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TRANSFER OF BOUNDARY CONDITIONS FOR POISSON'S EQUATION ON A CIRCLE

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Summary. The method of transfer of boundary conditions yields a universal frame into which most methods for solving boundary value problems for ordinary differential equations can be included. The purpose of this paper is to show a possibility to extend the idea of transfer of conditions to a particular twodimensional problem.

Keywords: numerical analysis, transfer of conditions, invariant imbedding, Dirichlet problem for Poisson's equation

AMS classification: 65L10, 34B05

1. INTRODUCTION

The invariant imbedding method has proved very efficient when solving boundary value problems for ordinary differential equations (cf., e.g., [1]). In the linear case, this method can be very simply derived from the idea of transferring boundary conditions (see, e.g., [2]). For the reader's convenience, let us recall the principles of the method of transfer of boundary conditions in the case of a model second-order differential equation

$$-\big(p(t)y'(t)\big)' + q(t)y(t) = f(t) \tag{1.1}$$

and a boundary condition

$$-\alpha_1 p(a)y'(a) + \beta_1 y(a) = \gamma_1 \tag{1.2}$$

since it is extremely simple in such a situation. Namely, it is obvious almost at the first glance that a function y satisfying in $[a, b]$ the differential equation (1.1) and, simultaneously, the condition (1.2) should satisfy in $[a,b]$ a differential equation of the first order. In other words, functions α, β and $\gamma\colon [a,b] \to \mathbf{R}$ must exist such that

$$(1.3) \qquad -\alpha(t)p(t)y'(t) + \beta(t)y(t) = \gamma(t)$$

holds for any $t \in [a,b]$. Thus, a linear condition of the type (1.2) prescribed at the left end point of the interval $[a,b]$ can be transferred to any other point of this interval. Note also that the fact that the condition (1.2) is given at the point a is not essential and that the same assertion holds if the condition of the above type is prescribed at any point of $[a,b]$. It is also important to note that the coefficients α, β and γ in (1.3) can be obtained as the result of solution of initial value problems for ordinary differential equations. The algorithm for solving a two-point boundary value problem for the equation (1.1), i.e., the problem of finding such a solution of (1.1) that satisfies the conditions of the type (1.2) at both end points of $[a,b]$, is now more or less clear: Both the boundary conditions are transferred to the same point $t = t_0$ of the interval $[a,b]$ thus yielding a system of two linear algebraic equations for determining the values $y(t_0)$ and $p(t_0)y'(t_0)$, i.e., the initial conditions for the solution sought.

Naturally, the functions α, β and γ in (1.3) are not determined uniquely (it is possible to multiply (1.3) by any continuous function different from zero, at least) so that the individual choice of them may influence substantially the properties of the algorithm just described. One possibility of a particular transfer of a boundary condition which leads to a stable numerical process is described in the following theorem (see [2]).

Theorem 1.1. *Let p, q and f be functions from $[a,b]$ to $\mathbf{R}$ such that $1/p$, q and f are Lebesgue integrable on $[a,b]$, and further let $p > 0$, $q \geqslant 0$ almost everywhere on $[a,b]$. Finally, let y be a solution of (1.1) in $[a,b]$*)* which satisfies the condition (1.2) with $\alpha_1 \geqslant 0$, $\beta_1 \geqslant 0$, $\alpha_1 + \beta_1 > 0$. Then:

(i) if $\alpha_1 > 0$, we have

$$(1.4) \qquad p(t)y'(t) + \eta(t)y(t) = \xi(t), \quad t \in [a,b],$$

and the functions η and ξ are uniquely determined as the solutions of initial value problems

$$(1.5) \qquad \eta'(t) = \frac{1}{p(t)}\eta^2(t) - q(t), \quad \eta(a) = \frac{\beta_1}{\alpha_1},$$

*) By a solution of (1.1) in $[a,b]$ we mean here and in the following text an absolutely continuous function such that also py' is absolutely continuous and that (1.1) is satisfied almost everywhere in $[a,b]$.

and

$$\xi'(t) = \frac{1}{p(t)}\eta(t)\xi(t) - f(t), \quad \xi(a) = -\frac{\gamma_1}{\alpha_1}; \tag{1.6}$$

(ii) if $\beta_1 > 0$, we have

$$\eta(t)p(t)y'(t) + y(t) = \xi(t), \quad t \in [a, b] \tag{1.7}$$

and the functions η and ξ are uniquely determined as the solutions of initial value problems

$$\eta'(t) = q(t)\eta^2(t) - \frac{1}{p(t)}, \quad \eta(a) = -\frac{\alpha_1}{\beta_1}, \tag{1.8}$$

and

$$\xi'(t) = q(t)\eta(t)\xi(t) - \eta(t)f(t), \quad \eta(a) = \frac{\gamma_1}{\beta_1}. \tag{1.9}$$

The aim of the present paper is to show a possibility of generalizing the above idea to twodimensional problems.

