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SPECIAL MOTIONS OF ROBOT-MANIPULATORS

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Summary. There exist many examples of closed kinematical chains which have a freedom of motion, but there are very few systematical results in this direction. This paper is devoted to the systematical treatment of 4-parametric closed kinematical chains and we show that the so called Bennet's mechanism is essentially the only 4-parametric closed kinematical chain which has the freedom of motion. According to [3] this question is connected with the problem of existence of asymptotic geodesic lines on robot-manipulators considered as submanifolds of a pseudo-Riemannian space. All computations were performed by the help of a formal manipulation system on a PC-computer.

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THE THEORETICAL PART

The motion of a p-parametric robot-manipulator determined by axes $X_1, \ldots, X_p$ is described by the matrix

$$ g(u_1, \ldots, u_p) = g_1(u_1) \cdot \ldots \cdot g_p(u_p), \quad g_i(u_i) = \exp(u_iX_i), $$

where $g_i(u_i)$ denotes either the revolution around the straight line determined by its Plücker coordinates $X_i$ or the translation in the direction determined by $X_i$. In what follows we shall for simplicity consider robot-manipulators with rotational links only and so $\exp(u_iX_i)$ means always the revolution around the straight line $X_i$ with the angle of revolution $u_i$.

The motion $\hat{g}(u_1, \ldots, u_p)$ of the effector space with respect to the base space is expressed by the formula

$$ \hat{g}(u_1, \ldots, u_p)(\hat{R}) = R \cdot g(u_1, \ldots, u_p), $$
where $\tilde{R}$ is an orthonormal frame in the effector space, $R$ is an orthonormal frame in the base space and we may suppose that $R = \tilde{R}$ at the point $[0,\ldots,0]$ of the parameter space (in the starting position of the robot-manipulator). This condition yields $g(0,\ldots,0) = E$.

The instantaneous position $Y_1,\ldots,Y_p$ of axes $X_1,\ldots,X_p$ of a robot-manipulator is given by the formula

\[(1) \quad Y_1 = X_1, \quad Y_2 = \text{Ad}(g_1)X_2, \quad \ldots, \quad Y_p = \text{Ad}(g_1\ldots g_{p-1})X_p,\]

where $\text{Ad}(g)X$ denotes the induced (adjoint) action of the matrix $g$ of a space congruence on a straight line determined by its Plücker coordinates $X$. We can also say that the induced action is the action of the group of space congruences in the space of screws.

Details concerning the above introduced formalism can be found for instance in [1] or [2]. The proof of the formula (1) is easy: Let us consider the motion $h(t) = g(u_1,\ldots,u_{i-1},t,u_{i+1},\ldots,u_p)$, which is the revolution around $Y_i$. Then $h'(t)h^{-1}(t) = \text{Ad}(g_1\ldots g_{i-1})X_i$.

A motion $g(t) = g_1(u_1(t))\ldots g_p(u_p(t))$ of the $p$-parametric robot-manipulator, determined by functions $u_1(t),\ldots,u_p(t)$ of one variable $t$ determines a one-parametric motion of the end-effector. Such a motion $g(t)$ of the robot-manipulator will be called a motion of a closed kinematical chain iff $g_1(u_1(t))\ldots g_p(u_p(t)) = E$. The basic properties of closed kinematical chains are described in [2], the definition complies with the intuitive meaning that during the motion of a closed kinematical chain the distance, angle and offset between the last and first axes of the robot-manipulator remain fixed.

Let a robot-manipulator be given by axes $X_1,\ldots,X_p$. A position of this robot-manipulator determined by the position $Y_1,\ldots,Y_p$ of its axes is called singular iff $\text{rank}(Y_1,\ldots,Y_p) < p$.

**Theorem 1.** Motions of closed kinematical chains are possible only in singular positions.

**Proof.** We have $g_1(t)\ldots g_p(t) = E$. The derivative of this equation with respect to $t$ yields $g'_1 \cdot g_2 \ldots g_p + \ldots + g_1 \ldots g_{p-1} \cdot g'_p = 0$. We obtain $Y_1v_1 + \ldots + Y_pv_p = 0$, where $v_i = u'_i(t)$ and we see that vectors $Y_1,\ldots,Y_p$ are linearly dependent.

