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MULTI-LEVEL METHOD IN EIGENVALUE PROBLEM

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Summary. One method for computing the least eigenvalue of a positive definite matrix A of order n is described.

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1. INTRODUCTION

The goal of this paper is to describe one possibility how to compute the least eigenvalue of a positive definite matrix A of order n. The method described is in fact the power method part of which is the choice of an appropriate starting approximation of the corresponding eigenvector. The starting approximation is determined by solution of a less dimensional generalized eigenvalue problem.

2. CONVERGENCE

Let us consider the space $\mathbb{R}_n$ with the usual scalar product (x, y) and the corresponding norm $\|x\| = \sqrt{(x,x)}$. Let $m < n$. Again the usual scalar product on $\mathbb{R}_m$ will be used. Let A be a positive definite operator on $\mathbb{R}_n$, $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_s$, $2 \leqslant s \leqslant n$ all mutually different eigenvalues of A. Let $p\colon \mathbb{R}_m \to \mathbb{R}_n$ be a linear injective operator, r the adjoint operator to p with respect to both scalar products.

Lemma 2.1. *Let*

$$\omega \in \Big(0, \frac{2}{\lambda_1 + \lambda_s}\Big), \quad \omega \neq \frac{1}{\lambda_j} \quad \textit{for all } j\ .$$

Then $\varrho(M) = 1 - \omega\lambda_1$ is the spectral radius of $M = I - \omega A$.

Proof. Due to the assumptions, $1 - \omega\lambda_1 > 0$. We must prove

$$|1 - \omega\lambda_j| < 1 - \omega\lambda_1 \quad \text{for } j = 2, \dots, s.$$

But this inequality is equivalent to

$$\omega\lambda_1 < \omega\lambda_j < 2 - \omega\lambda_1,$$
$$0 < \omega < \frac{2}{\lambda_1 + \lambda_j} \quad \text{for all } j.$$

The last inequality is the evident consequence of the assumption of Lemma 2.1 □

Definition 2.1. For any positive integer ν and any $v \in \mathbf{R}_m$, $v \neq 0$, let us define

$$Q(v, \nu) = \frac{\|M^{\nu+1}pv\|}{\|M^\nu pv\|},$$
$$\hat{Q}(\nu) = \max_{v \neq 0} Q(v, \nu),$$
$$\hat{V}(\nu) = \{\hat{v}, Q(\hat{v}, \nu) = \hat{Q}(\nu)\}.$$

Remark 2.1. $\hat{V}(\nu) \cup \{0\}$ is a linear space, for proof see Lemma 2.2.

Algorithm 2.1. for determining an approximation of λ_1 consists in the following steps: given ν, ξ positive integers

Step 1. determine $\hat{v}$ such that $Q(\hat{v}, \nu) = \hat{Q}(\nu)$—see Lemma 2.2.

Step 2. compute $\tilde{\lambda}_1 = \dfrac{1 - Q(\hat{v}, \nu + \xi)}{\omega}$.

ξ additional iterations are done in order to suppress remainders of components of $M^\nu p\hat{v}$ with high energy, i.e. $\|\cdot\|_A \gg \|\cdot\|$.

Definition 2.2. For two linear operators K, L on a linear space U let us define a generalized eigenvalue μ and a generalized eigenvector v corresponding to μ by

$$Kv = \mu Lv, \quad v \neq 0.$$

The set of all generalized eigenvalues will be denoted by $\sigma(K, L)$.

Remark 2.2. If K, L are positive definite operators on an euclidean space U then $\sigma(K, L)$ is real. For further properties of generalized eigenvalues see [1] p. 383, algorithms for computing can be found in [2].

Lemma 2.2. *$\hat{V}(\nu)$ is the set of all generalized eigenvectors corresponding to the maximal generalized eigenvalue from $\sigma(K_\nu, L_\nu)$, where*

$$K_\nu = rM^{2\nu+2}p, \quad L_\nu = rM^{2\nu}p.$$

Further this maximal generalized eigenvalue is equal to $\hat{Q}^2(\nu)$.

Proof. K_ν, L_ν are obviously positive definite, therefore

$$K_\nu v = \mu L_\nu v, \quad v \neq 0$$

if and only if

$$L_\nu^{-\frac{1}{2}} K_\nu L_\nu^{-\frac{1}{2}} x = \mu x, \quad x = L_\nu^{\frac{1}{2}} v$$

and hence

$$\sigma(K_\nu, L_\nu) = \sigma(L_\nu^{-\frac{1}{2}} K_\nu L_\nu^{-\frac{1}{2}}).$$

Further

$$Q^2(v,\nu) = \frac{(K_\nu v, v)}{(L_\nu v, v)} = \frac{(L_\nu^{-\frac{1}{2}} K_\nu L_\nu^{-\frac{1}{2}} x, x)}{(x,x)}$$

where $x = L_\nu^{\frac{1}{2}} v$. This equality completes the proof. □

Definition 2.3. For every $\lambda_i \in \sigma(A)$ let us denote by $H_i = \mathrm{Ker}(\lambda_i I - A)$, let $P_i\colon \mathbb{R}_n \to H_i$ be the orthogonal projector on H_i, $\mu_i = 1 - \omega\lambda_i$, $i = 1, 2, \ldots, s$.