2. Transfer of boundary conditions for Poisson's equation

Consider the differential equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} = f(r, \varphi) \tag{2.1}$$

in the circle $\Omega = \{(r, \varphi)\,;\, 0 \leqslant r < R, 0 \leqslant \varphi < 2\pi\}$ with the boundary condition

$$u(R, \varphi) = 0, \quad 0 \leqslant \varphi < 2\pi \tag{2.2}$$

and suppose that $f \in L_2(\Omega)$. Then it is possible to write the solution of (2.1), (2.2) in the form of a Fourier series

$$u(r, \varphi) = \frac{1}{2}a_0(r) + \sum_{k=1}^{\infty}[a_k(r)\cos k\varphi + b_k(r)\sin k\varphi], \tag{2.3}$$

and the Fourier coefficients $a_k(r)$ and $b_k(r)$ of u satisfy the differential equations

$$-(ra_k'(r))' + \frac{k^2}{r}a_k(r) = -rc_k(r), \quad r \in [0, R],\ k = 0, 1, \ldots, \tag{2.4}$$

$$-(rb_r'(r))' + \frac{k^2}{r}b_k(r) = -rd_k(r), \quad r \in [0, R],\ k = 0, 1, \ldots, \tag{2.5}$$

where $c_k(r)$ and $d_k(r)$ are Fourier coefficients of f, with the boundary conditions

$$\lim_{r\to 0} r a_0'(r) = a_0(R) = 0, \tag{2.6}$$

$$a_k(0) = a_k(R) = 0, \quad k = 1, 2, \ldots, \tag{2.7}$$

$$b_k(0) = b_k(R) = 0, \quad k = 1, 2, \ldots. \tag{2.8}$$

One way to obtain the transferred boundary conditions of the problem (2.1), (2.2) is now apparent and consists in applying the idea of transferring the boundary conditions to the ordinary differential equations (2.4), (2.5) with the boundary conditions (2.6) to (2.8). However, for the left end point of the interval $[0, R]$, we cannot proceed straightforward according to Theorem 1.1 since it is assumed there that the coefficients of the second-order differential equation under consideration are regular. In [3], Theorem 1.1 was generalized to the case that either $1/p$ or q, but only one of them, has a nonintegrable singularity. This is not the case here, either. Thus, we must begin with stating a theorem covering our actual situation with singularities both in $1/p$ and q. We start with three simple lemmas (cf. [3]).

Lemma 2.1. *Let p, q and f be functions from $[a, b]$ to $\mathbb{R}$ such that $q \in \mathscr{L}(a, b)$, $f \in \mathscr{L}(a, b)$. Suppose, further, that $1/p \in \mathscr{L}(a + \varepsilon, b)$ for any ε, $0 < \varepsilon < b - a$, and that $1/p \notin \mathscr{L}(a, b)$. Finally, let y be a solution of (1.1). Then*

$$p(a)y'(a) = 0. \tag{2.9}$$

P r o o f. Let us first recall that the solution of (1.1) is such an absolutely continuous function y that py' is also absolutely continuous and that (1.1) is satisfied almost everywhere. To prove (2.9) let us suppose that $p(a)y'(a) > 0$ $\big(p(a)y'(a) < 0\big)$. Then $1/\big(p(t)y'(t)\big)$ is measurable and bounded in a neighbourhood of the point a. The function y' is Lebesgue integrable in (a, b) as it is a derivative of an absolutely continuous function. Consequently, $\big(1/(py')\big)y' = 1/p$ is Lebesgue integrable in a neighbourhood of a. This contradiction proves the lemma. □