**Theorem 2.** Let $g_1(t)\ldots g_p(t) = E$ be a motion of a closed kinematical chain. Then the $p+1$-parametric robot-manipulator $g_1(u_1)\ldots g_p(u_p) \cdot g_{p+1}(u_{p+1})$, where $g_{p+1}(u_{p+1}) = g_1(u_{p+1})$, satisfies the equation $Y_1(t) = Y_{p+1}(t)$ during this motion, $u_{p+1}(t)$ can be arbitrary.
Proof. We have $Y_{p+1}(t) = \text{Ad}(g_1(t) \cdots g_p(t))X_{p+1} = X_{p+1} = X_1 = Y_1$, because $g_1(t) \cdots g_p(t) = E$. □

Theorem 3. Let a $(p + 1)$-parametric robot-manipulator satisfy the equation $Y_1 = Y_{p+1}$ during some motion $g(t)$ given by $u_i = u_i(t)$, $i = 1, \ldots, p + 1$. Then this motion is a motion of a closed kinematical chain with the last link an arbitrary screw motion around the last axis.

Proof. $Y_1 = Y_{p+1}$ implies $Y_{p+1} = \text{Ad}(g_1(t) \cdots g_p(t))X_{p+1} = X_1$. Let us write $X_1 = \text{Ad} \gamma X_{p+1}$ for some fixed congruence $\gamma$. Then we have $\text{Ad}(g_1(t) \cdots g_p(t))X_{p+1} = \text{Ad} \gamma X_{p+1}$. This yields $g_1(t) \cdots g_p(t) \cdot h(t) = \gamma$, where $h(t)$ is a screw motion around the axis $X_{p+1}$. Let us choose one position of the robot-manipulator determined by $t = t_0$. Then we have $\gamma = g_1(t_0) \cdots g_p(t_0) \cdot h(t_0)$ and we obtain the following equation:

$$g_1(t) \cdots g_p(t) \cdot h(t) = g_1(t_0) \cdots g_p(t_0) \cdot h(t_0).$$

This equation can be written in the form

$$g_1(t) \cdots g_p(t) h(t) h^{-1}(t_0) g_p^{-1}(t_0) \cdots g_1^{-1}(t_0) = E.$$

Let us denote

$$k_i(t) = g_1(t_0) \cdots g_{i-1}(t_0) \cdot g_i(t) \cdot g_i^{-1}(t_0) \cdot g_{i-1}^{-1}(t_0) \cdots g_1^{-1}(t_0),$$

$$m(t) = g_1(t_0) \cdots g_p(t_0) \cdot h(t) h(t_0)^{-1} \cdot g_p^{-1}(t_0) \cdots g_1^{-1}(t_0).$$

Then we have

$$k_1(t) \cdots k_p(t) m(t) = E$$

and $k_i(t)$ is a revolution around the axis $Z_i = \text{Ad}(g_1(t_0) \cdots g_{i-1}(t_0))X_i$, $m(t)$ is a screw motion around the axis $Z_{p+1} = \text{Ad}(g_1(t_0) \cdots g_p(t_0))X_{p+1}$. □

Theorem 4. The motion $u_i = u_i(t)$, $i = 1, \ldots, p$ of a $p$-parametric robot-manipulator is a screw motion around some axis $X_{p+1}$ iff the robot manipulator determined by $X_1, \ldots, X_{p+1}$ has a motion of a closed kinematical chain with the last link a screw motion.

Proof. Let $g_1(t) \cdots g_p(t) = h(t)$, where $h(t)$ is a screw motion around $X_{p+1}$. Then we have

$$g_1(t) \cdots g_p(t) \cdot h^{-1}(t) = E$$

and the statement follows from the previous considerations.
From Theorems 1, 2, 3, 4 we can deduce the following facts: If we find all solutions of the equation \( Y_1 = Y_{p+1} \), we can compute the motion \( h(t) \) for each such solution and we have solved the following problems:

a) For \( h(t) \) a general screw motion we have find all motions of \( p-1 \)-parametric robot-manipulators, which yield a screw motion of the end-effector (with rotation and translation as special cases). We have also obtained examples of \( p+1 \)-parametric closed kinematical chains with the freedom of motion such that they have a translation and rotation with the same axis.

b) If \( h(t) \) is a rotation, we can suppose \( h(t) = E \) by changing the representation of the motion as rotations around the same axis commute. We have found all closed kinematical chains with \( p \) links, which have a freedom of motion.

