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ON BELATED DIFFERENTIATION AND A CHARACTERIZATION OF HENSTOCK-KURZWEIL-ITO INTEGRABLE PROCESSES

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Abstract. The Henstock-Kurzweil approach, also known as the generalized Riemann approach, has been successful in giving an alternative definition to the classical Itô integral. The Riemann approach is well-known for its directness in defining integrals. In this note we will prove the Fundamental Theorem for the Henstock-Kurzweil-Itô integral, thereby providing a characterization of Henstock-Kurzweil-Itô integrable stochastic processes in terms of their primitive processes.

Keywords: belated differentiation, Henstock-Kurzweil-Itô integral, integrable processes

MSC 2000: 26A39, 60H05

1. Introduction

The generalized Riemann approach, more commonly known as the Henstock-Kurzweil approach, has been successful in giving an alternative definition to the classical Itô integral, see [1], [5], [7], [8], [9], [10]. The advantage of using the Henstock-Kurzweil approach has been its explicitness and intuitiveness in giving a direct definition of the integral rather than the classical non-explicit $L^2$-procedure.

It is also well-known from the classical non-stochastic integration theory that all integrable functions can be characterized in terms of their primitives, that is, a function $f$ is Lebesgue (Henstock-Kurzweil) integrable on a compact interval $[a, b]$ if and only if there exists a function $F$ which is absolutely continuous (respectively, generalized absolutely continuous) there such that $F' = f$ a.e. on $[a, b]$, where $F'$ is the usual derivative of $F$, see for example [4].

In this paper, we will define the “belated derivative” of a stochastic process and thereby characterize the class of all Henstock-Kurzweil-Itô integrable processes on $[a, b]$ by its primitive process.
2. Setting

Let $\Omega$ denote the set of all real-valued continuous functions on $[a, b]$ and $\mathbb{R}$ the set of all real numbers.

The class of all Borel cylindrical sets $B$ in $\Omega$, denoted by $\mathcal{C}$, is a collection of all sets $B$ in $\Omega$ of the form

$$B = \{w: (w(t_1), w(t_2), \ldots, w(t_n)) \in E\}$$

where $0 \leq t_1 < t_2 < \ldots < t_n \leq 1$ and $E$ is any Borel set in $\mathbb{R}^n$ ($n$ is not fixed). The Borel $\sigma$-field of $\mathcal{C}$ is denoted by $\mathcal{F}$, i.e., it is the smallest $\sigma$-field which contains $\mathcal{C}$.

Let $P$ be the Wiener measure defined on $(\Omega, \mathcal{F})$. Then $(\Omega, \mathcal{F}, P)$ is a probability space, that is, a measure space with $P(\Omega) = 1$.

A stochastic process $\{\varphi(t, \omega): t \in [a, b]\}$ on $(\Omega, \mathcal{F}, P)$ is a family of $\mathcal{F}$-measurable functions (which are called random variables) on $(\Omega, \mathcal{F}, P)$. Very often, $\varphi(t, \omega)$ is denoted by $\varphi_t(\omega)$. Now we shall consider a very special and important process, namely, the Brownian motion denoted by $W$.

Let $W = \{W_t(\omega)\}_{a \leq t \leq b}$ be a canonical Brownian motion, that is, it possesses the following properties:

1. $W_a(\omega) = 0$ for all $\omega \in \Omega$;
2. it has Normal Increments, that is, $W_t - W_s$ has a normal distribution with mean 0 and variance $t - s$ for all $t > s$ (which naturally implies that $W_t$ has a normal distribution with mean 0 and variance $t$);
3. it has Independent Increments, that is, $W_t - W_s$ is independent of its past, that is, of $W_u$, $0 \leq u < s < t$; and
4. its sample paths are continuous, i.e., for each $\omega \in \Omega$, $W_t(\omega)$ as a function of $t$ is continuous on $[a, b]$.

