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ONE-STEP METHODS FOR TWO-POINT BOUNDARY VALUE PROBLEMS IN ORDINARY DIFFERENTIAL EQUATIONS WITH PARAMETERS

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(Received May 11, 1990)

Summary. A general theory of one-step methods for two-point boundary value problems with parameters is developed. On nonuniform nets $h_n$, one-step schemes are considered. Sufficient conditions for convergence and error estimates are given. Linear or quadratic convergence is obtained by Theorem 1 or 2, respectively.

Keywords: One-step methods, two-point boundary value problems.

AMS classification: 65L10

1. INTRODUCTION.

We study the first order nonlinear system of ordinary differential equations

\[ y'(t) = f(t, y(t), \lambda), \quad t \in I = [a, b], \quad a < b, \]

with the boundary conditions

\[ y(a) = y_a \in \mathbb{R}^q, \]
\[ B_1 \lambda + B_2 y(b) = b_0 \in \mathbb{R}^p, \]

where $f: I \times \mathbb{R}^q \times \mathbb{R}^p \to \mathbb{R}^q$ is continuous and $\lambda \in \mathbb{R}^p$ is a parameter. Here $B_1$ is a matrix of dimension $p \times p$ and $B_2$ is a matrix of dimension $p \times q$. By a solution $(\varphi, \lambda)$ of the BVP(1–3) we mean a function $\varphi \in C^1(I, \mathbb{R}^q)$ and a parameter $\lambda \in \mathbb{R}^p$ that satisfy the BVP(1–3) $(C^1(I, \mathbb{R}^q)$ denotes the space of all continuous functions.
from $I$ into $\mathbb{R}^q$ with a continuous first derivative). Conditions under which (1-3) has a solution were determined in many papers (for example, see [4, 9, 10, 11]).

Indeed, $y(t) = y(t; \lambda)$. It is well known that if $f$ has continuous first order partial derivatives $f_y$ and $f_\lambda$ with respect to the second and third variables, then

$$\frac{\partial y(t; \lambda)}{\partial \lambda} = Y(t; \lambda),$$

where the $q \times p$ matrix $Y$ is the solution of

$$\begin{cases}
    Y'(t; \lambda) = f_y(t, y(t; \lambda), \lambda)Y(t; \lambda) + f_\lambda(t, y(t; \lambda), \lambda), & t \in I, \\
    Y(a; \lambda) = 0_{q \times p}.
\end{cases}$$

Let $y(t) = y(t; \lambda)$ be a solution of (1-2). It is also a solution of the BVP (1-3) provided (3) is satisfied, that is if $\lambda$ is a root of the equation

$$\Phi(\lambda) \equiv B_1 \lambda + B_2 y(b; \lambda) = b_0.$$  

Since

$$\Phi'(\lambda) = B_1 + B_2 y(b; \lambda),$$

Newton's method can be used for finding the root of (5).

In the present paper we discuss the numerical solution of the BVP (1-3) using a variable step size $h_n > 0$. On the interval $I$ we place a net of points $\{t_n\}$ with

$$t_0 = a, \quad t_{n+1} = t_n + h_n, \quad n = 0, 1, \ldots, N - 1 \quad \text{and} \quad t_N = b.$$

Our analysis refers to a family of such nets in which $N \to \infty$ while $h \to 0$ where $h = \max_{n=0,1,\ldots,N-1} h_n$. Now the numerical solution $(y_h, \lambda_{hj})$ of (1-3) at each point $t_n$ may be defined by

$$\begin{cases}
    y_h(t_0; \lambda_{hj}) = y_a, \\
    y_h(t_{n+1}; \lambda_{hj}) = y_h(t_n; \lambda_{hj}) + h_n F(t_n, h_n, y_h(t_n; \lambda_{hj}), \lambda_{hj}),
\end{cases}$$

or

$$\begin{cases}
    Y_h(t_0; \lambda_{hj}) = 0_{q \times p'}, \\
    Y_h(t_{n+1}; \lambda_{hj}) = [I + h_n F_y(t_n, h_n, y_h(t_n; \lambda_{hj}), \lambda_{hj})]Y_h(t_n; \lambda_{hj}) \\
    \quad + h_n F_\lambda(t_n, h_n, y_h(t_n; \lambda_{hj})\lambda_{hj}),
\end{cases}$$

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and

\[
\begin{cases}
\lambda_{h0} = \lambda_0 \in \mathbb{R}^p, \\
\lambda_{h,j+1} = \lambda_{hj} - [B_1 + B_2 Y_h(b; \lambda_{hj})]^{-1} [B_1 \lambda_{hj} + B_2 y_h(b; \lambda_{hj}) - b_0]
\end{cases}
\]

for \( n = 0, 1, \ldots, N - 1 \) and \( j = 0, 1, \ldots \). Here the increment function \( F \) has first order partial derivatives \( F_y \) and \( F_\lambda \) with respect to the third and fourth variables, respectively. Taking \( F = f \) we have the Euler scheme. Sometimes it is useful to write (9) in the following way:

\[
Y_h(t_n; \lambda_{hj}) = \sum_{i=0}^{n-1} \left( \prod_{r=i+1}^{n-1} A_{n+i-r,j} \right) B_{ij},
\]

where

\[
A_{nj} = I + h_n F_y(t_n, h_n, y_h(t_n; \lambda_{hj}), \lambda_{hj}),
\]

\[
B_{nj} = h_n F_\lambda(t_n, h_n, y_h(t_n; \lambda_{hj}), \lambda_{hj}).
\]