Lemma 2.3. *Let $v \in \mathbb{R}_m$ be such that $P_1 pv \neq 0$. Then*

$$\lim_{\nu\to\infty} Q(v,\nu) = \varrho(M).$$

Proof. Due to Lemma 2.1 $\varrho(M) = \mu_1$. Further

$$pv = \sum_{i=1}^{s} P_i pv$$

and

$$Q^2(v,\nu) = \frac{(M^{2\nu+2}pv, pv)}{(M^{2\nu}pv, pv)} = \frac{\sum_{i=1}^{s} \mu_i^{2\nu+2} \|P_i pv\|^2}{\sum_{i=1}^{s} \mu_i^{2\nu} \|P_i pv\|^2}$$

$$= \frac{\mu_1^2 (\|P_1 pv\|^2 + \sum_{i=2}^{s} \left(\frac{\mu_i}{\mu_1}\right)^{2\nu+2} \|P_i pv\|^2)}{\|P_1 pv\|^2 + \sum_{i=2}^{s} \left(\frac{\mu_i}{\mu_1}\right)^{2\nu} \|P_i pv\|^2} \to \mu_1^2.$$

□

Lemma 2.4. *Let there exist $v \in \mathbb{R}_m$ such that $P_1 pv \neq 0$. Then*

$$\lim_{\nu\to\infty} \hat{Q}(\nu) = \varrho(M).$$

Proof. To prove the statement we use the evident inequality

$$Q(v,\nu) \leqslant \hat{Q}(\nu) \leqslant \varrho(M)$$

and Lemma 2.3. □

Lemma 2.5. *If K is a selfadjoint operator on a Hilbert space U then*

$$\frac{\|K^2x\|}{\|Kx\|} \geqslant \frac{\|Kx\|}{\|x\|} \quad \text{for every } x \in U \setminus \mathrm{Ker}(K).$$

Proof.

$$\|Kx\|^2 = (K^2x, x) \leqslant \|K^2x\| \cdot \|x\|.$$

□

Theorem 1. *Let there exist $v \in \mathbb{R}_m$ such that $P_1 pv \neq 0$. Then for every $\hat{v} \in \hat{V}(\nu)$ and every positive integer ξ*

$$\lim_{\nu\to\infty} \frac{1 - Q(\hat{v}, \nu+\xi)}{\omega} = \lambda_1.$$

Proof. At first we will prove the inequality

$$\hat{Q}(\nu) \leqslant Q(\hat{v}, \nu+\xi)$$

for every $\hat{v} \in \hat{V}(\nu)$ and every positive integer ξ. Using Lemma 2.5 and induction we get

$$Q(\hat{v}, \nu+\xi) = \frac{\|M^{\nu+\xi+1}p\hat{v}\|}{\|M^{\nu+\xi}p\hat{v}\|} \geqslant \frac{\|M^{\nu+1}p\hat{v}\|}{\|M^{\nu}p\hat{v}\|} = \hat{Q}(\nu).$$

To complete the proof it is sufficient to apply Lemma 2.4. □

3. Rate of convergence

Definition 3.1. Let us denote by

$$S_\nu = R(M^\nu p), \qquad T_\nu = \mathrm{Ker}(rM^\nu).$$

R e m a r k 3.1. It is not difficult to see that T_ν is the orthogonal complement of S_ν.

Lemma 3.1. *For every $\hat{v} \in \hat{V}(\nu)$*

$$[M^2 - \hat{Q}^2(\nu)I]M^\nu p\hat{v} \in T_\nu.$$

P r o o f. Due to Lemma 2.2 for every $\hat{v} \in \hat{V}(\nu)$ we have

$$(rM^{2\nu+2}p)\hat{v} = \hat{Q}^2(\nu)(rM^{2\nu}p)\hat{v}$$

which is equivalent to

$$rM^\nu[M^2 - \hat{Q}^2(\nu)I]M^\nu p\hat{v} = 0.$$

□

Lemma 3.2. *Let $\hat{v} \in \hat{V}(\nu)$, let there exist an $\alpha > 0$ such that*

$$\|[M^2 - Q^2(\hat{v}, \nu + \xi)I]M^{\nu+\xi}p\hat{v}\| \leqslant \alpha\|M^{\nu+\xi}p\hat{v}\|.$$