**Theorem 5.** The only solution of the equation \( Y_1 = Y_5 \) is the Bennet's mechanism.

**Remark.** The Bennet's mechanism is the closed kinematical chain with four rotational links oriented in such a way that the following relations for Denavit-Hartenberg parameters (see below) are satisfied:

\[
d_i = 0, \ i = 1, \ldots, 4, \quad \alpha_1 = \alpha_3, \quad \alpha_2 = \alpha_4, \quad m_1 = m_3, \quad m_2 = m_4, \quad a_1^2 s_2^2 = a_2^2 s_1^2.
\]

The trivial cases of all axes parallel and of all axes passing through one point are considered as special cases of the Bennet's mechanism.

**Corollaries of Theorem 5.**

a) The Bennet's mechanism is the only \( 4 \)-parametrical closed kinematical chain with the freedom of motion.

b) If one of the links of the Bennet's mechanism is allowed to slide (to perform an arbitrary screw motion), nothing will change and the concerned link will remain rotational.

c) There exists not a \( 3 \)-parametrical robot-manipulator with rotational links, which can perform a translation or a screw motion of the end-effector apart from the revolutions around its axes.

**Remark.** Corollaries follow from Theorems 1 to 4. The Bennet's mechanism is known for a long time already, but its uniquennes was not shown before. The question of the classification of all closed kinematical chains with freedom of movement for 5 links remains open. The computations below show that the solution of such a problem will be extremely complicated even on the assumption that the formal manipulation with equations will be done on a computer as was the case also in the presented paper.
THE COMPUTATIONAL PART

We shall solve the equation \( Y_1 = Y_{p+1} \) for \( p = 4 \). For this purpose one has to compute the instantaneous position of axes of a robot-manipulator. This has been done in [2] for a 6-parametric robot manipulator and we shall use the result of this computation.

For symmetry and simplicity reasons it is convenient to choose as the reference frame the orthonormal frame located between axes \( X_3 \) and \( X_4 \) in symmetrical position (the origin is at the middle distance between \( X_3 \) and \( X_4 \), the \( z \) axis is perpendicular to both \( X_3 \) and \( X_4 \) and the direction of \( x \) and \( y \) axes is in the middle between \( X_3 \) and \( X_4 \)).

The computation yields for Plücker coordinates of axes \( Y_1, \ldots, Y_6 \), where \( Y_i = (y_i; z_i) \):

\[
y_4 = \begin{pmatrix} \kappa \\ \sigma \\ 0 \end{pmatrix}, \quad z_4 = \frac{1}{2} a_3 \begin{pmatrix} \kappa \\ -\sigma \\ 0 \end{pmatrix}, \quad y_5 = \begin{pmatrix} \kappa c_4 - \sigma c_4 s_4 \\ \sigma c_4 + \kappa c_4 s_4 \\ s_4 s_4 \end{pmatrix}, \quad z_5 = \begin{pmatrix} -\kappa g_4 + \sigma h_4 \\ -\sigma g_4 + \kappa h_4 \\ r_4 \end{pmatrix},
\]

where

\[
G_4 = s_4 \left( a_4 + \frac{1}{2} a_3 c_4 \right), \quad H_4 = s_4 s_4 d_4 - C_4 \left( \frac{1}{2} a_3 + a_4 c_4 \right), \quad R_4 = d_4 c_4 s_4 + a_4 s_4 c_4,
\]

\( \kappa = \cos(\frac{1}{2} \alpha_3), \ \sigma = \sin(\frac{1}{2} \alpha_3) \), and as usually \( C_i = \cos \alpha_i, \ S_i = \sin \alpha_i, \ c_i = \cos u_i, \ s_i = \sin u_i \).

