Vladimir D. Samodivkin
Minimal acyclic dominating sets and cut-vertices

*Mathematica Bohemica*, Vol. 130 (2005), No. 1, 81–88

Persistent URL: [http://dml.cz/dmlcz/134216](http://dml.cz/dmlcz/134216)

**Terms of use:**

© Institute of Mathematics AS CR, 2005

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use.*
MINIMAL ACYCLIC DOMINATING SETS AND CUT-VERTECIES

VLADMIR SAMODIVKIN, Sofia

(Received April 7, 2004)

Abstract. The paper studies minimal acyclic dominating sets, acyclic domination number and upper acyclic domination number in graphs having cut-vertices.

Keywords: cut-vertex, dominating set, minimal acyclic dominating set, acyclic domination number, upper acyclic domination number

MSC 2000: 05C69, 05C40

For the graph theory terminology not presented here, we follow Haynes et al. [3]. All our graphs are finite and undirected with no loops or multiple edges. We denote the vertex set and the edge set of a graph $G$ by $V(G)$ and $E(G)$, respectively. The subgraph induced by $S \subseteq V(G)$ is denoted by $\langle S, G \rangle$. For any vertex $v$ of $G$ its open neighborhood $N(v, G)$ is $\{x \in V(G); vx \in E(G)\}$ and its closed neighborhood $N[v, G]$ is $N(v, G) \cup \{v\}$. For a set $S \subseteq V(G)$ its open neighborhood $N(S, G)$ is $\bigcup_{v \in S} N(v, G)$, its closed neighborhood $N[S, G]$ is $N(S, G) \cup S$. A subset of vertices $A$ in a graph $G$ is said to be acyclic if $\langle A, G \rangle$ contains no cycles. Note that the property of being acyclic is a hereditary property, that is, any subset of an acyclic set is itself acyclic. A dominating set in a graph $G$ is a set of vertices $D$ such that every vertex of $G$ is either in $D$ or is adjacent to an element of $D$. A dominating set $D$ is a minimal dominating set if no proper subset $D' \subset D$ is a dominating set. The set of all minimal dominating sets of a graph $G$ is denoted by $\text{MDS}(G)$. The domination number $\gamma(G)$ of a graph $G$ is the minimum cardinality taken over all dominating sets of $G$. The literature on this subject has been surveyed and detailed in the two books by Haynes et al. [4], [5].

A given graph invariant can often be combined with another graph theoretical property $P$. Harary and Haynes [3] defined the conditional domination number $\gamma(G : P)$ as the smallest cardinality of a dominating set $S \subseteq V(G)$ such that the
subgraph $\langle S, G \rangle$ induced by $S$ has property $P$. One of the many possible properties imposed on $S$ is:

$P_{ad}$: $\langle S, G \rangle$ has no cycles.

The conditional domination number $\gamma(G : P_{ad})$ is called the \textit{acyclic domination number} and is denoted by $\gamma_a(G)$. The concept of acyclic domination in graphs was introduced by Hedetniemi et al. [6]. An acyclic dominating set $D$ is a \textit{minimal acyclic dominating set} if no proper subset $D' \subset D$ is an acyclic dominating set. The \textit{upper acyclic domination number} $\Gamma_a(G)$ is the maximum cardinality of a minimal acyclic dominating set of $G$. The set of all minimal acyclic dominating sets of a graph $G$ is denoted by $\text{MD}_aS(G)$. For every vertex $x$ of a graph $G$ let $\text{MD}_aS(x, G) = \{D \in \text{MD}_aS(G); x \in D\}$.

Let us introduce the following assumption

(*) a graph $H$ is the union of two connected graphs $H_1$ and $H_2$ having exactly one common vertex $x$ and $|V(H_i)| \geq 2$ for $i = 1, 2$.

In this paper we deal with minimal acyclic dominating sets, acyclic domination number and upper acyclic domination number in graphs having cut-vertices. Observe that domination and some of its variations in graphs having cut-vertices has been the topic of several studies—see for example [1, 7, 5 Chapter 16].

1. \textbf{Minimal acyclic dominating sets}

In this section we begin an investigation of minimal acyclic dominating sets in graphs having cut-vertices.