Lemma 2.2. *Let p, q and f be functions from $[a, b]$ to $\mathbb{R}$ such that $f \in \mathscr{L}(a, b)$, $1/p$, $q \in \mathscr{L}(a + \varepsilon, b)$ for any ε, $0 < \varepsilon < b - a$, and let $1/p \notin \mathscr{L}(a, b)$, $q \notin \mathscr{L}(a, b)$. Finally, let y be a solution of (1.1). Then*

$$y(a) = p(a)y'(a) = 0. \tag{2.10}$$

P r o o f. The validity of $p(a)y'(a) = 0$ is proved exactly in the same way as in Lemma 2.1. To prove $y(a) = 0$ suppose that $y(a) > 0$ $\big(y(a) < 0\big)$. Thus, $1/y$

is continuous and bounded in a neighbourhood of a. Further, from (1.1) and the assumption that $f \in \mathscr{L}(a,b)$ the relation $qy \in \mathscr{L}(a,b)$ follows. Hence, $qy(1/y) \in \mathscr{L}(a, a+\delta)$ for some positive δ, which contradicts the assumption that $q \notin \mathscr{L}(a,b)$. The lemma is proved. □

Lemma 2.3. *Let P and Q be functions from $[a,b]$ to $\mathbf{R}$ such that $P \in \mathscr{L}(a,b)$, $Q \in \mathscr{L}(a+\varepsilon, b)$ for any ε, $0 < \varepsilon < b-a$, and let $Q \notin \mathscr{L}(a,b)$. Further, let Q be nonpositive almost everywhere in $[a,b]$. Then there exists one and only one absolutely continuous function $\xi\colon [a,b] \to \mathbf{R}$ such that*

$$\xi'(t) = Q(t)\xi(t) + P(t) \tag{2.11}$$

almost everywhere in $[a,b]$ and

$$\xi(a) = 0. \tag{2.12}$$

Proof. The existence part of the assertion of the lemma follows from the observation that the function φ given by

$$\varphi(t) = \int_a^t P(s)\exp\Big(\int_s^t Q(u)\,\mathrm{d}u\Big)\,\mathrm{d}s \tag{2.13}$$

is a solution of (2.11) satisfying (2.12) since

$$\left|P(s)\exp\Big(\int_s^t Q(u)\,\mathrm{d}u\Big)\right| \leqslant |P(s)| \in \mathscr{L}(a,b). \tag{2.14}$$

The uniqueness follows from the relation $d\big[(\xi_1-\xi_2)^2\big]/dt = 2Q(\xi_2-\xi_1)^2 \leqslant 0$, which is obviously satisfied almost everywhere on $[a,b]$ for any two solutions of (2.11). □

Remark 2.1. Lemmas 2.1 and 2.2 imply that the boundary conditions for functions a_k and b_k at the point $r=0$ (cf. (2.6) to (2.8)) were chosen in a natural way as they are satisfied automatically.

Remark 2.2. If $Q \in \mathscr{L}(a,b)$ then the assertion of Lemma 2.3 is obviously true even without the assumption $Q \leqslant 0$.

Theorem 2.1. *Let p and q be functions from $[a,b]$ to $\mathbf{R}$ such that $p > 0$, $q \geqslant 0$ almost everywhere on $[a,b]$, $1/p$, $q \in \mathscr{L}(a+\varepsilon,b)$ for any ε, $0<\varepsilon<b-a$, $1/p \notin \mathscr{L}(a,b)$, $q \notin \mathscr{L}(a,b)$. Further, let η be a nonpositive absolutely continuous function which satisfies the differential equation*

$$\eta' = \frac{1}{p}\eta^2 - q \tag{2.15}$$

almost everywhere in $[a,b]$. Then any absolutely continuous function y fulfilling the equation

$$-(py')' + qy = f \tag{2.16}$$

with $f \in \mathcal{L}(a,b)$ satisfies

$$\eta(t)y(t) + p(t)y'(t) = \xi(t) \tag{2.17}$$

for any $t \in [a,b]$. At the same time, the function ξ is determined by the differential equation

$$\xi' = \frac{1}{p}\eta\xi - f \tag{2.18}$$

with the initial condition

$$\xi(a) = 0. \tag{2.19}$$

P r o o f. We immediately obtain from the assumptions of the theorem that $(1/p)\eta \in \mathcal{L}(a+\varepsilon, b)$ and $(1/p)\eta \leqslant 0$ almost everywhere in $[a,b]$. Thus, Lemma 2.3 guarantees that the differential equation (2.18) and the initial condition (2.19) really define the function ξ uniquely. Further, put