\[
y_6 = \begin{pmatrix} -\kappa l_5 - \sigma (c_4 M_5 - s_4 F_5) \\ -\sigma l_5 + \kappa (c_4 M_5 - s_4 F_5) \\ s_4 M_5 + c_4 F_5 \end{pmatrix}, \quad z_6 = \begin{pmatrix} \kappa [B_5 - \frac{1}{2} a_3 (c_4 M_5 - s_4 F_5)] - \sigma (c_4 A_5 - s_4 P_5 - \frac{1}{2} a_3 L_5) \\ \sigma [B_5 - \frac{1}{2} a_3 (c_4 M_5 - s_4 F_5)] + \kappa (c_4 A_5 - s_4 P_5 - \frac{1}{2} a_3 L_5) \\ s_4 A_5 + c_4 P_5 \end{pmatrix},
\]

where

\[
M_5 = C_4 S_5 c_5 + S_4 C_5, \quad L_5 = S_4 S_5 c_5 - C_4 C_5, \quad F_5 = s_5 S_5, \\
B_5 = -a_4 M_5 - a_5 (S_4 C_5 c_5 + C_4 S_5) + d_5 S_4 F_5, \\
A_5 = -a_4 L_5 + a_5 (C_4 C_5 c_5 - S_4 S_5) - F_5 (d_4 + C_4 d_5), \\
P_5 = a_5 C_5 s_5 + d_5 S_5 c_5 + d_4 M_5.
\]
We used the Denavit-Hartenberg parameters:

- \( \alpha_i \) → the angle from \( X_i \) to \( X_{i+1} \),
- \( a_i \) → the distance from \( X_i \) to \( X_{i+1} \),
- \( d_i \) → the offset between \( X_{i-1}, X_i \) and \( X_i, X_{i+1} \),
- \( u_i \) → the angle of revolution around the axis \( X_i \).

The formulas for \( Y_1, Y_2, Y_3 \) are obtained from formulas for \( Y_6, Y_5, Y_4 \) by the following substitution:

\[
\alpha \rightarrow -\alpha_{6-i}, \quad a_i \rightarrow -a_{6-i}, \quad u_i \rightarrow u_{7-i}, \quad d_i \rightarrow d_{7-i}.
\]

Now we are going to solve the equation \( Y_2 = Y_6 \). We obtain 6 equations for six Plücker coordinates, from which only 4 are independent (\( y_2 \) is a unit vector, \( y_2 \) and \( z_2 \) are perpendicular). The angles of revolution \( u_3, u_4, u_5 \) are the unknowns, at least one of them must be different from a constant. This follows that in general we obtain two equations as solvability conditions. The equations \( Y_2 = Y_6 \) can be written as follows:

\[
\begin{align*}
\kappa(C_2 + L_5) + \sigma(-S_2c_3 + M_5c_4 - F_5s_4) &= 0, \\
\sigma(-C_2 + L_5) + \kappa(-S_2c_3 - M_5c_4 - F_5s_4) &= 0, \\
S_2s_3 - F_5c_4 - M_5s_4 &= 0, \\
\kappa[-G_2 + B_5 - \frac{1}{2}a_3(M_5c_4 - F_5s_4)] + \sigma(-H_5 - \frac{1}{2}a_3L_5 + A_5c_4 - P_5s_4) &= 0, \\
\sigma[G_2 + B_5 - \frac{1}{2}a_3(M_5c_4 - F_5s_4)] + \kappa(-H_2 + \frac{1}{2}a_3L_5 - A_5c_4 + P_5s_4) &= 0, \\
R_2 - P_5 - A_5s_4 &= 0,
\end{align*}
\]

where \( G_2, H_2, R_2 \) are defined analogically to \( G_4, H_4, R_4 \) using (6).

After making suitable linear combinations in (7) we obtain for \( S_3 \neq 0 \):

\[
\begin{align*}
r_1 &\equiv -C_3M_5c_4 + F_5C_3s_4 + L_5S_3 - S_2c_3 = 0, \\
r_2 &\equiv S_3M_5c_4 - F_5S_3s_4 + C_2 + C_3L_5 = 0, \\
r_3 &\equiv -F_5c_4 - M_5s_4 + S_2s_3 = 0, \\
r_4 &\equiv -A_3C_3c_4 + C_3P_5s_4 - B_5S_3 - C_2a_2c_3 + S_2d_3s_3 = 0, \\
r_5 &\equiv A_5S_3c_4 - S_3P_5s_4 - B_5C_3 - S_2a_2 - S_2a_3c_3 = 0, \\
r_6 &\equiv -P_5c_4 - A_5s_4 + S_2c_3d_3 + C_2a_2s_3 = 0
\end{align*}
\]