The following lemma will be used in the sequel, without specific reference.

\textbf{Lemma A} [5, Lemma 2.1]. For any graph $G$, $\text{MD}_aS(G) \subseteq \text{MDS}(G)$.

\textbf{Theorem 1.1.} Let $H_1, H_2$ and $H$ be graphs satisfying (*). Let $M \in \text{MD}_aS(x, H)$ and $M_j = M \cap V(H_j)$, $j = 1, 2$. Then one of the following holds:

(i) $M_j \in \text{MD}_aS(x, H_j)$ for $j = 1, 2$;

(ii) there are $l$ and $m$ such that $\{l, m\} = \{1, 2\}$, $M_l \in \text{MD}_aS(x, H_l)$, and $M_m - \{x\}$ is the unique subset of $M_m$ which belongs to $\text{MD}_aS(H_m)$.

\textbf{Proof.} Since $x \in M$ then $M_j$ is an acyclic dominating set of $H_j$, $j = 1, 2$. Let there be $i \in \{1, 2\}$ such that $M_i \not\in \text{MD}_aS(x, H)$. Suppose $M_j \not\in \text{MD}_aS(x, H_j)$ for $j = 1, 2$. Then there is a vertex $u_1 \in M_1$ and a vertex $u_2 \in M_2$ such that $M_j - \{u_j\}$ is an acyclic dominating set of $H_j$, $j = 1, 2$. Hence $(M_1 - \{u_1\}) \cup (M_2 - \{u_2\}) = M - ((\{u_1\} \cup \{u_2\})$ is an acyclic dominating set of $H$—a contradiction. So, without loss of generality let $M_1 \not\in \text{MD}_aS(x, H_1)$ and $M_2 \in \text{MD}_aS(x, H_2)$. Hence there is a
vertex $u \in M_1$ such that $M_1 - \{u\}$ is an acyclic dominating set of $H_1$. If $u \neq x$ then $M - \{u\}$ is an acyclic dominating set of $H$, which is a contradiction. Hence $u = x$ and $M_1 - \{x\}$ is an acyclic dominating set of $H_1$. Suppose $M_1 - \{x\} \notin M_{D_aS}(H_1)$. Then there is a vertex $w \in M_1 - \{x\}$ such that $M_1 - \{x, w\}$ is an acyclic dominating set of $H_1$. But then $M - \{w\}$ is an acyclic dominating set of $H$—a contradiction. Therefore $M_1 - \{x\} \notin M_{D_aS}(H_1)$. Let $v \in M_1 - \{x\}$. Suppose $M_1 - \{v\}$ is an acyclic dominating set of $H_1$. Then $M - \{v\}$ is an acyclic dominating set of $H$—a contradiction. \hfill \Box

**Theorem 1.2.** Let $H_1, H_2$ and $H$ be graphs satisfying $(\ast)$. Let $M \in M_{D_aS}(H)$, $x \notin M$ and $M_j = M \cap V(H_j)$, $j = 1, 2$. Then one of the following holds:

(i) $M_j \in M_{D_aS}(H_j)$ for $j = 1, 2$;
(ii) there are $l$ and $m$ such that $\{l, m\} = \{1, 2\}$, $M_l \in M_{D_aS}(H_l)$, $M_m \in M_{D_aS}(H_m - x)$ and $M_m$ is no dominating set in $H_m$.

**Proof.** Clearly, there is $i \in \{1, 2\}$ such that $M_i$ is an acyclic dominating set of $H_i$. Without loss of generality let $i = 1$. Suppose $M_1 \notin M_{D_aS}(H_1)$. Then there is $u \in M_1$ such that $M_1 - \{u\}$ is an acyclic dominating set of $H_1$ and then $M - \{u\}$ is an acyclic dominating set of $G$—a contradiction. So $M_1 \in M_{D_aS}(H_1)$. Analogously, if $M_2$ is an acyclic dominating set of $H_2$, then $M_2 \in M_{D_aS}(H_2)$. Now, let $M_2$ be not an acyclic dominating set of $H_2$. Then $M_2$ is an acyclic dominating set of $H_2 - x$. Suppose $M_2 \notin M_{D_aS}(H_2 - x)$. Then there is $v \in M_2$ such that $M_2 - \{v\}$ is an acyclic dominating set of $H_2 - x$ and hence $M - \{v\}$ is an acyclic dominating set of $H$—a contradiction. \hfill \Box