Assume for a moment that \( p = q \) and the matrix \( B_1 + B_2 \) is nonsingular. In such a situation we can determine another sequence \( \{\lambda_{hj}^*\} \) by

\[
\lambda_{h,j+1}^* = \lambda_{hj}^* - (B_1 + B_2)^{-1} [B_1 \lambda_{hj}^* + B_2 y_h(b; \lambda_{hj}^*) - b_0], \quad j = 0, 1, \ldots
\]

It means that in this case we do not need the approximate solution \( Y_h \) of (4). Now the method (8,11) is convergent to the solution \((\varphi, \lambda)\) of the BVP (1–3) if we suppose among other that the condition

\[
||(B_1 + B_2)^{-1} B_2|| \left[ 1 + \frac{M_2}{M_1} (\exp(M_1(b - a)) - 1) \right] < 1
\]

holds where \( M_1, M_2 > 0 \) are Lipschitz constants of \( F \) with respect to the last two variables. This was obtained in [5] for the constant step size \( h \). The condition (12) does not differ too much from the corresponding Keller result [7] (see also [2, 12]).

The condition (12) is superfluous for the convergence of the method (8–10). Assuming that the derivatives \( F_y \) and \( F_\lambda \) satisfy the Lipschitz condition we can prove the convergence of (8–10) if \( \lambda_0 \) is not too far from \( \lambda \). The location of \( \lambda_0 \) is one of the problems in computations. The estimates of errors are given, too. The result of this paper extends the corresponding Keller result [8] to boundary value problems with parameters.
2. Definitions

We introduce the usual definitions.

**Definition 1.** We say that the method (8–10) is convergent to the solution \((\varphi, \lambda)\) of the BVP (1–3) if

\[
\lim_{N \to \infty} \max_{j=0,1,\ldots,N} ||y_h(t_n; \lambda_{hj}) - \varphi(t_n)|| = 0
\]

\[
\lim_{h \to 0} ||\lambda_{hj} - \lambda|| = 0.
\]

**Definition 2.** We say that the method (8–10) is consistent with the problem (1–3) on the solution \((\varphi, \lambda)\) if there exist functions \(\gamma_1, \gamma_2 : I \times H \to \mathbb{R}_+ = [0, \infty), H = [0, h^*], h^* > 0\) such that

(i) \[
\left\| h_n F(t_n, h_n, \varphi(t_n), \lambda) + \varphi(t_n) - \varphi(t_{n+1}) \right\| \leq \gamma_1(t_n, h_n),
\]

(ii) \[
\left\| (I + h_n F_y(t_n, h_n, \varphi(t_n), \lambda)) Y(t_n; \lambda) + h_n F_\lambda(t_n, h_n, \varphi(t_n), \lambda) - Y(t_{n+1}; \lambda) \right\|
\]

\[
\leq \gamma_2(t_n, h_n)
\]

for \(n = 0, 1, \ldots, N - 1\) and

(iii) \[
\lim_{h \to 0} \bar{\gamma}_s(h) = 0, \quad \bar{\gamma}_s(h) = \sum_{i=0}^{N-1} \gamma_s(t_i, h_i), \quad s = 1, 2, \quad h = \max_i h_i,
\]

where \(Y\) is the bounded solution of the IVP (4).

The method (8–10) is said to be \(H\)-consistent with (1–3) on \((\varphi, \lambda)\) if only the conditions (i) and (iii) (for \(s = 1\)) are satisfied!

**Remark 1.** Because \((\varphi, \lambda)\) and \(Y\) are solutions of (1–3) and (4), respectively, the conditions (i) and (ii) can also be written in the following way:

\[
\left\| h_n F(t_n, h_n, \varphi(t_n), \lambda) - \int_{t_n}^{t_{n+1}} f(\tau, \varphi(\tau), \lambda) \, d\tau \right\| \leq \gamma_1(t_n, h_n),
\]

\[
\left\| h_n \left[ F_y(t_n, h_n, \varphi(t_n), \lambda) Y(t_n; \lambda) + F_\lambda(t_n, h_n, \varphi(t_n), \lambda) \right.ight.

\[
- \int_{t_n}^{t_{n+1}} \left[ f_y(\tau, \varphi(\tau), \lambda) Y(\tau; \lambda) + f_\lambda(\tau, \varphi(\tau), \lambda) \right] d\tau \right\| \leq \gamma_2(t_n, h_n).
\]
It is known that our method is consistent with (1-3) on \((\varphi, \lambda)\) if

\[
\lim_{h \to 0} F(t, h, y, X) = f(t, y, X), \\
\lim_{h \to 0} F_y(t, h, y, X) = f_y(t, y, X), \\
\lim_{h \to 0} F_\lambda(t, h, y, X) = f_\lambda(t, y, X)
\]

for all \((t, y, \lambda) \in I \times \mathbb{R}^q \times \mathbb{R}^p\).