A stochastic process $\{\varphi_t(\omega): t \in [a, b]\}$ is said to be adapted to the standard filtering space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ if $\varphi_t$ is $\mathcal{F}_t$-measurable for each $t \in [a, b]$. We always assume that $W = \{W_t(\omega)\}$ is adapted to $\{\mathcal{F}_t\}$. For example, if $\{\mathcal{F}_t\}$ is the smallest $\sigma$-field generated by $\{W_s(\omega): s \leq t\}$, then $W = \{W_t(\omega)\}$ is adapted to $\{\mathcal{F}_t\}$.

A stochastic process $X = \{X_t(\omega): t \in [a, b]\}$ on the standard filtering space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ is called a martingale if

1. $X$ is adapted to $\{\mathcal{F}_t\}$, that is, $X_t$ is $\mathcal{F}_t$-measurable for each $t \in [a, b]$;
2. $\int_{\Omega} |X_t| \, dP$ is finite for almost all $t \in [a, b]$; and
3. $E(X_t|\mathcal{F}_s) = X_s$ for all $t \geq s$, where $E(X_t|\mathcal{F}_s)$ is the conditional expectation of $X_t$ given $\mathcal{F}_s$. 

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If in addition
\[ \sup_{t \in [a, b]} \int_{\Omega} |X_t|^2 \, dP \]
is finite, we say that \( X \) is an \( L_2 \)-martingale.

In the following we define \( E(f) \) to be \( \int_{\Omega} f \, dP \) for any random variable \( f \).

It is well-known, see for example [6, P239], that the following assertions hold. The details are given for the convenience of readers who are not familiar with stochastic analysis.

(i) 
\[ E[X_s] = E[E[X_t | \mathcal{F}_s]] = E[X_t] \]
for any \( t \geq s \), that is, \( E[X_s] \) is a constant for all \( s \in [a, b] \).

(ii) For any \( a \leq u < v \leq s < t \leq b \), we have
\[ E[(X_t - X_s)(X_v - X_u)] = 0, \]
that is, a martingale has orthogonal increments.

(iii) From (ii) we get
\[ E\left| \sum (X_v - X_u)^2 \right| = (D) \sum E(X_v - X_u)^2 \]
for any partial partition \( D = \{[u, v]\} \) of \([a, b]\).

(iv) For any \( u < v \) we have
\[ E[X_v X_u] = E[E[X_v X_u | \mathcal{F}_u]] = E[X_u E[X_v | \mathcal{F}_u]] = E[X_u^2] \]
and hence
\[ E(X_v - X_u)^2 = E(X_v^2 - X_u^2). \]

It is also well-known, see for example [6, P28], that a canonical Brownian motion is a martingale. In fact, it is an \( L_2 \)-martingale with \( E(W_t^2) = t \), see property 2 of a Brownian motion.
3. Differentiation

In this section we define our belated derivative and state its basic properties.

**Definition 1.** Let \( F = \{F_t: t \in [a, b]\} \) be an \( L^2 \)-martingale. A stochastic process \( F \) is said to be *belated differentiable* at \( t \in [a, b] \) if there exists a random variable \( f_t \) such that for any \( \varepsilon > 0 \), there exists a positive number \( \delta(t) \) on \([a, b]\) such that whenever \([t, v] \subset [t, t + \delta(t))\), we have

\[
E(\left|f_t(W_v - W_t) - (F_v - F_t)\right|^2) \leq \varepsilon E(W_v - W_t)^2 = \varepsilon|v - t|.
\]

The random variable \( f_t \) is called the *belated derivative* of \( F \) at the point \( t \). We will denote \( f_t \) by \( D^\beta F_t \) in our subsequent presentation. It is also easily checked that the belated derivative of \( F \) is defined uniquely up to a set of probability measure zero. The proof is omitted.

The \( L^2 \)-martingale \( F \) is said to be belated differentiable at \( t \in [a, b] \) if \( f_t \) in the above definition exists.

**Remark.** The word belated is used in the above definition because the point of differentiation \( t \) is always the left end point of the interval \([t, v]\). This is motivated by the use of belated division in the definition of Henstock-Kurzweil-Itô integrals, see [1].