**Theorem 1.3.** Let $H_1, H_2$ and $H$ be graphs satisfying $(\ast)$. Let $M_j \in M_{D_aS}(H_j)$ for $j = 1, 2$ and $x \notin M_1 \cup M_2$. Then one of the following holds:

(i) $M_1 \cup M_2 \in M_{D_aS}(H)$;
(ii) there are $l \in \{1, 2\}$ and $u \in V(H_l)$ such that $\{u\} = N(x, H_l) \cap M_l$, $M_l - \{u\} \in M_{D_aS}(H_l - x)$ and $(M_1 \cup M_2) - \{u\} \in M_{D_aS}(H)$.

**Proof.** Let $M = M_1 \cup M_2$. Then $M$ is an acyclic dominating set of $H$. Suppose $M \notin M_{D_aS}(H)$. Hence, there is a vertex $u \in M$ such that $M - \{u\}$ is an acyclic dominating set of $H$. Without loss of generality let $u \in V(H_1)$. Then $M_1 - \{u\}$ is no acyclic dominating set of $H_1$ and hence $M_1 - \{u\}$ is an acyclic dominating set of $H_1 - x$. Therefore $\{u\} = N(x, H_1) \cap M_1$. Suppose $M_1 - \{u\} \notin M_{D_aS}(H_1 - x)$. Then there is a vertex $v \in M_1 - \{u\}$ such that $M_1 - \{u, v\}$ is an acyclic dominating set of $H_1 - x$. Hence $M_1 - \{v\}$ is an acyclic dominating set of $H_1$—a contradiction. So $M_1 - \{u\} \in M_{D_aS}(H_1 - x)$. Suppose $M - \{u\} \notin M_{D_aS}(H)$. Hence there is a vertex $w, w \in M - \{u\}$ that $M - \{u, w\}$ is an acyclic dominating set of $H$. If $w \in V(H_1)$,
then $M_1 - \{u, w\}$ is an acyclic dominating set of $H_1 - x$—a contradiction. Therefore $w \in V(H_2)$ and then $M_2 - \{w\}$ is an acyclic dominating set of $H_2$—a contradiction. So $M - \{u\} \notin \text{MD}_a S(H)$.

\[\square\]

**Theorem 1.4.** Let $H_1, H_2$ and $H$ be graphs satisfying $(\ast)$. Let $M_j \in \text{MD}_a S(x, H_j)$ for $j = 1, 2$. Then $M_1 \cup M_2 \in \text{MD}_a S(x, H)$.

**Proof.** Let $M = M_1 \cup M_2$. Obviously $M$ is an acyclic dominating set of $H$. Suppose $M \notin \text{MD}_a S(H)$. Then there is a vertex $u \in M$ such that $M - \{u\}$ is an acyclic dominating set of $H$. First, let $u \neq x$ and without loss of generality let $u \in V(H_1) - \{x\}$. Then $M_1 - \{u\}$ is an acyclic dominating set of $H_1$—a contradiction. Secondly, let $u = x$. Now, there is $i \in \{1, 2\}$ such that $M_i - \{x\}$ is an acyclic dominating set of $H_i$, which is a contradiction. So $M \notin \text{MD}_a S(H)$ and since $x \in M$ we have $M \in \text{MD}_a S(x, H)$.

\[\square\]

**Theorem 1.5.** Let $H_1, H_2$ and $H$ be graphs satisfying $(\ast)$. Let $M_1 \in \text{MD}_a S(x, H_1)$, $M_2 \in \text{MD}_a S(H_2)$, $x \notin M_2$ and $M = M_1 \cup M_2$. Then one of the following holds:

(i) $M \in \text{MD}_a S(H)$;

(ii) $M_1 - \{x\} \in \text{MD}_a S(H_1 - x)$ and $M - \{x\} \in \text{MD}_a S(H)$;

(iii) there is $U \subseteq M_2$ such that $(M_2 - U) \cup \{x\} \in \text{MD}_a S(H_2)$ and $M - U \in \text{MD}_a S(H)$;

(iv) no subset of $M$ is an acyclic dominating set of $H$.