3. CONVERGENCE

We are now in a position to establish the main convergence theorems and the associated error estimates.

Let

\[
0 \leq z_{n+1} \leq D[Az_n^2 + Bz_n + C], \quad A, B, C, D > 0, \quad n = 0, 1, \ldots
\]

We will need the following lemma.

**Lemma 1** (see [6]). Assume that there exists \(d\) such that

\[
DB < d < 1, \quad 4\bar{p}^2AC < 1, \quad \text{where} \quad \bar{p} = \frac{D}{d - DB}.
\]

If \(z_0 \leq \epsilon = DC/(1 - d) \leq 1/(\bar{p}A)\) then

\[
z_n \leq d^n\epsilon + DC \frac{1 - d^n}{1 - d'} \quad n = 0, 1, \ldots
\]

**Remark 2.** It is easy to see that \(z_n \leq \epsilon, \quad n = 0, 1, \ldots\)

**Proof of Lemma 1** [6]. We can write

\[
Q(z) = D[Az^2 + Bz + C] = Dq(z) + dz, \quad \text{where} \quad q(z) = Az^2 - z/\bar{p} + C.
\]

The quadratic function \(q\) has two distinct positive zeros \(z_-\) and \(z_+\), where \(z_+ > z_- > 0\). The function \(Q\) is increasing for \(z > 0\) so if \(z_0 \leq \epsilon\) then \(q(z) \leq C\) for \(0 \leq z \leq \epsilon\) and by induction \(z_n \leq \epsilon\) for \(n = 0, 1, \ldots\) Now

\[
z_{n+1} \leq DC + dz_n, \quad n = 0, 1, \ldots,
\]

and hence we have our estimate for \(z_n\). 

\[\Box\]
Now we can formulate the theorem.

**Theorem 1.** Let the following assumptions be satisfied:

1° there exists a unique solution \((\varphi, \lambda)\) of the BVP (1–3),

2° the function \(F: I \times H \times \mathbb{R}^q \times \mathbb{R}^p \rightarrow \mathbb{R}^q\) is continuous and has first order partial derivatives \(F_y\) and \(F_{\lambda}\) with respect to the third and fourth variables, respectively,

3° there exist constants \(L_1, L_2, K_1, K_2, K_3 \geq 0\) and functions \(\varepsilon_1, \varepsilon_2: I \times H \rightarrow \mathbb{R}_+\) such that for \((t, h, x, \bar{x}, \mu, \bar{\mu}) \in I \times H \times \mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^p \times \mathbb{R}^p\) we have

\[
\begin{align*}
(i) \quad &\|F_y(t, h, x, \mu)\| \leq L_1, \quad \|F_{\lambda}(t, h, x, \mu)\| \leq L_2; \\
(ii) \quad &\|F_y(t, h, x, \mu) - F_y(t, h, \bar{x}, \mu)\| \leq K_1\|x - \bar{x}\| + \varepsilon_1(t, h); \\
(iii) \quad &\|F_{\lambda}(t, h, x, \mu) - F_{\lambda}(t, h, \bar{x}, \mu)\| \leq K_2\|x - \bar{x}\| + K_3\|\mu - \bar{\mu}\| + \varepsilon_2(t, h),
\end{align*}
\]

and

\[
\lim_{h \to 0} \delta_s(h) = 0, \quad \delta_s(h) = \sum_{i=0}^{N-1} h_i \varepsilon_s(t_i, h_i), \quad s = 1, 2, \quad h = \max h_i,
\]

where the matrix norm is consistent with the vector norm (see [12]);

4° the method (8–10) is \(H\)-consistent with the BVP(1–3) on the solution \((\varphi, \lambda)\);

5° the matrix \(B_1 + B_2 Y_h(b; \lambda_{hj})\) is nonsingular for \(j = 0, 1, \ldots\) and there exists a constant \(D > 0\) such that

\[
\|(B_1 + B_2 Y_h(b; \lambda_{hj}))^{-1} B_2\| \leq D, \quad j = 0, 1, \ldots
\]

Then for sufficiently small \(\bar{h}\) there exists a positive constant \(d < 1\) such that the method (8–10) is convergent to the solution \((\varphi, \lambda)\) of the BVP (1–3) provided

\[
\|\lambda_0 - \lambda\| \leq u_0(h) = \sup_{x \in \bar{h}} \frac{DC(x)}{1 - d}, \quad h \leq \bar{h}.
\]

Moreover, the estimates

\[
\|\lambda_{hj} - \lambda\| \leq u_j(h), \quad j = 0, 1, \ldots
\]

\[
\max_{n=0, \ldots, N} \|y_h(t_n; \lambda_{hj}) - \varphi(t_n)\| \leq c[L_2(b - a)u_j(h) + \bar{\gamma}_1(h)], \quad j = 0, 1, \ldots
\]

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hold for $h = \max_i h_i \leq \bar{h}$ with

$$u_j(h) = d^j \| \lambda_0 - \lambda \| + DC(h) \frac{1 - d^j}{1 - d}, \quad j = 1, 2, \ldots$$

and

$$C(h) = c\bar{\gamma}_1(h) \left[ \frac{K_1}{2} (b - a) c^2 \bar{\gamma}_1(h) + c \delta_1(h) + 1 \right], \quad c = \exp(L_1(b - a)).$$