Then there exists $\mu_l \in \sigma(M)$ such that

$$|Q^2(\hat{v}, \nu + \xi) - \mu_l^2| \leqslant \alpha.$$

P r o o f. As

$$\|[M^2 - Q^2(\hat{v}, \nu + \xi)I]M^{\nu+\xi}p\hat{v}\| \geqslant \lambda_{min}(M^2 - Q^2(\hat{v}, \nu + \xi)I)\|M^{\nu+\xi}p\hat{v}\|$$

hence

$$\lambda_{min}(M^2 - Q^2(\hat{v}, \nu + \xi)I) \leqslant \alpha.$$

Since $\{\mu_l^2 - Q^2(\hat{v}, \nu+\xi)\}$ is the spectrum of $M^2 - Q^2(\hat{v}, \nu+\xi)I$ the lemma is proved.

□

Notation. Let U be a linear space, T a subspace of U. For every linear operator X on U we will denote by X_T the restriction of X on T.

Lemma 3.3. *Let $\hat{v} \in \hat{V}(\nu)$. Let us denote by*

$$b = \frac{\|M^{\nu+2}p\hat{v}\|}{\|M^{\nu}p\hat{v}\|}.$$

Then for every positive integer ξ

$$\frac{\|M^{\xi}[M^2 - \hat{Q}^2(\nu)I]M^{\nu}p\hat{v}\|^2}{\|M^{\nu+\xi}p\hat{v}\|^2} \leqslant \|M^{\xi}_{T_\nu}\|^2 \frac{b^2 - \hat{Q}^4(\nu)}{b^{\xi}},$$

$$b^2 \geqslant \hat{Q}^4(\nu).$$

P r o o f. Due to Lemma 3.1

$$[M^2 - \hat{Q}^2(\nu)I]M^{\nu}p\hat{v} \in T_{\nu},$$
$$M^{\nu}p\hat{v} \in S_{\nu} = T_{\nu}^{\perp}.$$

Therefore

$$\|[M^2 - \hat{Q}^2(\nu)I]M^{\nu}p\hat{v}\|^2 = \|M^{\nu+2}p\hat{v}\|^2 - \hat{Q}^4(\nu)\|M^{\nu}p\hat{v}\|^2 = [b^2 - \hat{Q}^4(\nu)]\|M^{\nu}p\hat{v}\|^2$$

which implies

$$b^2 \geqslant \hat{Q}^4(\nu).$$

Further

$$(3.1) \qquad \|M^{\xi}[M^2 - \hat{Q}^2(\nu)I]M^{\nu}p\hat{v}\|^2 \leqslant \|M^{\xi}_{T_\nu}\|^2 \, (b^2 - \hat{Q}^4(\nu))\|M^{\nu}p\hat{v}\|^2.$$

Using Lemma 2.5 and induction we will prove

$$(3.2) \qquad \frac{\|M^{\nu+\xi}p\hat{v}\|}{\|M^{\nu}p\hat{v}\|} \geqslant b^{\frac{\xi}{2}}.$$

For $\xi = 2$ (3.2) becomes the equality. Let us suppose (3.2) holds for ξ then

$$\frac{\|M^{\nu+\xi+1}p\hat{v}\|}{\|M^{\nu}p\hat{v}\|} \geqslant b^{\frac{\xi}{2}} \frac{\|M^{\nu+\xi+1}p\hat{v}\|}{\|M^{\nu+\xi}p\hat{v}\|}.$$

Let us set

$$A_1 = \frac{\|M^{\nu+1}p\hat{v}\|}{\|M^{\nu}p\hat{v}\|}, \; A_2 = \frac{\|M^{\nu+2}p\hat{v}\|}{\|M^{\nu+1}p\hat{v}\|}, \; A_3 = \frac{\|M^{\nu+\xi+1}p\hat{v}\|}{\|M^{\nu+\xi}p\hat{v}\|}$$

then

$$b = A_1A_2, \; A_3 \geqslant A_1, \; A_3 \geqslant A_2.$$

Therefore

$$A_3 \geqslant \sqrt{A_1A_2} = \sqrt{b}$$

and (3.2) is proved. Combining inequalities (3.1) and (3.2) we get the statement. □

Following inequalities are useful for estimating the operator norm $\|M^{\xi}_{T_\nu}\|$.

Lemma 3.4.

1. $\|M^{\xi}_{T_\nu}\| \leqslant \|M_{T_\nu}\| \cdot \|M_{T_{\nu-1}}\| \cdots \|M_{T_{\nu-\xi-1}}\|$,
2. *for every positive integer* i $\|M_{T_i}\| \leqslant \|M_{T_0}\|$.