**Proof.** Let $M \notin \text{MD}_a S(H)$ and let there exist $M_3 \subset M$ such that $M_3 \in \text{MD}_a S(H)$. First, let $x \notin M_3$. Then $M_1 - \{x\}$ is an acyclic dominating set of $H_1 - x$. Suppose $M_1 - \{x\} \notin \text{MD}_a S(H_1 - x)$. Now, there is a vertex $v \in M_1 - \{x\}$ that $M_1 - \{x, v\}$ is an acyclic dominating set of $H_1 - x$. Hence $M_1 - \{v\}$ is an acyclic dominating set of $H_1$—a contradiction. So, $M_1 - \{x\} \in \text{MD}_a S(H_1 - x)$ and $M - \{x\}$ is an acyclic dominating set of $H$. Now, suppose $M - \{x\} \notin \text{MD}_a S(H)$. Then there is a vertex $w \in M - \{x\}$ such that $M - \{x, w\}$ is an acyclic dominating set of $H$. If $w \in V(H_1)$ then $M_1 - \{x, w\}$ is an acyclic dominating set of $H_1 - x$—a contradiction. If $w \in V(H_2)$, then $M_2 - \{w\}$ is an acyclic dominating set of $H_2$—a contradiction. So $M - \{x\} \in \text{MD}_a S(H)$. Secondly, let $x \in M_3$. Let $U = M - M_3$. If there is $u \in U \cap M_1$, then $M_1 - \{u\}$ is an acyclic dominating set of $H_1$—a contradiction. Hence, $U \subseteq M_2$. Then $(M_2 - U) \cup \{x\} = M_3 \cap V(H_2)$ is an acyclic dominating set of $H_2$. Since $M$ is no minimal acyclic dominating set of $H$ we have $U \neq \emptyset$ and hence $M_2 - U$ is no dominating set of $H_2$. If there is $v \in M_2 - U$ such that $(M_2 - (U \cup \{v\}) \cup \{x\}$ is an acyclic dominating set of $H_2$ then $M_3 - \{v\}$ is an acyclic dominating set of $H$—a contradiction. Hence $(M_2 - U) \cup \{x\}$ is a minimal acyclic dominating set of $H_2$. \[\square\]
2. $\Gamma_a$-sets and $\gamma_a$-sets

In this section we present some results concerning the acyclic domination number and the upper acyclic domination number of graphs having cut-vertices.

Let $\mu(G)$ be a numerical invariant of a graph $G$ defined in such a way that it is the minimum or maximum number of vertices of a set $S \subseteq V(G)$ with a given property $P$. A set with the property $P$ and with $\mu(G)$ vertices in $G$ is called a $\mu$-set of $G$. Fricke et al. [2] define a vertex $v$ of a graph $G$ to be

(i) $\mu$-good, if $v$ belongs to some $\mu$-set of $G$ and
(ii) $\mu$-bad, if $v$ belongs to no $\mu$-set of $G$.

**Theorem 2.1.** Let $H_1$, $H_2$ and $H$ be graphs satisfying (*).

1. Let $x$ be a $\Gamma_a$-good vertex of a graph $H$. Then $\Gamma_a(H) \leq \Gamma_a(H_1) + \Gamma_a(H_2)$. If $\Gamma_a(H) = \Gamma_a(H_1) + \Gamma_a(H_2)$, $M$ is a $\Gamma_a$-set of $H$ and $x \in M$, then there are $l$ and $m$ such that \( \{l, m\} = \{1, 2\} \). If $\Gamma_a(H_1) + \Gamma_a(H_2) - 1 = \Gamma_a(H)$, $M_j$ is a $\Gamma_a$-set of $H_j$, $j = 1, 2$ and \{x\} = $M_1 \cap M_2$ then $M_1 \cup M_2$ is a $\Gamma_a$-set of $H$.