Next we shall state the standard properties of belated differentiation.

**Theorem 2.** Let \( X \) and \( Y \) be two \( L^2 \)-martingales which are belated differentiable at \( t \in [a, b] \) and let \( \alpha \in \mathbb{R} \). Then

(a) \( X + Y \) is belated differentiable at \( t \) and

\[
D^\beta(X + Y)_t = (D^\beta X)_t + (D^\beta Y)_t,
\]

(b) \( \alpha X \) is belated differentiable at \( t \) and

\[
(D^\beta(\alpha X))_t = \alpha(D^\beta X)_t.
\]

**Proof.** The proof of Theorem 2 is straightforward and hence omitted. \( \square \)

**Example 3.** Let \( X = \{X_t: t \in [0, 1]\} \) be the stochastic process \( X_t = \frac{1}{2}W_t^2 - \frac{1}{2}t \), where \( W \) is the Brownian motion, over the standard filtering space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P) \). Then it is easy to verify that \( X \) is in fact an \( L^2 \)-martingale with respect to the standard filtering space. Furthermore, it can be proved that

\[
D^\beta X_t = W_t
\]

for all \( t \in [a, b] \).
That $X$ is a martingale follows from the fact that

$$E(W_b^2 - W_a^2 | \mathcal{F}_a) = b - a$$

where $0 \leq a \leq b$. Furthermore,

$$E(|X_t|^2) = \frac{1}{2}t^2 \leq \frac{1}{2}b^2$$

for all $t \in [a, b]$, thereby showing that $X$ is in fact an $L^2$-martingale. We next show that $D\beta X_t = W_t$ for all $t \in [a, b]$.

Given $\varepsilon > 0$, let $\delta(t) \leq 2\varepsilon$ for all $t \in [a, b]$. Consider a $\delta$-fine interval-point pair $([t, v], t)$ such that $[t, v] \subset [t, t + \delta(t)]$ so that $|v - t| \leq 2\varepsilon$. Then

$$E(W_t(W_v - W_t) - (X_v - X_t))^2 = E\left(\frac{1}{2}(W_v - W_t)^2 + \frac{1}{2}(t - v)^2\right)^2$$

$$= E\left(\frac{1}{2}(W_v - W_t)^2 - (v - t)^2\right)^2$$

$$= \frac{1}{4}E[(W_v - W_t)^4 - 2(W_v - W_t)^2(v - t) + (v - t)^2]$$

$$= \frac{1}{2}(v - t)^2 \leq \frac{1}{2} \cdot 2\varepsilon(v - t) = \varepsilon(v - t),$$

which completes our proof.

By Definition 1, belated differentiation is defined for $L^2$-martingales in our context. If we were to allow the belated differentiation to be defined for more general stochastic processes, we could even have $D\beta(\frac{1}{2}W_t^2) = W_t$. However, in this sense, the anti-derivative of $W_t$ would not be uniquely defined. Hence we restrict ourselves to the belated differentiation of $L^2$-martingales.

**Definition 4.** A stochastic process $X = \{X_t: t \in [a, b]\}$ on $(\Omega, \mathcal{F}, P)$ is said to be $AC^2$ on $[a, b]$ if given any $\varepsilon > 0$, there exists $\eta > 0$ such that

$$E\left(\sum_{i=1}^n (X_{v_i} - X_{u_i})^2\right) \leq \varepsilon$$

for any finite collection $D = \{[u_i, v_i]\}_{i=1}^n$ of non-overlapping intervals for which $\sum_{i=1}^n |v_i - u_i| \leq \eta$. 

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Example 5. The stochastic process $X = \{X_t: t \in [a, b]\}$, where

$$X_t = \frac{1}{2} W_t^2 - \frac{1}{2} t$$

in Example 3, is $AC^2$ on $[a, b]$. The proof is easy and hence omitted.

4. ANTIDERIVATIVE AND HENSTOCK-ITO INTEGRAL

In this section we will characterize the class of all Henstock-Itô adapted processes in terms of their derivatives.