$$\varphi = \eta y + py' - \xi \tag{2.20}$$

so that φ is absolutely continuous in $[a,b]$. By a direct computation we easily find that

$$\varphi' = \frac{1}{p}\eta\varphi \tag{2.21}$$

almost everywhere on $[a,b]$. From Lemma 2.2 and from the condition (2.19) we have $\varphi(a) = 0$. Since $(1/p)\eta \in \mathcal{L}(a+\varepsilon, b)$ and $(1/p)\eta \leqslant 0$, Lemma 2.3 implies that $\varphi \equiv 0$ on $[a,b]$. The theorem is proved. □

Now we have all that is needed to formulate and prove the theorem on the transfer of boundary conditions for the problem (2.1), (2.2).

Theorem 2.2. *Let $u(r,\varphi)$ be the solution of (2.1), (2.2) with $f \in \mathcal{L}_2(a,b)$. Then for any r, $0 < r \leqslant R$ and φ, $0 \leqslant \varphi < 2\pi$ we have*

$$\begin{aligned} u(r,\varphi) - \frac{1}{2\pi}\int_0^{2\pi} u(r,\psi)\,\mathrm{d}\psi + \frac{r}{2\pi}\int_0^{2\pi} \ln 2\big[1 - \cos k(\varphi-\psi)\big]\, u'_r(r,\psi)\,\mathrm{d}\psi \\ = \frac{1}{2\pi}\int_0^r\int_0^{2\pi} \varrho f(\varrho,\psi) \ln\left[1 - 2\frac{\varrho}{r}\cos(\varphi-\psi) + \left(\frac{\varrho}{r}\right)^2\right] \mathrm{d}\varrho\,\mathrm{d}\psi. \text{ *)} \end{aligned} \tag{2.22}$$

*) The symbol u'_r denotes the partial derivative of u with respect to the variable r.

Proof. The identity (2.22) will be derived by transferring the boundary conditions for the differential equations (2.4), (2.5) prescribed at the point $r = 0$ to an arbitrary point of the interval $(0, R]$ by means of Theorem 2.1. In our particular situation, the equation (2.15) has the form

$$\eta' = \frac{1}{r}\eta^2 - \frac{k^2}{r}, \quad k = 1, 2, \ldots \tag{2.23}$$

and a function satisfying the assumptions of Theorem 2.1 is, obviously, $\eta = -k$. The equation (2.18) from Theorem 2.1 is

$$\xi' = -\frac{1}{r}k\xi + rc_k(r), \quad k = 1, 2, \ldots \tag{2.24}$$

or

$$\xi' = -\frac{1}{r}k\xi + rd_k(r), \quad k = 1, 2, \ldots, \tag{2.25}$$

respectively. The corresponding solutions with zero initial conditions are

$$\xi(r) = \int_0^r \left(\frac{\varrho}{r}\right)^k \varrho c_k(\varrho)\,\mathrm{d}\varrho, \quad k = 1, 2, \ldots \tag{2.26}$$

and

$$\xi(r) = \int_0^r \left(\frac{\varrho}{r}\right)^k \varrho d_k(\varrho)\,\mathrm{d}\varrho, \quad k = 1, 2, \ldots. \tag{2.27}$$

Thus, according to (2.17), we have

$$a_k(r) = \frac{r}{k}a_k'(r) - \frac{1}{k}\int_0^r \left(\frac{\varrho}{r}\right)^k \varrho c_k(\varrho)\,\mathrm{d}\varrho, \quad k = 1, 2, \ldots \tag{2.28}$$

and

$$b_k(r) = \frac{r}{k}b_k'(r) - \frac{1}{k}\int_0^r \left(\frac{\varrho}{r}\right)^k \varrho d_k(\varrho)\,\mathrm{d}\varrho, \quad k = 1, 2, \ldots. \tag{2.29}$$