2. Let $x$ be a $\Gamma_a$-good vertex of graphs $H_1$ and $H_2$. Then $\Gamma_a(H_1) + \Gamma_a(H_2) - 1 \leq \Gamma_a(H)$. If $\Gamma_a(H_1) + \Gamma_a(H_2) - 1 = \Gamma_a(H)$, $M_j$ is a $\Gamma_a$-set of $H_j$, $j = 1, 2$ and \{x\} $= M_1 \cap M_2$ then $M_1 \cup M_2$ is a $\Gamma_a$-set of $H$.

3. Let $x$ be a $\Gamma_a$-bad vertex of a $H_1$ and $H_2$. Then $\Gamma_a(H) \geq \Gamma_a(H_1) + \Gamma_a(H_2) - 1$. If $\Gamma_a(H) = \Gamma_a(H_1) + \Gamma_a(H_2)$ and $M_j$ is a $\Gamma_a$-set of $H_j$, $j = 1, 2$ then there are $l \in \{1, 2\}$ and $u \in V(H_l)$ such that \{u\} $= N(x, H_l) \cap M_l$ and $M_1 \cup M_2 - \{u\}$ is a $\Gamma_a$-set of $H$.

4. Let $x$ be a $\Gamma_a$-bad vertex of $H$. Then $\Gamma_a(H) \leq \max\{\Gamma_a(H_1) + \Gamma_a(H_2), \Gamma_a(H_1 - x) + \Gamma_a(H_2), \Gamma_a(H_1) + \Gamma_a(H_2 - x)\}$.

**Proof.**

1. Let $M$ be a $\Gamma_a$-set of $H$, $x \in M$ and $M \cap V(H_j) = M_j$, $j = 1, 2$.

Case $M_j \in \text{MD}_aS(x, H_j)$, $j = 1, 2$: Then $\Gamma_a(H) = |M| = |M_1| + |M_2| - 1 \leq \Gamma_a(H_1) + \Gamma_a(H_2) - 1$.

Case there are $l, m$ such that \{l, m\} = \{1, 2\}, $M_l \in \text{MD}_aS(x, H_l)$ and $M_m - \{x\} \in \text{MD}_aS(H_m)$: We have $\Gamma_a(H) = |M| = |M_l| + |M_m - \{x\}| \leq \Gamma_a(H_l) + \Gamma_a(H_m)$.

If $\Gamma_a(H) = \Gamma_a(H_1) + \Gamma_a(H_2)$, then $|M_l| = \Gamma_a(H_l)$ and $|M_m - \{x\}| = \Gamma_a(H_m)$. Hence $M_l$ is a $\Gamma_a$-set of $H_l$ and $M_m - \{x\}$ is a $\Gamma_a$-set of $H_m$.

There are no other possibilities because of Theorem 1.1.

2. Let $M_j$ be a $\Gamma_a$-set of $H_j$, $j = 1, 2$ and \{x\} $= M_1 \cap M_2$. It follows from Theorem 1.4 that $M_1 \cup M_2 \in \text{MD}_aS(x, H)$. Hence $\Gamma_a(H) \geq |M_1 \cup M_2| = |M_1| + |M_2| - 1 = \Gamma_a(H_1) + \Gamma_a(H_2) - 1$. If $\Gamma_a(H) = \Gamma_a(H_1) + \Gamma_a(H_2) - 1$ then $|M_1 \cup M_2| = \Gamma_a(H)$. Hence $M_1 \cup M_2$ is a $\Gamma_a$-set of $H$.

3. Let $M_j$ be a $\Gamma_a$-set of $H_j$, $j = 1, 2$ and $M = M_1 \cup M_2$. If $M \in \text{MD}_aS(H)$ then $\Gamma_a(H) \geq |M| = |M_1| + |M_2| = \Gamma_a(H_1) + \Gamma_a(H_2)$. Otherwise it follows from
Theorem 1.3 that there are $l \in \{1, 2\}$ and $u \in V(H_l)$ such that \(\{u\} = N(x, H_l) \cap M_l\) and $M - \{u\} \in \text{MD}_{aS}(H)$. Hence $\Gamma_a(H) \geq |M - \{u\}| = |M_1| + |M_2| - 1 = \Gamma_a(H_1) + \Gamma_a(H_2) - 1$. If $\Gamma_a(H) = \Gamma_a(H_1) + \Gamma_a(H_2) - 1$ then $|M - \{u\}| = \Gamma_a(H)$. Hence $M - \{u\}$ is a $\Gamma_a$-set of $H$.