Proof. Put

$$v_n^j = \tilde{y}_h(t_n; \lambda_{hj}) - \varphi(t_n), \quad V_n^j = \| v_n^j \|,$$

$$z_n^j = \lambda_{hj} - \lambda, \quad Z_n^j = \| z_n^j \|,$$

$$w_n^j = Y_h(t_n; \lambda_{hj}) z_n^j - v_n^j, \quad W_n^j = \| w_n^j \|,$$

$$C_n = h_n F(t_n, h_n, \varphi(t_n), \lambda) + \varphi(t_n) - \varphi(t_{n+1}).$$

The mean value theorem yields the relation

(16) $$v_{n+1}^j = v_n^j + h_n \left[ F(t_n, h_n, \tilde{y}_h(t_n; \lambda_{hj}) \lambda_{hj}) ight.$$ \n
$$- F(t_n, h_n, \varphi(t_n), \lambda_{hj})$$ \n
$$+ F(t_n, h_n, \varphi(t_n), \lambda_{hj}) - F(t_n, h_n, \varphi(t_n), \lambda) + C_n$$ \n
$$= \left[ I + h_n \int_0^1 F_y(t_n, h_n, \varphi(t_n) + \tau v_n^j, \lambda_{hj}) d\tau \right] v_n^j$$ \n
$$+ h_n \int_0^1 F_\lambda(t_n, h_n, \varphi(t_n), \lambda + \tau z_n^j) d\tau z_n^j + C_n,$$

$$n = 0, 1, \ldots, N - 1,$$

or

$$V_{n+1}^j \leq (1 + h_n L_1) V_n^j + h_n L_2 Z_n^j + \gamma_1(t_n, h_n), \quad n = 0, 1, \ldots, N - 1.$$  

Hence we get

$$V_n^j \leq \sum_{i=0}^{n-1} \left( \prod_{r=i+1}^{n-1} (1 + h_r L_1) \right) (h_i L_2 Z_h^i + \gamma_1(t_i, h_i))$$

for $n = 0, 1, \ldots, N$, \quad $j = 0, 1, \ldots$ (here $\sum_r^s = 0$, $\prod_r^s = 1$, if $r > s$, or

(17) $$V_n^j \leq c[(b - a) L_2 Z_h^i + \bar{\gamma}_1(h)], \quad n = 0, 1, \ldots, N.$$
Now we need some relation for $z_i^j$. By the definition (10) we have
\begin{equation}
(18) \quad z_i^{j+1} = (B_1 + B_2 Y_h(b; \lambda_h))^{-1} B_2 w_{N}^j, \quad j = 0, 1, \ldots
\end{equation}

By (9) it is easy to see
\begin{equation}
w_n^{j+1} = A_n j w_n^j + A_n j v_n^j - v_{n+1}^j + B_n j z_h^j, \quad n = 0, 1, \ldots, N - 1,
\end{equation}

where $A_n j$ and $B_n j$ are defined in (9'). According to 3° and (16), the last relation implies
\begin{equation}
W_{n+1}^j \leq (1 + h_n L_1) W_n^j + b_n^j
\end{equation}

with
\begin{equation*}
b_n^j = h_n \left[ \frac{K_1}{2} (V_n^j)^2 + K_2 V_n^j Z_h^j + \frac{K_3}{2} (Z_h^j)^2 \right] + \gamma_1 (t_n, h_n) + h_n \left[ \varepsilon_1 (t_n, h_n) V_n^j + \varepsilon_2 (t_n, h_n) Z_h^j \right]
\end{equation*}
for $n = 0, 1, \ldots, N - 1$ and $W_0^j = 0$.

Using now (17) we have
\begin{equation}
W_n^j \leq \sum_{i=0}^{n-1} \left( \prod_{r=i+1}^{n-1} (1 + h_r L_1) \right) b_i^j, \quad n = 0, 1, \ldots, N - 1, \quad j = 0, 1, \ldots,
\end{equation}

and hence
\begin{equation}
(19) \quad W_n^j \leq A (Z_h^j)^2 + B(h) Z_h^j + C(h), \quad j = 0, 1, \ldots,
\end{equation}

where
\begin{align*}
A &= c (b - a) \left\{ \frac{K_1}{2} (c (b - a) L_2)^2 + K_2 c (b - a) L_2 + \frac{K_3}{2} \right\}, \\
B(h) &= c \left\{ (b - a) c [K_1 c (b - a) L_2 + K_2] \hat{v}_1 (h) + c (b - a) L_2 \delta_1 (h) + \delta_2 (h) \right\}.
\end{align*}

Combining this with (18) we see that
\begin{equation}
(20) \quad Z_h^{j+1} \leq D [A (Z_h^j)^2 + B(h) Z_h^j + C(h)], \quad j = 0, 1, \ldots
\end{equation}

Now for a sufficiently small $\bar{h}$ there exists a positive constant $d < 1$ such that
\begin{equation}
(21) \quad \begin{cases}
DB(h) < d < 1, \\
4\tilde{p}^2 (h) A C(h) < 1, & \tilde{p}(h) = D/(d - DB(h)), \\
DC(h) A \tilde{p}(h) + d' \leq 1
\end{cases}
\end{equation}
hold for $h = \max h_i \leq \bar{h}$. Hence by Lemma 1 we can get (14) and (15) for $h \leq \bar{h}$.