Let $\delta$ be a positive function on $[a, b]$. A finite collection $D$ of interval-point pairs $\{([\xi_i, v_i], \xi_i), i = 1, 2, \ldots, n\}$ is called a $\delta$-fine belated partial division of $[a, b]$ if

1. $\{[\xi_i, v_i], i = 1, 2, \ldots, n\}$ is a collection of non-overlapping subintervals of $[a, b]$;
2. $[\xi_i, v_i] \subset [\xi_i, \xi_i + \delta(\xi_i)]$ for each $i = 1, 2, 3, \ldots, n$.

In the sequel we will denote $\{([\xi_i, v_i], \xi_i), i = 1, 2, 3, \ldots, n\}$ by $\{(\xi, v, \xi)\}$.

Definition 6 (See [1, Definition 2]). Let $f = \{f_t: t \in [a, b]\}$ be an adapted process on the standard filtering space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$. Then $f$ is said to be Henstock-Kurzweil-Itô integrable on $[a, b]$ if there exists a process $F = \{F_t: t \in [a, b]\}$ which is an $L^2$-martingale and $AC^2$ on $[a, b]$ such that for any $\varepsilon > 0$ there exists a positive function $\delta$ on $[a, b]$ such that

$$E\left( (D) \sum \{ f_\xi (W_v - W_u) - (F_v - F_u) \}^2 \right) \leq \varepsilon$$

whenever $D = \{(\xi, v, \xi)\}$ is a $\delta$-fine belated partial division of $[a, b]$.

It follows from Vitali’s Covering Lemma that given any positive function $\delta$ there exists a belated partial division of $[a, b]$ covering this interval up to a set of arbitrarily small positive measure, hence the uniqueness of the integral process $F$ follows.

It was also proved in [1] that the standard properties of integrals (such as uniqueness of the integral, additivity of the integral, integrability over subintervals) hold true for the Henstock-Kurzweil-Itô integral. The proofs are similar to the classical integration theory, see [2], [3], [4]. In fact, it has been proved in Theorem 9 of [1] that the integral defined by this new approach is equivalent to the classical Itô integral.

We have a class of stochastic processes which are Henstock-Kurzweil-Itô integrable on $[a, b]$. 

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Let $L^2$ denote the class of all adapted stochastic processes $\phi = \{\phi_t : t \in [a, b]\}$ on the standard filteringspace $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ such that

$$\int_a^b E|\phi(t, \omega)|^2 \, dt$$

is finite. Then any adapted process from $L^2$ is Henstock-Kurzweil-Itô integrable on $[a, b]$.

In fact, $L^2$ is the class of all classical Itô integrable functions. We have proved in [1] that if $f$ is classical Itô integrable, then $f$ is also Henstock-Kurzweil-Itô integrable and the two integrals coincide.

**Theorem 8.** Let an adapted process $f$ be Henstock-Kurzweil-Itô integrable on $[a, b]$ and let $F_t = \int_a^t f_s \, dW_s$. Then $D\beta F_t = f_t$ a.e. on $[a, b)$.  

**Proof.** The idea of this proof is motivated by that of the Henstock integration theory. We need to show that the set of points $B$ of $[a, b)$ for which $D\beta F_t$ does not exist or is unequal to $f$ is of Lebesgue measure zero. Let $t \in B$. By definition, there exists $\gamma(t) > 0$ such that for any positive number $\delta(t)$, there exists $[t, v] \subset [t, t+\delta(t)]$ such that

$$(1) \quad E(|f_t(W_v - W_t) - (F_v - F_t)|^2) > \gamma(t)(v-t).$$

From the definition of the Henstock-Kurzweil-Itô integral (see Definition 6), given $\varepsilon > 0$, there exists a positive function $\beta$ on $[a, b]$ such that whenever $D = \{(\xi_i, v_i)\}_{i=1}^n$ is a $\beta$-fine belated partial division of $[a, b]$, we have

$$(2) \quad E\left(\sum |f_{\xi}[W_v - W_{\xi}] - (F_v - F_{\xi})|^2\right) \leq \varepsilon.$$