4. Let $M$ be a $\Gamma_a$-set of $H$ and $M_j = M \cap V(H_j)$, $j = 1, 2$. If $M_j \in \text{MD}_{aS}(H_j)$, $j = 1, 2$ then $\Gamma_a(H) = |M| = |M_1| + |M_2| \leq \Gamma_a(H_1) + \Gamma_a(H_2)$. Otherwise it follows from Theorem 1.2 that $M_l \in \text{MD}_{aS}(H_l)$ and $M_m \in \text{MD}_{aS}(H_m - x)$ for some $l, m$ such that \(\{l, m\} = \{1, 2\}\). Hence $\Gamma_a(H) = |M| = |M_1| + |M_m| \leq \Gamma_a(H_l) + \Gamma_a(H_m - x)$. \qed

**Theorem 2.2.** Let $G$ be a graph of order at least two. Then for each vertex $v \in V(G)$ we have $\gamma_a(G) - 1 \leq \gamma_a(G - v) \leq |V(G)| - 1$. If $v \in V(G)$ and $\gamma_a(G) - 1 = \gamma_a(G - v)$ then

(i) $v$ is a $\gamma_a$-good vertex of the graph $G$;

(ii) if $v$ is not isolated and $u \in N(v, G)$ then $u$ is a $\gamma_a$-bad vertex of the graph $G - v$.

**Proof.** Clearly $\gamma_a(G - v) \leq |V(G - v)| = |V(G)| - 1$. Assume $\gamma_a(G - v) < \gamma_a(G)$. Then for an arbitrary $\gamma_a$-set $M$ of the graph $G - v$ we have $N[M, G] = V(G) - \{v\}$ and then $N(v, G) \cap M = \emptyset$. Hence $M \cup \{v\}$ is an acyclic dominating set of $G$ and then $\gamma_a(G) \leq |M \cup \{v\}| = |M| + 1 = \gamma_a(G - v) + 1 \leq \gamma_a(G)$. Therefore $\gamma_a(G) - 1 = \gamma_a(G - v)$ and $M \cup \{v\}$ is a $\gamma_a$-set of $G$. Hence $v$ is a $\gamma_a$-good vertex of $G$. Since $N(v, G) \cap M = \emptyset$ we conclude that each vertex belonging to $N(v, G)$ is a $\gamma_a$-bad vertex of $G - v$. \qed

**Theorem 2.3.** Let $H_1$, $H_2$ and $H$ be graphs satisfying $(\ast)$. Then

1. $\gamma_a(H) \geq \gamma_a(H_1) + \gamma_a(H_2) - 1$.

2. Let $x$ be a $\gamma_a$-bad vertex of the graph $H$, $\gamma_a(H) = \gamma_a(H_1) + \gamma_a(H_2) - 1$ and let $M$ be a $\gamma_a$-set of $H$. Then there are $l, m$ such that \(\{l, m\} = \{1, 2\}\), $M \cap V(H_l)$ is a $\gamma_a$-set of $H_l$, $M \cap V(H_m)$ is a $\gamma_a$-set of $H_m - x$ and $\gamma_a(H_m - x) = \gamma_a(H_m) - 1$.

3. Let $x$ be a $\gamma_a$-good vertex of $H$, $\gamma_a(H) = \gamma_a(H_1) + \gamma_a(H_2) - 1$, let $M$ be a $\gamma_a$-set of $H$ and $x \in M$. Then $M \cap V(H_j)$ is a $\gamma_a$-set of $H$, $j = 1, 2$.

4. Let $x$ be a $\gamma_a$-good vertex of graphs $H_1$ and $H_2$. Then $\gamma_a(H) = \gamma_a(H_1) + \gamma_a(H_2) - 1$. If $M_j$ is a $\gamma_a$-set of $H_j$, $j = 1, 2$ and \(\{x\} = M_1 \cap M_2\) then $M_1 \cup M_2$ is a $\gamma_a$-set of the graph $H$.

