

Applications of Mathematics

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Applications of Mathematics, Vol. 39 (1994), No. 6, 445--447

Persistent URL: <http://dml.cz/dmlcz/134270>

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ON KALMBACH MEASURABILITY

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(Received August 16, 1993)

Summary. In this note we show that, for an arbitrary orthomodular lattice L , when μ is a faithful, finite-valued outer measure on L , then the Kalmbach measurable elements of L form a Boolean subalgebra of the centre of L .

Keywords: Kalmbach measurability, Boolean algebra, orthomodular lattice

AMS classification: 28B, D6E, D6C

1. INTRODUCTION

In non-commutative measure theory, which is being developed because of the desire to investigate the mathematical foundations of quantum mechanics, see [1], [3], [6] and [7], one replaces the notion of a Boolean algebra by the notion of an orthomodular lattice.

In [5] Kalmbach considers outer measures defined on an orthomodular lattice L and extends the Caratheodory notion of measurability to this general setting. She proves that when L is a dimension lattice then the Kalmbach measurable elements form a Boolean algebra.

In this note we show that, for an arbitrary orthomodular lattice L , when μ is a faithful, finite-valued outer measure on L , then the Kalmbach measurable elements of L form a Boolean subalgebra of the centre of L . Throughout this note L will be an orthomodular lattice. Our standard references for orthomodular lattices are [2], [4].

We shall define a function $\mu: L \rightarrow [0, +\infty)$ to be a *finitely additive outer measure* if the following conditions are satisfied:

- (i) $\mu(0) = 0$,
- (ii) $\mu(p \vee q) \leq \mu(p) + \mu(q)$ whenever $p \perp q$,
- (iii) if $p \leq q$ then $\mu(p) \leq \mu(q)$.

In [5] a stronger condition than (ii) is imposed. The outer measures defined in [5] are clearly finitely additive outer measures. The converse is false, in general. If $\mu(p) = 0$ implies $p = 0$ then μ is said to be *faithful*. If $\mu(p) < +\infty$ for all p , then μ is said to be *finite valued*.

2. KALMBACH MEASURABLE SETS

Let μ be an outer measure on L . Then, see [5], $f \in L$ is said to be *Kalmbach measurable* (with respect to μ) if

$$(*) \quad \mu(x) = \mu(x \wedge (x' \vee f)) + \mu(x \wedge (x' \vee f'))$$

for each x in L .

More generally, whenever μ is a function from L to an abelian group G , we may define $f \in L$ to be *Kalmbach measurable* if $(*)$ holds for each x in L . The reader whose primary interest is in real valued measures may interpret G as the additive group of reals.

Theorem. *Let L be an orthomodular lattice and let G be an abelian group. Let μ be a G -valued function on L such that $\mu(a) = 0$ precisely when $a = 0$. Then the Kalmbach measurable elements of L form a Boolean subalgebra, B , of the centre of L . Furthermore the restriction of μ to B is additive.*

Proof. Let f be a Kalmbach measurable element and let e be an element of L . We put

$$e_0 = (e' \vee f') \wedge (e' \vee f) \wedge e$$

and we observe that

$$\begin{aligned} e'_0 \vee f' &= (e \wedge f) \vee (e \wedge f') \vee e' \vee f' = 1, \\ e'_0 \vee f &= (e \wedge f) \vee (e \wedge f') \vee e' \vee f = 1. \end{aligned}$$

From the Kalmbach measurability of f it follows that

$$\mu(e_0) = \mu(e_0 \wedge (e'_0 \vee f')) + \mu(e_0 \wedge (e'_0 \vee f)) = \mu(e_0) + \mu(e_0).$$

Then $\mu(e_0) = 0$ and hence $e_0 = 0$.

We now write the upper commutator of e and f :

$$\begin{aligned} &(e' \vee f') \wedge (e' \vee f) \wedge (e \vee f) \wedge (e \vee f') = \\ &\left(((e' \vee f') \wedge (e' \vee f) \wedge e) \vee ((e' \vee f') \wedge (e' \vee f) \wedge f) \right) \wedge \\ &\left(((e' \vee f') \wedge (e' \vee f) \wedge e) \vee ((e' \vee f') \wedge (e' \vee f) \wedge f') \right) = \\ &(e' \vee f') \wedge (e' \vee f) \wedge f \wedge f' = 0. \end{aligned}$$

Thus e and f commute and we have proved that every Kalmbach measurable element is in the centre of L . It follows that an element p of L is Kalmbach measurable if

- (i) p is central,
- (ii) $\mu(x) = \mu(x \wedge p) + \mu(x \wedge p')$ for every x in L .

Suppose f and g are both Kalmbach measurable. Then $f \vee g$ is central and x , f and g are mutually commutative. The equality

$$\mu(x \wedge (f \vee g)) + \mu(x \wedge (f \vee g)') = \mu(x)$$

can be proved as in classical measure theory. □

Corollary. *Let L be an orthomodular lattice. Let $\mu: L \rightarrow [0, +\infty)$ be a faithful finite additive outer measure. Then the elements of L which are Kalmbach measurable with respect to μ form a Boolean sublattice of the centre of L .*

References

- [1] *E. Beltrametti, G. Cassinelli*: The logic of quantum mechanics. Addison Wesley, Reading MA, 1981.
- [2] *L. Beran*: Orthomodular Lattices. Algebraic approach. Academia, Prague and Reidel, Dordrecht, 1984.
- [3] *S. Gudder*: Stochastic methods in quantum mechanics. North-Holland, Amsterdam, 1979.
- [4] *G. Kalmbach*: Orthomodular lattices. Academic Press, London, 1983.
- [5] *G. Kalmbach*: Quantum measure spaces. Foundations of Phys. *20* (1990), 801–821.
- [6] *P. Pták, S. Pulmannova*: Orthomodular structures as quantum logics. Kluwer, Dordrecht, 1991.
- [7] *V. S. Varadarajan*: Geometry of quantum theory. Springer-Verlag, Berlin, 1985.

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