Now we consider a special $D$ such that each $[\xi_i, v_i]$ satisfies (1) and (2). Let us denote $B_m = \{t \in [a, b] : \gamma(t) \geq \frac{1}{m}\}$, $m = 1, 2, 3, \ldots$, and fix $B_m$. Suppose each $\xi_i \in B_m$. Then by (1) and (2), we have

$$\sum_{i=1}^n (v_i - \xi_i) \leq m\varepsilon.$$

Let $\mathcal{G}$ be the family of collections of intervals $[\xi, v]$ induced from all $\beta$-fine belated partial divisions with $\xi \in B_m$ satisfying (1). Then $\mathcal{G}$ covers $B_m$ in Vitali’s sense. Applying the Vitali Covering Theorem, there exists a finite collection of intervals $\{[\xi_i, v_i], i = 1, 2, 3, \ldots, q\}$ such that

$$\mu(B_m) \leq \sum_{i=1}^q |v_i - \xi_i| + \varepsilon \leq (m + 1)\varepsilon.$$

Hence $\mu(B_m) = 0$ and so $\mu(B) = 0$. Thus our proof is completed. \qed
Theorem 9. Let $f$ be an adapted process on $[a, b]$ such that
(i) $F$ is an $L^2$-martingale with $F_a = 0$ a.e.;
(ii) $F$ has the $AC^2$ property;
(iii) $D\beta F_t = f_t$ a.e. on $[a, b]$; then $f$ is Henstock-Kurzweil-Itô integrable on $[a, b]$ with $F_t = \int_a^t f_s \, dW_s$.

The reader is reminded that (iii) means that $D\beta F_t(\omega) = f_t(\omega)$ for almost all $\omega \in \Omega$ for a.e. $t \in [a, b)$.

Proof. Let $D\beta F_t = f_t$ for all $t \in [a, b)$ except possibly for a set $B$ which has Lebesgue measure zero. Let $\xi \in [a, b) \setminus B$. Given $\varepsilon > 0$, there exists a positive function $\delta$ on $[a, b)$ such that whenever $(\xi, v)$ is $\delta$-fine, we have

$$E(|f_\xi(W_v - W_\xi) - (F_v - F_\xi)|^2) \leq \varepsilon |v - \xi|.$$  

Let $D = \{(\xi_i, v_i, \xi_i), i = 1, 2, 3, \ldots, n\}$ be a $\delta$-fine belated partial division of $[a, b]$ with all $\xi_i \in [a, b) \setminus B$. Then

$$E\left(\left|\sum_{i=1}^n f_{\xi_i}(W_{v_i} - W_{\xi_i}) - (F_{v_i} - F_{\xi_i})\right|^2\right)$$

$$= E\left(\sum_{i=1}^n |f_{\xi_i}(W_{v_i} - W_{\xi_i}) - (F_{v_i} - F_{\xi_i})|^2\right) \text{ by (i)}$$

$$\leq \varepsilon \sum_{i=1}^n |v_i - \xi_i| \leq \varepsilon (b - a).$$

Thus if $B = \varnothing$, it is clear from the above that $f$ is Itô integrable with $F_t = \int_a^t f_s \, dW_s$. In general, $B$ is nonempty with $\mu(B) = 0$.

Now let

$$B_m = \{t \in [a, b) : m - 1 < E[f_t^2] \leq m\},$$

where $\mu(B_m) = 0$ and $B = \bigcup_{m=1}^{\infty} B_m$.

Since $F$ has the $AC^2$ property, given any positive integer $m$, there exists $\eta_m > 0$ with $\eta_m \leq (\varepsilon/2^m)^2 \cdot m^{-2}$ such that whenever $\{(u_i, v_i)\}$ is a finite collection of disjoint left-open subintervals of $[a, b]$ with $\sum |v_i - u_i| \leq \eta_m$, we have

$$E\left(\left|\sum [F_{v_i} - F_{u_i}]\right|^2\right) \leq \left(\frac{\varepsilon}{2^m}\right)^2.$$  

Take an open set $G_m \supset B_m$ such that $\mu(G_m) \leq \eta_m$.  