5. Let $x$ be a $\gamma_a$-bad vertex of graphs $H_1$ and $H_2$. Then $\gamma_a(H) = \gamma_a(H_1) + \gamma_a(H_2)$.

**Proof.** 1: Let $M$ be a $\gamma_a$-set of $H$ and $M_i = M \cap V(H_i)$, $i = 1, 2$.

Case $x \notin M$: If $M_j \in \text{MD}_{aS}(H_j)$ for $j = 1, 2$ then $\gamma_a(H) = |M| = |M_1| + |M_2| \geq \gamma_a(H_1) + \gamma_a(H_2)$. Otherwise it follows by Theorem 1.2 that there are $l, m$ such that \(\{l, m\} = \{1, 2\}\), $M_l \in \text{MD}_{aS}(H_l)$ and $M_m \in \text{MD}_{aS}(H_m - x)$.
\[ \gamma_a(H) = |M| = |M_l| + |M_m| \geq \gamma_a(H_l) + \gamma_a(H_m - x). \]
Now, Theorem 2.2 yields \[ \gamma_a(H) \geq \gamma_a(H_1) + \gamma_a(H_2) - 1. \]

**Case** \( x \in M \) and \( M_j \in \text{MD}_aS(H_j), \ j = 1, 2 \): It follows that \( \gamma_a(H) = |M| = |M_l| + |M_2| - 1 \geq \gamma_a(H_1) + \gamma_a(H_2) - 1. \)

**Case** \( x \in M \) and there are \( l, m \) such that \( \{l, m\} = \{1, 2\}, M_l \in \text{MD}_aS(H_l) \) and \( M_m - \{x\} \in \text{MD}_aS(H_m): \) We have \( \gamma_a(H) = |M| = |M_l| + |M_m - \{x\}| \geq \gamma_a(H_1) + \gamma_a(H_m). \)

There are no other possibilities because of Theorem 1.1.

2: Let \( M \cap V(H_i) = M_i, i = 1, 2. \) From the proof of 1 we have that there are \( l, m \) such that \( \{l, m\} = \{1, 2\}, M_l \in \text{MD}_aS(H_l), M_m \in \text{MD}_aS(H_m - x), |M_l| = \gamma_a(H_l) \) and \( |M_m| = \gamma_a(H_m - x) = \gamma_a(H_m) - 1. \) Hence the result follows.

3: It follows from the proof of 1 that \( M \cap V(H_i) \in \text{MD}_aS(H_i) \) and \( |M \cap V(H_i)| = \gamma_a(H_i) \) for \( i = 1, 2. \) Hence \( M \cap V(H_i) \) is a \( \gamma_a \)-set of \( H_i, i = 1, 2. \)

4: Let \( M_j \) be a \( \gamma_a \)-set of \( H_j, j = 1, 2 \) and \( \{x\} = M_1 \cap M_2. \) It follows from Theorem 1.4 that \( M_1 \cup M_2 \in \text{MD}_aS(H). \) Hence \( \gamma_a(H) \leq |M_1 \cup M_2| = |M_l| + |M_2| - 1 = \gamma_a(H_1) + \gamma_a(H_2) - 1. \) Now from 1 we have that \( \gamma_a(H) = \gamma_a(H_1) + \gamma_a(H_2) - 1. \) Then \( |M_1 \cup M_2| = \gamma_a(H). \) Therefore \( M_1 \cup M_2 \) is a \( \gamma_a \)-set of \( H. \)

5: Suppose \( \gamma_a(H) = \gamma_a(H_1) + \gamma_a(H_2) - 1. \) If \( x \) is a \( \gamma_a \)-good vertex of \( H \) then by 2 there exists \( m \in \{1, 2\} \) such that \( \gamma_a(H_m - x) = \gamma_a(H_m) - 1. \) Hence by Theorem 2.2 \( x \) is a \( \gamma_a \)-good vertex of \( H_m \)—a contradiction. If \( x \) is a \( \gamma_a \)-good vertex of \( H, M \) is a \( \gamma_a \)-set of \( H \) and \( x \in M \) then by 3 we have \( M \cap V(H_s) \) is a \( \gamma_a \)-set of \( H_s, s = 1, 2. \)

But then \( x \) is a \( \gamma_a \)-good vertex of \( H_s, s = 1, 2, \) which is a contradiction.