The proof is completed.
Remark 3. Let \( p = q = 1 \) and
\[
F_y(t, h, x, \mu) = h^\alpha(|\sin(x)|)^{1/2} + \xi(t, h, \mu),
\]
where \( \alpha > 0 \) and \( \xi : I \times H \times \mathbb{R} \to \mathbb{R} \). The function \( F_y \) does not satisfy the Lipschitz condition with respect to the third variable but it satisfies (ii) with \( K_1 = 0 \) and \( \varepsilon(t, h) = 2h^\alpha \). Hence \( \delta_1(h) \leq 2h^\alpha(b - a) \) and \( \delta_1(h) \to 0 \) as \( h \to 0 \).

Now we try to formulate some conditions which guarantee that 5° of Theorem 1 holds. We have

Lemma 2. Let the assumptions 1° – 3° of Theorem 1 hold with (ii) replaced by
\[
||F_y(t, h, x, \mu) - F_y(t, h, \bar{x}, \bar{\mu})|| \leq K_1 ||x - \bar{x}|| + K_0 ||\mu - \bar{\mu}|| + \varepsilon_1(t, h), \quad K_1, K_0 > 0.
\]
Let the method (8–10) be consistent with the BVP(1–3) on the solution \((\varphi, \lambda)\). Moreover, let the matrix \( B_1 + B_2 Y(b; \lambda) \) be nonsingular and
\[
||(B_1 + B_2 Y(b; \lambda))^{-1}|| \leq \beta_1, \quad ||B_2|| \leq \beta_2.
\]
Then for sufficiently small \( h \leq \bar{h} \) the condition 5° of Theorem 1 holds if \( \lambda_0 \) is not too far from \( \lambda \).

Proof. Put
\[
Q_n(u) = B_1 + B_2 Y_h(b; u) \quad Q(u) = B_1 + B_2 Y(b; u).
\]
Note that for \( j = 0, 1, \ldots \)
\[
Q_h(\lambda_{hj}) = Q(\lambda) \{ I + Q^{-1}(\lambda) [Q_h(\lambda_{hj}) - Q(\lambda)] \}
\]
and
\[
Q_h(\lambda_{hj}) - Q(\lambda) = B_2 q^j_N,
\]
where
\[
q^j_n = Y_h(t_n; \lambda_{hj}) - Y(t_n; \lambda), \quad n = 0, 1, \ldots, N, \quad j = 0, 1, \ldots
\]
Now we need an estimate for \( q^j_N \). By the definition of \( Y_n \) we have
\[
q^j_{n+1} = [I + h_n F_y(t_n, h_n, y_h(t_n; \lambda_{hj}), \lambda_{hj})] [Y_h(t_n; \lambda_{hj}) - Y(t_n; \lambda)] + Y(t_n; \lambda)
+ h_n [F_y(t_n, h_n, y_h(t_n; \lambda_{hj}), \lambda_{hj}) - F_y(t_n, h_n, \varphi(t_n), \lambda)] Y(t_n; \lambda)
+ h_n F_y(t_n, h_n, \varphi(t_n), \lambda) Y(t_n; \lambda) + h_n F_{\lambda}(t_n, h_n, \varphi(t_n), \lambda) - Y(t_{n+1}; \lambda)
+ h_n [F_{\lambda}(t_n, h_n, y_h(t_n; \lambda_{hj}), \lambda_{hj}) - F_{\lambda}(t_n, h_n, \varphi(t_n), \lambda)].
\]
Our assumptions yield
\[ Q_{n+1}^j \leq (1 + h_n L_1) Q_n^j + h_n \left[ K_1 V_n^j + K_0 Z_h^j + \varepsilon_1(t_n, h_n) \right] Y_b + \gamma_2(t_n, h_n) \]
\[ + h_n \left[ K_2 V_n^j + K_3 Z_h^j + \varepsilon_2(t_n, h_n) \right], \quad Q_n^j = ||q_n^j||, \]
where $Y$ is bounded by $Y_b, V_n^j$ and $Z_h^j$ are defined in the proof of Theorem 1. Now using the estimate (17) we get
\[ Q_{n+1}^j \leq (1 + h_n L_1) Q_n^j + h_n \left[ P_1 Z_h^j + P_2 \tilde{g}_1(h) + Y_b \varepsilon_1(t_n, h_n) + \varepsilon_2(t_n, h_n) \right] + \gamma_2(t_n, h_n) \]
for $n = 0, 1, \ldots, N - 1, j = 0, 1, \ldots$, where $P_1$ and $P_2$ are some nonnegative constants. Hence
\[ Q_N^j \leq c(b - a) P_1 Z_h^j + \eta(h), \]
and for $\beta = \beta_1 \beta_2$ we have
\[ \|Q^{-1}(\lambda) [Q_h(\lambda h_j) - Q(\lambda)] \| \leq \beta Q_N^j \leq c \beta(b - a) P_1 Z_h^j(h) + \beta \eta(h), \]
where
\[ \eta(h) = c \left[ (b - a) P_2 \tilde{g}_1(h) + Y_b \delta_1(h) + \beta_2(h) + \tilde{g}_2(h) \right]. \]
Let
\[ ||\lambda_0 - \lambda|| \leq \varrho = \sup_{h \leq \tilde{h}} DC(h)/(1 - d) \quad \text{and} \quad c \beta(b - a) P_1 \varrho \leq \alpha_1 < 1, \]
where $\tilde{h}$ is sufficiently small that (21) holds. It means that there exists $\alpha$ such that for sufficiently small $h < \tilde{h}$ we get
\[ c \beta(b - a) P_1 \varrho + \beta \eta(h) \leq \alpha < 1. \]
By Lemma 4.4.14([12]), p. 180 we conclude that $I + Q^{-1}(\lambda)[Q_h(\lambda_0) - Q(\lambda)]$ is nonsingular. Now by (22), $Q_h(\lambda_0)$ is also nonsingular and
\[ \|Q_h^{-1}(\lambda_0)\| \leq \frac{\beta_1}{1 - \alpha}. \]
Hence the condition 5° of Theorem 1 is true for $j = 0$ with $D = \beta/(1 - \alpha)$.

