diameter-vertex-invariant graph of diameter \( d \geq 3 \) has at least \( 2d + 2 \) vertices.

To obtain a d.v.i. graph with \( 2d + 2 \) vertices is sufficient to take \( K_2 \) instead of \( H \) in Figure 5.

![Figure 5](image)

**Theorem 3.4.** Let \( G \) be a diameter-adding-invariant graph with \( n \) vertices and diameter \( d \geq 3 \). Then for all \( v \in V(G) \)

1. \( \deg(v) \leq n - \frac{3}{2}d + 2 \) if \( d \) is even,
2. \( \deg(v) \leq n - \frac{3}{2}(d + 1) + 3 \) if \( d \) is odd.

These bounds are sharp.

**Proof.** Consider a diametral \( u-v \) path and the cycle \( F \) of length \( d + 1 \) in the graph \( G + uv \) formed by the \( u-v \) path and the edge \( uv \). The eccentricity of every vertex \( w \) in the subgraph \( F \) is \( \lfloor \frac{1}{2}d \rfloor \). Also \( d_F(s, t) = d_{G+uv}(s, t) \) for all \( s, t \in F \). Moreover, since \( G \) is d.a.i., there are at least two vertices \( x, y \in V(G+uv) \) such that \( d_{G+uv}(x, y) = d \).

**Case 1:** \( x \in F \)

Let \( z \) be the last joint vertex of the \( x-y \) geodesic and of the cycle \( F \). One can prove that \( d_{G+uv}(z, y) \geq \lfloor \frac{1}{2}d \rfloor \). For every \( a \in V(G + uv) \) we have:

1. \( a \) is adjacent to at most 3 successive vertices of \( F \). Otherwise \( d_G(u, v) < d(G) \).
2. \( a \) is adjacent to at most 3 successive vertices of any \( z-y \) geodesic. Otherwise \( d_{G+uv}(x, y) < d(G) \).
3. \( a \) is adjacent to at most 4 vertices of the cycle \( F \) and of some \( z-y \) geodesic together. (Only if \( a \) is adjacent to \( z \) and its neighbours.) Otherwise \( d_{G+uv}(x, y) < d(G) \).

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(4) if $a = z$ then it is adjacent to at most 3 vertices of the cycle $F$ and of some $z-y$ geodesic together.

Case 2: $x \notin F$, $y \notin F$

It is clear that the $x-y$ geodesic contains at most $\lceil \frac{1}{2}d \rceil$ vertices of cycle $F$. If two vertices $b, c$ belong to $F$ and to the $x-y$ geodesic, then some $b-c$ geodesic belongs to $F$. For every $a \in V(G + uv)$ we have:

1. $a$ is adjacent to at most 3 successive vertices of $F$. Otherwise $d(u, v)_G < d(G)$.
2. $a$ is adjacent to at most 3 successive vertices of any $x-y$ geodesic. Otherwise $d_{G+uv}(x, y) < d(G)$.
3. If the cycle $F$ and the $x-y$ geodesic have $\lceil \frac{1}{2}d \rceil$ vertices in common, then $a$ is adjacent to at most 4 vertices of the cycle $F$ and the $x-y$ geodesic together. If the cycle $F$ and the $x-y$ geodesic have $\lceil \frac{1}{2}d \rceil - i$ vertices in common, then $a$ is adjacent to at most $4 + i$ vertices of the cycle $F$ and the $x-y$ geodesic together. Otherwise $d_{G+uv}(x, y) < d(G)$.
4. If $a$ belongs both to $x-y$ geodesic and to the cycle $F$ then it is adjacent to at most 3 vertices of the cycle $F$ and the $x-y$ geodesic together.

Thus $a$ is adjacent to at most $n - 1 - (d + 1 + \lceil \frac{1}{2}d \rceil - 4)$ vertices which is the same as the bounds.

To obtain a graph which certifies that the bounds are the best possible it is sufficient to take the graphs $I_1$ ($I_2$) and substitute some central vertex with the graph $K_{n-3d/2}$ (or $K_{n-(3d+1)/2}$).

The next bound follows immediately from the proof of the previous theorem.

**Theorem 3.5.** Diameter-adding-invariant graph of diameter $d$ has at least

1. $\frac{3}{2}d + 1$ vertices if $d$ is even,
2. $\frac{1}{2}(3d + 1)$ vertices if $d$ is odd.

**References**


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