Hence \( \gamma_a(H) \geq \gamma_a(H_1) + \gamma_a(H_2). \)

Let \( M_j \) be a \( \gamma_a \)-set of \( H_j, j = 1, 2 \) and \( M = M_1 \cup M_2. \)

**Case** there are \( l \in \{1, 2\} \) and \( u \in V(H_l) \) such that \( \{u\} = N(x, H_l) \cap M_l, M_l - \{u\} \in \text{MD}_aS(H_l - x) \) and \( M - \{u\} \in \text{MD}_aS(H). \) Let \( \{m\} = \{1, 2\} - \{l\}. \) Hence \( \gamma_a(H) \leq |M - \{u\}| = |M_l - \{u\}| + |M_m| = |M_l| - 1 + |M_m| = \gamma_a(H_1) + \gamma_a(H_2) - 1, \) which is a contradiction.

**Case** \( M \in \text{MD}_aS(H): \) Then \( \gamma_a(H_1) + \gamma_a(H_2) \leq \gamma_a(H) \leq |M| = |M_l| + |M_2| = \gamma_a(H_1) + \gamma_a(H_2). \) Hence \( \gamma_a(H) = \gamma_a(H_1) + \gamma_a(H_2) \) and then \( |M| = \gamma_a(H). \) Therefore \( M \) is a \( \gamma_a \)-set of \( H. \)

The result now follows because of Theorem 1.3.

**Remark 2.4.** In [1] Brigham, Chinn and Dutton obtained that, in the above notation, \( \gamma(H_1) + \gamma(H_2) \geq \gamma(H) \geq \gamma(H_1) + \gamma(H_2) - 1. \)

Observe that if \( m \) is a positive integer then there exists a graph \( H \) (in the above notation) such that \( m = \gamma_a(H) - \gamma_a(H_1) - \gamma_a(H_2). \) Indeed, let \( n \) and \( p \) be integers, \( m + 1 \leq n \leq p, V(H) = \{x, y, z; a_1, \ldots, a_{m+1}; b_1, \ldots, b_p; c_1, \ldots, c_p\}, E(H) = \{xy, xz, yz; xa_1, \ldots, xa_{m+1}; yb_1, \ldots, yb_p; zc_1, \ldots, zc_p\}, H_1 = \langle \{x; a_1, \ldots, a_{m+1}\}, H \rangle \)
and $H_2 = \langle \{x, y, z; b_1, \ldots, b_n; c_1, \ldots, c_p\}, H \rangle$. Then $\gamma_a(H) = 3 + m$, $\gamma_a(H_1) = 1$ and $\gamma_a(H_2) = 2$. Hence $m = \gamma_a(H) - \gamma_a(H_1) - \gamma_a(H_2)$.

**Theorem 2.5.** Let $G$ be a connected graph with blocks $G_1, G_2, \ldots, G_n$. Then $\gamma_a(G) \geq \sum_{i=1}^{n} \gamma_a(G_i) - n + 1$.

**Proof.** We proceed by induction on the number of blocks $n$. The statement is immediate if $n = 1$. Let the blocks of $G$ be $G_1, G_2, \ldots, G_n, G_{n+1}$ and without loss of generality let $G_{n+1}$ contain only one cut-vertex of $G$. Hence Theorem 2.3 implies that $\gamma_a(G) \geq \gamma_a(G_{n+1}) + \gamma_a(Q) - 1$ where $Q = \left\langle \bigcup_{i=1}^{n} V(G_i), G \right\rangle$. The result now follows from the inductive hypothesis. \hfill \Box

**References**


**Author’s address:** Vladimir Samodivkin, Department of Mathematics, University of Architecture, Civil Engineering and Geodesy, Hristo Smirnenski 1 Blv., 1046 Sofia, Bulgaria, e-mail: vlsam_fte@uacg.bg.