Put $u_0(h) = \varrho$. By (20) and Remark 2 we have $Z_h^1 \leq \varrho$. Moreover, (24) yields
\[ \|Q^{-1}(\lambda)[Q_h(\lambda h_1) - Q(\lambda)]\| \alpha < 1. \]
It means that $I + Q^{-1}(\lambda)[Q_h(\lambda h_1) - Q(\lambda)]$ is nonsingular and
\[ \|Q_h^{-1}(\lambda h_1)\| \leq \frac{\beta_1}{1 - \alpha}, \]
and hence the condition 5° of Theorem 1 is true for $j = 1$. Now by induction with respect to $n$ we can prove that 5° holds.

This completes the proof. □
Theorem 1 says that under some assumptions the method (8-10) converges to 
\((\varphi, \lambda)\) provided that \(\lambda_0\) is not far from \(\lambda\). This convergence is linear. Under a little stronger assumptions we can get quadratic convergence of (8-10). To this end \(\lambda_0\) must be nearer to \(\lambda\) than it was in Theorem 1. We have

**Theorem 2.** Assume that the assumptions of Lemma 2 are satisfied with 
\(\varepsilon_1(t, h) = \varepsilon_2(t, h) = 0, \; t \in \mathcal{I}, \; h \in \mathcal{H} \). Then

\[
\|\lambda_{h,j+1} - \lambda_{h,j}\| \leq T \|Q_{h,j}^{-1}\| \|\lambda_{h,j} - \lambda_{h,j-1}\|^2, \quad j = 1, 2, \ldots
\]

where

\[
T_0 = c(b - a)[K_2(b - a)L_2c + K_3]/2 + c(b - a)^2L_2[K_1(b - a)L_2c + K_0]/2,
\]
\[
T = \|B_2\|T_0, \quad Q_{h,j} = B_1 + B_2 Y_h(b; \lambda_{h,j}).
\]

Moreover, for a sufficiently small \(\bar{h}\) and \(\|\lambda_{h_1} - \lambda_{h_0}\| \leq \varepsilon < 1/(TD)\) the method (8-10) is convergent to \((\varphi, \lambda)\) and the estimates (14-15) hold for \(h = \max \{h_i \leq \bar{h}\}\) with

\[
u_j(h) = \frac{1}{TD}(TDe)^{2j-1} + m(h), \quad j = 1, 2, \ldots,
\]
\[u_0(h) = m(h),\]

where \(\|Q_{h,j}^{-1}\| \leq D\) and

\[
m(h) = 2 \frac{C(h)}{x_h + (x_h^2 - 4AC(h))^{1/2}}, \quad x_h = \frac{1 - DB(h)}{D}.
\]

**Proof.** Let

\[
k_{nj} = y_h(t_n; \lambda_{h,j}) - y_h(t_n; \lambda_{h,j-1}),
\]
\[
\bar{A}_{nj} = I + h_n \int_0^1 F_y(t_n, h_n, y_n(t_n; \lambda_{h,j-1}) + \tau k_{nj}, \lambda_{h,j-1} + \tau(\lambda_{h,j} - \lambda_{h,j-1})) \, d\tau,
\]
\[
\bar{B}_{nj} = h_n \int_0^1 F_\lambda(t_n, h_n, y_n(t_n; \lambda_{h,j-1}) + \tau k_{nj}, \lambda_{h,j-1} + \tau(\lambda_{h,j} - \lambda_{h,j-1})) \, d\tau.
\]

for \(n = 0, 1, \ldots, N, \quad j = 1, 2, \ldots\). Then we have

\[
\left\| \prod_{r=i+1}^{n} \bar{A}_{n+r-i,j} \right\| \leq \prod_{r=i+1}^{n} (1 + h_{n+1-r}L_1) \leq c, \quad i = 0, 1, \ldots, n - 1, \; n = 1, 2, \ldots, N.
\]
Moreover, for \( n = 0, 1, \ldots, N \) we have

\[
k_{n+1,j} = k_n + h_n \left[ F(t_n, h_n, y_h(t_n; \lambda_{h_{j-1}}), \lambda_{h_{j-1}}) - F(t_n, h_n y_h(t_n; \lambda_{h_{j-1}}), \lambda_{h_{j-1}}) \right],
\]

and by the mean value theorem this yields

\[
k_{n+1,j} = \overline{A}_{nj} k_n + \overline{B}_{nj}(\lambda_{h_{j-1}} - \lambda_{h_{j-1}}), \quad n = 0, 1, \ldots, N - 1, \quad j = 1, 2, \ldots.
\]