Let \( S_2 \neq 0 \). From \( r_1 \) and \( r_3 \) we obtain

\[
\begin{align*}
c_3 &= \frac{1}{S_2}(-C_3M_5c_4 + F_5C_3s_4 + L_5S_3), \quad s_3 = \frac{1}{S_2}(F_5c_4 + M_5s_4).
\end{align*}
\]
Substitution and combination with \( r_2 \) yields:

\[
(10) \quad r_4 = (F_b d_3 - C_3) S_2 S_3 c_4 + (C_3 P_5 + M_5 d_3) S_2 S_3 s_4 - B_5 S_2 S_3^2 \\
- C_2^2 C_3 a_2 - C_2 L_5 a_2 - C_2 S_2 S_3 a_3 = 0, \\
r_5 \equiv A_5 S_2^2 c_4 - P_5 S_3^2 s_4 - B_5 C_3 S_3 - S_2 S_3 a_2 - C_2 C_3 a_3 - L_5 a_3 = 0, \\
r_6 \equiv (-P_5 S_2 + C_2 F_5 a_2) S_3 c_4 + (-A_5 S_2 + C_2 M_5 a_2) S_3 s_4 + C_2 C_3 S_2 d_3 \\
+ L_5 S_2 d_3 = 0.
\]

We consider equations \( r_2, r_5 \) and \( r_6 \) as linear equations in \( c_4 \) and \( s_4 \). They can have common solution only if their determinant is equal to zero; we shall write

\[
(11) \quad \det |r_2, r_5, r_6| = 0.
\]

Similarly we have

\[
(12) \quad \det |r_2, r_4, r_5| = 0.
\]

Remark. Let

\[
(13) \quad a_1 \cos \varphi + b_1 \sin \varphi + c_1 = 0, \quad a_2 \cos \varphi + b_2 \sin \varphi + c_2 = 0,
\]

be two equations for unknown angle \( \varphi \). Equations (13) have a common solution iff

\[
(14) \quad \det \begin{pmatrix} b_1 & c_1 \\ b_2 & c_2 \end{pmatrix}^2 + \det \begin{pmatrix} a_1 & c_1 \\ a_2 & c_2 \end{pmatrix}^2 - \det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}^2 = 0.
\]

Equations (11), (12) and (14) for \( r_2 \) and \( r_5 \) are algebraic in \( \cos u_5 \) and \( \sin u_5 \). (We have to take into account the identity \( \cos^2 u_5 + \sin^2 u_5 = 1 \), we can change (11), (12) and (14) in such a way that there are linear in \( \cos u_5 \).) If \( u_5 \) is constant, we can leave out the axis \( Y_5 \) and we obtain a solution of the equation \( Y_2 = Y_5 \). It is easy to see from the expression for \( Y_5 \) in (2) that the equation \( Y_2 = Y_5 \) has no nontrivial solutions. Therefore we can suppose that (11), (12) and (14) must be satisfied on some interval and so they must be identically zero.

The expanded form of (11) and (14) is very long and complicated, (11) has 197 terms, (14) has 304 terms and therefore we shall not write this equations here in full. We shall solve these equations step by step by choosing suitable coefficients at various powers of \( \cos u_5 \) and \( \sin u_5 \).

During the computations we shall use also the following simple fact:
To each solution of the equation \( Y_2 = Y_6 \) we obtain a new solution of this equation by the substitution \( Y_2 \to Y_6, Y_3 \to Y_5, Y_5 \to Y_3, Y_6 \to Y_2, Y_4 \) remains.
Now we are ready to solve equations (11) and (14). The coefficients at the highest power in (14) yield: At $c_5 s_5^2$:

(15) 
$$-2S_3^4S_4^4S_5^4a_4d_4 = 0,$$

at $s_5^4$:

(16) 
$$S_3^3S_4^2S_5^4(S_4^2a_3^2 - S_3^2a_4^2 + S_3^2S_4^2d_4^2) = 0.$$

\[ \text{A)} \text{ Let } S_5 = 0. \text{ At } s_5^2 \text{ in (14) we obtain } S_4 = 0, \text{ this yields } M_5 = F_5 = 0. \text{ From } r_2 \text{ and } r_5 \text{ we compute } c_4 \text{ and } c_5 \text{ and substitute into (9). We obtain } s_3 = 0 \text{ and therefore there is no solution in this case. If } r_2 \text{ and } r_5 \text{ are linearly dependent, we obtain } c_3 = 0 \text{ from } r_1. \]

\[ \text{B)} \text{ Let } S_5 \neq 0, S_4 = 0. \text{ The coefficient at } s_5^2 \text{ in (14) and (11) leads to } C_3 = C_5 = 0. \text{ The substitution into (9) yields } c_3 = 0 \text{ and we have again no solution.} \]

\[ \text{C)} \text{ Let } S_5S_4 \neq 0. \text{ Then from (15) and (16) we obtain } d_4 = 0, a_4 = a_3mS_4S_3^{-1}, m^2 = 1. \]

\[ C_1) \text{ Let } a_3 = 0. \text{ Then (14) at } c_5s_5 \text{ yields } d_5(-C_2S_5a_2 + C_5S_2a_5) = 0. \]

\[ C_1a). \text{ Let } d_5 = 0. \text{ At } c_5s_5 \text{ in (11) we obtain } a_5d_3 = 0. \]

\[ ao) \text{ Let } a_5 \neq 0. \text{ Then } d_3 = 0. \text{ From } s_5^2 \text{ in (13) we obtain} \]

$$a_2 = -2C_2C_5S_2^{-1}S_5^{-1}a_2a_5 - a_5^2 + a_5^2S_2^{-2} + a_5^2S_5^{-2}.$$  

Substitution into the coefficient at $s_5^2$ in (11) yields $C_5s_2a_2 = C_2S_5a_5$. The coefficient at $c_5s_5$ in (12) yields $C_2S_5a_2 = C_5S_2a_5$. This yields $S_2^2 = S_5^2$, $a_2 = a_5^2$. Similarly we obtain $C_3S_2a_2 = C_4S_5a_5$ which yields a solution.

\[ ao) \] Let $a_5 = 0$. From the coefficient at $s_5^2$ in (14) we obtain $a_2 = 0$, from the equation $r_2$ we obtain $d_3 = 0$, we obtain a trivial solution with all axes passing through one point.

\[ C_1(\beta) \text{ From (12) at } c_5s_5 \text{ we obtain } C_5d_5 + C_2d_3 = 0. \]

\[ a) \text{ Let } C_5 \neq 0. \text{ Then from (11) at } s_5^2 \text{ we obtain } S_2^2 = S_5^2 \text{ and (12) yields } s_4 = 0 \text{ and we have no solution in this case.} \]

\[ b) \text{ Let } C_5 = 0. \text{ From the coefficient at } s_5^2 \text{ in (11) we obtain } C_2 = 0, \text{ remaining coefficients in (11) lead to } s_4 = 0 \text{ and we have no solution}. \]

\[ C_2) \text{ Let } a_3 \neq 0. \text{ The coefficient at } s_5^2 \text{ in (11) yields } S_3a_4d_3 + S_4a_3d_5 = 0, \text{ so } d_5 = -d_3m. \text{ The coefficient at } s_5^2 \text{ in (14) yields } a_3d_3(C_2 - C_5m) = 0. \]

\[ a) \text{ Let } C_2 - C_5m \neq 0. \text{ Then } d_3 = 0. \text{ From coefficients at } c_5s_5^2 \text{ in (11) and (14) we obtain} \]

$$a_5 = a_3S_3S_3^{-1}(C_5 - C_2m)^{-1}(C_2C_4C_5 + C_3S_2^2 - C_4m),$$

$$a_2 = -a_3S_2S_3^{-1}(C_5 - C_2m)^{-1}(C_2C_3C_5 + C_4S_5^2 - C_3m).$$
Let $C_5 = -C_2m$. Then from coefficients at $s_1$ in (11) and (14) we obtain $S_3^2 = S_4^2$, $S_2^2 = S_3^2$, the coefficient at $c_5$ in (11) yields $C_4 = -C_3m$ and we have a solution.