Proof. From the definition of T_ν it follows $x \in T_\nu$ implies $M^i x \in T_{\nu-i}$. Further

$$\frac{\|M^{\xi}x\|}{\|x\|} = \frac{\|M(M^{\xi-1}x)\|}{\|M^{\xi-1}x\|} \cdots \frac{\|Mx\|}{\|x\|}.$$

Therefore

$$\|M^{\xi}_{T_\nu}\| \leqslant \|M_{T_{\nu-\xi+1}}\| \cdots \|M_{T_\nu}\|.$$

Now let $x \in T_i$ then $M^i x \in T_0$ and using Lemma 2.5 we get

$$\frac{\|M(M^i x)\|}{\|M^i x\|} \geqslant \frac{\|Mx\|}{\|x\|}$$

which yields 2. □

Corollary.

$$\|M^{\xi}_{T_\nu}\| \leqslant \|M_{T_{\nu-\xi+1}}\|^{\xi} \leqslant \|M_{T_0}\|^{\xi}.$$

Lemma 3.5. *There exists* $\mu_l \in \sigma(M)$ *such that*

$$|\mu_l^2 - Q^2(\hat{v}, \nu+\xi)| \leqslant \|M^{\xi}_{T_\nu}\| \, \frac{\sqrt{\varrho(M)^2 - Q^2(v,\nu)}}{Q^{\xi-1}(v,\nu)}$$

for every $v \in \mathbb{R}_m$, $v \neq 0$.

Proof. Using Lemmas 3.2 and 3.3 we get

$$|\mu_l^2 - Q^2(\hat{v}, \nu+\xi)| \leqslant \|M^{\xi}_{T_\nu}\| \, \frac{\sqrt{b^2 - \hat{Q}^4(\nu)}}{b^{\frac{\xi}{2}}}.$$

Due to Lemma 3.3

$$b \geqslant \hat{Q}^2(\nu) \geqslant Q^2(v,\nu)$$

for every $v \in \mathbb{R}_m$, $v \neq 0$. Further

$$b = \frac{\|M(M^{\nu+1})p\hat{v}\|}{\|M^{\nu+1}p\hat{v}\|} \cdot \frac{\|M^{\nu+1}p\hat{v}\|}{\|M^{\nu}p\hat{v}\|} \leqslant \varrho(M)\hat{Q}(\nu).$$

Combining these inequalities we get

$$|\mu_l^2 - Q^2(\hat{v}, \nu + \xi)| \leqslant \|M_{T_\nu}^\xi\| \frac{\sqrt{\varrho(M)^2 \hat{Q}^2(\nu) - \hat{Q}^2(\nu) Q^2(v, \nu)}}{Q^{\xi-1}(v, \nu) \hat{Q}^2(\nu)}$$

and

$$|\mu_l^2 - Q^2(\hat{v}, \nu + \xi)| \leqslant \|M_{T_\nu}^\xi\| \frac{\sqrt{\varrho(M)^2 - Q^2(v, \nu)}}{Q^{\xi-1}(v, \nu)}.$$

□

Remark 3.2.

1. $\varrho(M)$ in the estimates above can be replaced by 1.
2. The estimate in Lemma 3.5 is the better, the better approximation of an eigenvector belonging to $\varrho(M)$ pv is.

Convergence of the algorithm (2.1) was proved in Theorem 1. Estimates of the rate of convergence are given in the following theorem.

Theorem 2. *There exists $\lambda_l \in \sigma(A)$ such that*

$$|\tilde{\lambda}_1 - \lambda_l| \leqslant \|M_{T_\nu}^\xi\| \cdot \frac{\sqrt{(1 - \omega\lambda_1)^2 - Q^2(v, \nu)}}{\omega Q^{\xi-1}(v, \nu) |1 - \omega\lambda_s + Q(v, \nu)|} \tag{3.3}$$

for every $v \in \mathbb{R}_m$ for which the right hand side is defined.

Proof. We will use Lemma 3.5. As $\mu_l = 1 - \omega\lambda_l$

$$\begin{aligned} |\mu_l^2 - Q^2(\hat{v}, \nu + \xi)| &= \omega \Big| \frac{1}{\omega}(1 - Q(\hat{v}, \nu + \xi)) - \lambda_l \Big| \cdot |1 - \omega\lambda_l + Q(\hat{v}, \nu + \xi)| \\ &\geqslant \omega |\tilde{\lambda}_1 - \lambda_l| \cdot |1 - \omega\lambda_s + Q(v, \nu)| \end{aligned}$$

which yields the statement. □

Remark 3.3.

1. λ_s in (3.3) can be estimated using the Gershgorin theorem.
2. λ_1 in (3.3) can be estimated by 0.

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