Hence

\[
k_n = \sum_{i=0}^{n-1} \left( \prod_{r=i+1}^{n-1} \overline{A}_{n+i-r,j} \right) \overline{B}_{ij}(\lambda_{h_{j-1}} - \lambda_{h_{j-1}}), \quad n = 0, 1, \ldots, N, \quad j = 1, 2, \ldots,
\]
or

\[
\|k_n\| \leq c(b - a)L_2\|\lambda_{h_{j-1}} - \lambda_{h_{j-1}}\|, \quad n = 0, 1, \ldots, N, \quad j = 1, 2, \ldots.
\]

We can also get an estimate for \( B_{ij} - B_{ij} \), where \( B_{ij} \) is defined in (9'). We have now

\[
\|B_{ij} - B_{ij}\| \leq h_i \int_0^1 [K_2(1 - \tau)||k_{ij}|| + K_3(1 - \tau)||\lambda_{h_{j-1}} - \lambda_{h_{j-1}}||] d\tau
\]

\[
\leq \frac{h_i}{2}[K_2(b - a)L_2c + K_3]\|\lambda_{h_{j-1}} - \lambda_{h_{j-1}}||,
\]

\( i = 0, 1, \ldots, N, \quad j = 1, 2, \ldots \)

and

\[
(27) \quad \left\| \sum_{i=0}^{N-1} \left( \prod_{r=i+1}^{N-1} \overline{A}_{N+i-r,j} \right) [B_{ij} - B_{ij}] \right\|
\]

\[
\leq \frac{c}{2}(b - a)[K_2(b - a)L_2c + K_3]\|\lambda_{h_{j-1}} - \lambda_{h_{j-1}}||, \quad j = 1, 2, \ldots.
\]

Put

\[
\xi_{ij} = \prod_{r=i+1}^{N-1} \overline{A}_{N+i-r,j} - \prod_{r=i+1}^{N-1} \overline{A}_{N+i-r,j}, \quad i = 0, 1, \ldots, N - 2, \quad j = 1, 2, \ldots,
\]

\[
\xi_{N-1,j} = 0_{q \times q}.
\]

We will prove that

\[
(28) \quad \|\xi_{N-s,j}\| \leq K\|\lambda_{h_{j-1}} - \lambda_{h_{j-1}}|| \sum_{i=N-s+1}^{N-1} \prod_{r=N-s+1}^{N-1} (1 + h_r L_1)h_i,
\]

\( s = 1, 2, \ldots, N, \quad j = 1, 2, \ldots, \)

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where
\[ K = \frac{1}{2}[K_1(b-a)L_2 + K_0]. \]

Indeed, it is true for \( s = 1 \). For \( s = 2 \) we have
\[
\|\xi_{N-2,j}\| = \|\bar{A}_{N-1,j} - A_{N-1,j}\|
\leq h_{N-1} \int_0^1 [K_1(1-\tau)\|\kappa_{N-1,j}\| + K_0(1-\tau)\|\lambda_h - \lambda_{h,j-1}\|] \, d\tau
\leq h_{N-1} K\|\lambda_h - \lambda_{h,j-1}\|.
\]

so (28) is true for \( s = 2 \).

Now we assume that (28) is satisfied for some \( s < N \). Then we see that
\[
\|\xi_{N-s-1,j}\| = \|\bar{A}_{N-1,j} \times \ldots \times \bar{A}_{N-s+1,j} \bar{A}_{N-s,j} - A_{N-1,j} \times \ldots \times A_{N-s+1,j} A_{N-s,j}
- A_{N-1,j} \times \ldots \times A_{N-s+1,j} A_{N-s,j}
+ A_{N-1,j} \times \ldots \times \bar{A}_{N-s+1,j} \bar{A}_{N-s,j}\|
\leq \|\bar{A}_{N-1,j} \times \ldots \times \bar{A}_{N-s+1,j}\| \|\bar{A}_{N-s,j} - A_{N-s,j}\| + \|\xi_{N-s,j}\| \|A_{N-s,j}\|
\leq \prod_{r=N-s+1}^{N-1} (1 + h_r L_1) K h_{N-3}\|\lambda_{h,j} - \lambda_{h,j-1}\|
+ (1 + h_{N-s} L_1) K\|\lambda_{h,j} - \lambda_{h,j-1}\| \sum_{i=N-s+1}^{N-1} \prod_{r=N-s+1}^{N-1} (1 + h_r L_1) h_i
= K\|\lambda_{h,j} - \lambda_{h,j-1}\| \sum_{i=N-s}^{N-1} \prod_{r=N-s, r \neq i}^{N-1} (1 + h_r L_1) h_i.
\]