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Fix a positive integer \( m \). Let \( D = \{ ((\xi_i, v_i], \xi_i) \} \) be a \( \beta \)-fine belated partial division of \([a, b)\) such that \( \xi_i \in B_m \) for all \( i \). Then we have

\[
E \left( \left| \sum_i f_{\xi_i} (W_{v_i} - W_{\xi_i}) - (F_{v_i} - F_{\xi_i}) \right|^2 \right) \\
\leq 2E \left( \left| \sum_i f_{\xi_i} (W_{v_i} - W_{u_i}) \right|^2 \right) + 2E \left( \left| \sum_i (F_{v_i} - F_{\xi_i}) \right|^2 \right) \\
\leq 2 \sum_i E \left[ f_{\xi_i}^2 \right] (v_i - u_i) + 2 \left( \frac{\varepsilon}{2^m} \right)^2 \leq 4 \left( \frac{\varepsilon}{2^m} \right)^2.
\]

So, considering any \( \beta \)-fine belated partial division over \([a, b]\), denoted by \( D_1 = \{ ((\xi_i, v_i], \xi_i) \} \), we have

\[
E \left( \left| \sum_{i} f_{\xi_i} (W_{v_i} - W_{\xi_i}) - (F_{v_i} - F_{\xi_i}) \right|^2 \right) \\
\leq 2E \left( \left| \sum_{\xi \in [a,b]\setminus B} f_{\xi} (W_{v_i} - W_{\xi_i}) - (F_{v_i} - F_{\xi_i}) \right|^2 \right) \\
+ 2E \left( \left| \sum_{m=1}^{\infty} \sum_{\xi_i \in B_m} f_{\xi_i} (W_{v_i} - W_{\xi_i}) - (F_{v_i} - F_{\xi_i}) \right|^2 \right) \\
\leq 2\varepsilon (b - a) + 2\varepsilon,
\]

thereby showing that \( f \) is Itô integrable with \( F_t = \int_a^t f_t \, dW_t \). \( \square \)

Combining Theorems 8 and 9, we have the following characterization of all Henstock-Kurzweil-Itô integrable stochastic processes:

**Theorem 10.** Let \( f \) be an adapted process on \([a, b]\). Then \( f \) is Henstock-Kurzweil-Itô integrable on \([a, b]\) if and only if there exists an \( L^2 \)-martingale \( F \) on \([a, b]\) with \( F_a = 0 \) a.s. and \( AC^2 \) on \([a, b]\) such that \( D_\beta F_t = f_t \) almost everywhere on \([a, b]\).

**Example 11.** From Example 3, \( X_t = \frac{1}{2} W_t^2 - \frac{1}{2} t \) is an \( L^2 \)-martingale on \([a, b]\). Hence the process

\[
F_t = \frac{1}{2} W_t^2 - \frac{1}{2} W_a^2 - \frac{1}{2} (t - a),
\]

where \( F_a = 0 \), is an \( L^2 \)-martingale on \([a, b]\). It can be also easily verified that \( F \) is \( AC^2 \) on \([a, b]\). Furthermore, it was shown that

\[
D_\beta X_t = W_t
\]
on \([a, b]\), hence
\[
D_\beta F_t = W_t
\]
on \([a, b]\). By Theorem 10, we have
\[
\int_a^b W_t \, dW_t = F_b = \frac{1}{2} \left( W_b^2 - W_a^2 \right) - \frac{1}{2} (b - a).
\]

Example 12. Let \(f \in \mathcal{L}_2\), the class of all classical Itô integrable adapted processes on the standard filtering space. Then there exists an \(L^2\)-martingale \(F\) on \([a, b]\) which is also \(AC^2\) on \([a, b]\), such that \(D_\beta F_t = f_t\) a.e. on \([a, b]\).

References


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