aβ) Let $S_2^2 \neq S_3^2$. We compute the coefficient at $s_2^2$ in (11) and we obtain an equation of the type

$$AC_3C_4 + B = 0,$$

where

$$A = -(S_2^2 + S_3^2)(C_5 - C_2m), B = (C_5 - C_2m)(S_2^2 - 2S_3^2 - 2S_4^2 + S_5^2) - (S_3^2 + S_4^2)(C_2S_5^2 - C_5S_2^2m).$$

From it we obtain the equation $A^2(1 - S_3^2)(1 - S_4^2) - B^2 = 0$, which is of the type $KC_2C_5 + L = 0$, where $K$ and $L$ are polynomials in $S_2^2, S_3^2, S_4^2, S_5^2$.

The coefficient at $s_2^2$ in (14) is an equation of the type $PC_3C_4 + Q = 0$. Substitution from previous equations leads to the equation

$$(S_2^2 - S_3^2)(S_2^2 S_3^2 - S_4^2 S_5^2) = 0,$$

which yields $S_2^2 = S_2^2 S_3^2 S_4^2$. Substitution into the equation $AC_3C_4 + B = 0$ yields $$(S_2^2 - S_3^2)(S_2^2 - S_3^2) = 0.$$ Because $S_2^2 = S_3^2$ leads to $S_2^2 = S_2^2$, we must have $S_2^2 = S_2^2, S_3^2 = S_3^2$.

The coefficient at $s_2^2$ in (13) now yields

$$(S_2^2 - 1)(S_3^2 - 1)(S_2^2 - S_3^2)(C_4C_5 - C_2C_3) = 0.$$ $S_2^2 = S_3^2$ leads to $S_2^2$, which is impossible. The only possibility is $C_4C_5 = C_2C_3$ as the other possibilities are special cases of this one. We obtain a solution.

b) Let $C_5 = C_2m$. The coefficient at $c_5s_2^2$ in (14) yields $a_2S_2 = S_5a_5m$, the coefficient at $c_5s_2^2$ in (11) yields $C_4 = C_3m$. Now we solve equations $r_2$ and $r_5$ for $c_4$ and $s_4$ and we obtain $s_4 = 0$. In the case that equations $r_2$ and $r_5$ are linearly dependent, we obtain the trivial case with all axes passing through one point. This shows that in this case we also have no solution.

D) Let $S_2 = 0$, $S_3 \neq 0$. We obtain equations

$$(17) \quad F_5c_4 + M_5s_4 = 0, \quad -C_3M_5c_4 + F_5C_3s_4 + L_5S_3 = 0.$$ Let at first $S_5 = 0$. Then $M_5 = S_4C_5 = 0$, and therefore $S_4 = 0$. We must have $L_5 = C_4C_5 = 0$, which is impossible. Therefore we must have $S_5 \neq 0$. If $S_4 \neq 0$, we have one of the previous cases for the inverse motion. Therefore we can suppose $S_4 = 0$. (17) implies $C_2C_3 = C_4C_5$. We compute $c_3$ and $s_3$ from equations $r_4$ and $r_6$ in (8) and consider the equation $c_3^2 + s_3^2 = 1$. This equation yields $a_4 = 0$ which is impossible and there is no solution in this case.

E) Let $S_2 = S_3 = 0$. Then $C_2 + C_3L_5 = 0$, so $C_2 = C_3C_4C_5$ and $S_4 = S_5 = 0$, a solution. All axes are parallel and we have a trivial solution.
F) Let $s_2 \neq 0$, $s_3 = 0$. If $s_4 \neq 0$, we obtain one of the previous cases by taking the inverse motion. So we can suppose $s_4 = 0$. We obtain $c_2 + l_5 = 0$, which yields $c_2 = c_4 c_5$. Therefore $s_5 \neq 0$. From equations $-\frac{1}{2} a_3 (m_5 c_4 - f_5 s_4) + b_5 - g_2 = 0$ and $m_5 c_4 - f_5 s_4 + s_2 c_3 = 0$ we obtain $a_4 c_4 s_5 = 0$, which is impossible and we have no solution in this case.

References


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