Hence (28) is true for any value of \( s = 1, 2, \ldots, N, \quad j = 1, 2, \ldots \). Moreover, from (28) we may get the estimate
\[
\|\xi_{N-s,j}\| \leq K\|\lambda_{h,j} - \lambda_{h,j-1}\| \sum_{i=N-s+1}^{N-1} \prod_{r=N-s+1}^{N-1} (1 + h_r L_1) h_i
\leq cK(b-a)\|\lambda_{h,j} - \lambda_{h,j-1}\|, \quad s = 1, 2, \ldots, N, \quad N = 1, 2, \ldots
\]

and hence
\[
(29) \quad \left\| \sum_{i=0}^{N-1} \left( \prod_{r=i+1}^{N-1} A_{N+i-r,j} - \prod_{r=i+1}^{N-1} A_{N+i-r,j} \right) B_{ij} \right\| \leq \sum_{i=0}^{N-1} \|\xi_{ij}\| \|B_{ij}\|
\leq cK(b-a)^2 L_2\|\lambda_{h,j} - \lambda_{h,j-1}\|, \quad j = 1, 2, \ldots.
\]
By the definition of $\lambda_{h,j+1}$ and by (9') we have

\[\begin{align*}
(30) \quad |\lambda_{h,j+1} - \lambda_{hj}| &= |Q_{hj}^{-1}||B_1(\lambda_{hj} - \lambda_{h,j-1}) \\
&+ B_2 k_{Nj} - Q_{h,j-1}(\lambda_{hj} - \lambda_{h,j-1})| \\
&= |Q_{hj}^{-1}||\lambda_{hj} - \lambda_{h,j-1}| \\
&\times \left| B_1 + B_2 \sum_{i=0}^{N-1} \left( \prod_{r=i+1}^{N-1} A_{N+i-r,j} \right) \bar{B}_{ij} - Q_{h,j-1} \right| \\
&= |Q_{hj}^{-1}||\lambda_{hj} - \lambda_{h,j-1}| \left| B_2 \sum_{i=0}^{N-1} \left( \prod_{r=i+1}^{N-1} A_{N+i-r,j} \right) \bar{B}_{ij} \\
&- \sum_{i=0}^{N-1} \left( \prod_{r=i+1}^{N-1} A_{N+i-r,j} \right) B_{ij} \right|.
\end{align*}\]

Using (27) and (29) we find

\[\begin{align*}
(31) \quad \left| \sum_{i=0}^{N-1} \left( \prod_{r=i+1}^{N-1} A_{N+i-r,j} \right) \bar{B}_{ij} - \sum_{i=0}^{N-1} \left( \prod_{r=i+1}^{N-1} A_{N+i-r,j} \right) B_{ij} \right| \\
&\leq \left| \sum_{i=0}^{N-1} \left( \prod_{r=i+1}^{N-1} A_{N+i-r,j} \right) (\bar{B}_{ij} - B_{ij}) \right| \\
&+ \left| \sum_{i=0}^{N-1} \left( \prod_{r=i+1}^{N-1} A_{N+i-r,j} - \prod_{r=i+1}^{N-1} A_{N+i-r,j} \right) B_{ij} \right| \\
&\leq T_0|\lambda_{hj} - \lambda_{h,j-1}|, \quad j = 1, 2,\ldots
\end{align*}\]

Combining (27), (30) and (31) we have (26).

By Lemma 2 we know that for sufficiently small $h$ the matrix $Q_{hj}$ is nonsingular and $|Q_{hj}^{-1}| \leq D$. It means that

\[|\lambda_{h,j+1} - \lambda_{hj}| \leq T D |\lambda_{hj} - \lambda_{h,j-1}|^2, \quad j = 1, 2,\ldots\]

and

\[|\lambda_{h,j+1} - \lambda_{hj}| \leq \frac{1}{TD} (TD |\lambda_{h1} - \lambda_{h0}|)^{2^j}, \quad j = 0, 1,\ldots\]

We see that all assumptions of Theorem 1 are satisfied, so (20) yields

\[Z_{h}^{j+1} \leq D[A(Z_h^j)^2 + B(h)Z_h^j + C(h)] = Dp_h(Z_h^j) + Z_h^j,\]

where

\[p_h(z) = Az^2 - x_hz + C(h).\]
The quadratic function $p_h$ has two distinct zeros $z^+_h$ and $z^+_h$ where $z^+_h > z^+_h > 0$. If $\|\lambda_{h_0} - \lambda\| \leq \min_{h \leq h} [z^+_h, \max DC(h)/(1 - d)]$ then $\|\lambda_{h_j} - \lambda\| \leq z^+_h, j = 1, 2, \ldots$. Hence

$$\|\lambda_{h,j+1} - \lambda\| \leq \|\lambda_{h,j+1} - \lambda_{h,j}\| + \|\lambda_{h,j} - \lambda\| \leq \frac{1}{TD}(TD\|\lambda_{h,1} - \lambda_{h,0}\|^{2^j} + z^+_h, \quad j = 0, 1, \ldots$$

so we have (14). The rest follows from Theorem 1.

This completes the proof. \hfill \Box

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