Substitute now into the right-hand terms of the above identities according to the formulae

$$a_k'(r) = \frac{1}{\pi}\int_0^{2\pi} u_r'(r, \psi)\cos k\psi\,\mathrm{d}\psi, \quad k = 1, 2, \ldots, \tag{2.30}$$

$$b_k'(r) = \frac{1}{\pi}\int_0^{2\pi} u_r'(r, \psi)\sin k\psi\,\mathrm{d}\psi, \quad k = 1, 2, \ldots, \tag{2.31}$$

$$c_k(\varrho) = \frac{1}{\pi}\int_0^{2\pi} f(\varrho, \psi)\cos k\psi\,\mathrm{d}\psi, \quad k = 1, 2, \ldots, \tag{2.32}$$

$$d_k(\varrho) = \frac{1}{\pi}\int_0^{2\pi} f(\varrho, \psi)\sin k\psi\,\mathrm{d}\psi, \quad k = 1, 2, \ldots. \tag{2.33}$$

The result is

$$(2.34)\qquad \begin{aligned} a_k(r) = \frac{r}{\pi k} \int_0^{2\pi} u'_r(r,\psi)\cos k\psi \,\mathrm{d}\psi \\ - \frac{1}{\pi k}\int_0^r \int_0^{2\pi} (\frac{\varrho}{r})^k \varrho f(\varrho,\psi)\cos k\psi \,\mathrm{d}\psi\,\mathrm{d}\varrho, \quad k = 1,2,\ldots \end{aligned}$$

and

$$(2.35)\qquad \begin{aligned} b_k(r) = \frac{r}{\pi k} \int_0^{2\pi} u'_r(r,\psi)\sin k\psi \,\mathrm{d}\psi \\ - \frac{1}{\pi k}\int_0^r \int_0^{2\pi} (\frac{\varrho}{r})^k \varrho f(\varrho,\psi)\sin k\psi \,\mathrm{d}\psi\,\mathrm{d}\varrho, \quad k = 1,2,\ldots. \end{aligned}$$

Multiplying (2.34) by $\cos k\varphi$ and (2.35) by $\sin k\varphi$ and summing up the relations for $k = 1,2,\ldots$ we obtain, after simple manipulations,

$$(2.36)\qquad \begin{aligned} &\sum_{k=1}^{\infty}[a_k(r)\cos k\varphi + b_k(r)\sin k\varphi] \\ &= \frac{r}{\pi}\int_0^{2\pi} u'_r(r,\psi)\sum_{k=1}^{\infty}\frac{1}{k}\cos k(\varphi-\psi)\,\mathrm{d}\psi - \\ &- \frac{1}{\pi}\int_0^r\int_0^{2\pi} \varrho f(\varrho,\psi)\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{\varrho}{r}\right)^k \cos k(\varphi-\psi)\,\mathrm{d}\psi\,\mathrm{d}\varrho. \end{aligned}$$

Observing now that

$$(2.37)\qquad \begin{aligned} &\sum_{k=1}^{\infty}[a_k(r)\cos k\varphi + b_k(r)\sin k\varphi] \\ &= u(r,\varphi) - \tfrac{1}{2}a_0(r) = u(r,\varphi) - \frac{1}{2\pi}\int_0^{2\pi} u(r,\psi)\,\mathrm{d}\psi \end{aligned}$$

and using the well-known formulae

$$(2.38)\qquad \sum_{k=1}^{\infty}\frac{1}{k}\cos kx = \frac{1}{2}\ln\frac{1}{2(1-\cos x)}, \quad 0 < x < 2\pi$$

and

$$(2.39)\qquad \sum_{k=1}^{\infty}\frac{1}{k}p^k\cos kx = \ln\frac{1}{(1-2p\cos x + p^2)^{1/2}}, \quad 0 < x < 2\pi,\ p^2 \leqslant 1$$

(see, e.g., [4]), we obtain (2.22). The theorem is proved. □

Similarly, transferring the right-hand boundary conditions for the Fourier coefficients of the solution, now according to Theorem 1.1, we obtain the relation

$$u(r,\varphi) - \frac{1}{2\pi}\int_0^{2\pi} u(r,\psi)\,\mathrm{d}\psi + \int_0^{2\pi} K(r,\psi)u'_r(r,\psi)\,\mathrm{d}\psi = L(r,\varphi) \tag{2.40}$$

where K and L are known functions.

The equations (2.22) and (2.40) now represent an analogue of the transfer of boundary conditions (the first being the trivial condition from the point $r = 0$ and the second the Dirichlet condition from the circumference $r = R$). It should be remarked that the result of transferring a point boundary condition (Dirichlet condition) is a global (integral) boundary condition.

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